Symmetry in Physics, Problem Sheet 1

For simplicity, we work in the natural system of units where $\hbar = c = 1$.

1. Show that any linear operator $\hat{A}$ can be written in the form

   $$\hat{A} = \hat{A}_R + i\hat{A}_I \,,$$

   where $\hat{A}_R$ and $\hat{A}_I$ are hermitian operators.

2. A linear operator is *normal* if and only if $[\hat{A}, \hat{A}^\dagger] = 0$. Show that, if $\hat{A}$ is normal, it has a complete orthonormal set of eigenstates.

   Hint: Use the decomposition of $\hat{A}$ introduced in question 1.

3. In this problem we study a simplified model of the propagation of K^0 and $\overline{K^0}$. We start by representing their states as kets in Dirac notation:

   $$|K^0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,, \qquad |\overline{K^0}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \,.$$

   The Schrödinger equation for a superposition of of K^0 and $\overline{K^0}$

   $$|\psi(t)\rangle = \psi_1(t)|K^0\rangle + \psi_2(t)|\overline{K^0}\rangle = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}$$

   has the form

   $$i\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \,,$$

   where $\hat{H}$ has the general form

   $$\hat{H} = \hat{M} - \frac{i}{2}\hat{\Gamma} \,.$$

   where $\hat{M}$ and $\hat{\Gamma}$ are two hermitian operators. We assume that $\hat{M}$ and $\hat{\Gamma}$ have the form

   $$\hat{M} = \begin{pmatrix} m_K & M \\ M & m_K \end{pmatrix} \,, \qquad \hat{\Gamma} = \begin{pmatrix} \gamma & \Gamma \\ \Gamma & \gamma \end{pmatrix} \,,$$

   with M and Γ real numbers.

   (a) Compute the eigenvalues of $M_{1,2} - i/2\,\Gamma_{1,2}$ of $\hat{H}$.

   (b) We identify K_1 with the particle with the larger width $\Gamma_1 > \Gamma_2$ (i.e. the shorter lifetime). From experimental data we have

   $$\begin{aligned} &\frac{M_1 + M_2}{2} \simeq 500\,\text{MeV} \,, \\ &\Delta M \equiv M_1 - M_2 = -3.5 \times 10^{-6}\,\text{eV} \,, \\ &\Gamma_1 \simeq 1.1 \times 10^{10}\,\text{s}^{-1} \,, \\ &\Gamma_2 \simeq 1.9 \times 10^{7}\,\text{s}^{-1} \,. \end{aligned}$$

   Introducing $\Delta\Gamma \equiv \Gamma_1 - \Gamma_2$, what conclusions can we infer for m_K, γ, M and Γ?

(c) If we perform a CP transformation on K^0 and $\overline{K^0}$ we obtain

$$CP|K^0\rangle = -|\overline{K^0}\rangle\,, \qquad CP|\overline{K^0}\rangle = -|K^0\rangle\,.$$

It is found experimentally that the state with the largest width is CP-even. Use this information to ultimately constrain M and Γ, and determine $|K_1\rangle$ and $|K_2\rangle$.

(d) At the time $t = 0\,\mathrm{s}$ the system is a beam composed of K^0 particles. What is the probability of finding a $\overline{K^0}$ at a time $t > 0\,\mathrm{s}$? Express your result in terms of Γ_1, Γ_2 and ΔM. Can you insert the actual numbers and plot the probability as a function of t?

4. We know from data that CP is not conserved in kaon decays. Then, the actual states $|K_L\rangle$ and $|K_S\rangle$ that we observe are linear combinations of $|K_1\rangle$ and $|K_2\rangle$, as follows

$$|K_S\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}\left(|K_1\rangle + \epsilon|K_2\rangle\right)\,,$$
$$|K_L\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}\left(|K_2\rangle + \epsilon|K_1\rangle\right)\,,$$

where ϵ is a *complex* number, and $|\epsilon| \simeq 2.3 \times 10^{-3}$.

(a) Are $|K_S\rangle$ and $|K_L\rangle$ orthonormal?

(b) Let us denote with $\mathcal{A}(K^0 \to \pi^0\pi^0)$ the amplitude for the decay process $K^0 \to \pi^0\pi^0$. We define similarly the amplitudes $\mathcal{A}(K^0 \to \pi^+\pi^-)$, $\mathcal{A}(\overline{K^0} \to \pi^0\pi^0)$, $\mathcal{A}(\overline{K^0} \to \pi^+\pi^-)$, and for all other processes. We assume that the interaction giving these decays conserves CP, which implies (why?)

$$\mathcal{A}(K^0 \to \pi\pi) = -\mathcal{A}(\overline{K^0} \to \pi\pi)\,.$$

We define the two ratios of amplitudes

$$\eta_{00} \equiv \frac{\mathcal{A}(K_L \to \pi^0\pi^0)}{\mathcal{A}(K_S \to \pi^0\pi^0)}\,, \qquad \eta_{+-} \equiv \frac{\mathcal{A}(K_L \to \pi^+\pi^-)}{\mathcal{A}(K_S \to \pi^+\pi^-)}\,.$$

Express η_{00} and η_{+-} in terms of ϵ and comment on your result.

(c) We now consider the semileptonic decays $K^0 \to \pi^- e^+ \nu_e$ and $\overline{K^0} \to \pi^+ e^- \bar{\nu}_e$. The same processes with opposite sign of the lepton charges are forbidden. Consider the ratio

$$\delta_L = \frac{|\mathcal{A}(K_L \to \pi^- e^+ \nu_e)|^2 - |\mathcal{A}(K_L \to \pi^+ e^- \bar{\nu}_e)|^2}{|\mathcal{A}(K_L \to \pi^- e^+ \nu_e)|^2 + |\mathcal{A}(K_L \to \pi^+ e^- \bar{\nu}_e)|^2}\,,$$

and express it in terms of ϵ.