Symmetry in Physics, Problem Sheet 2

For simplicity, we work in the natural system of units where $\hbar = c = 1$.

1. A unitary operator $\hat{U}$ satisfies $\hat{U}^\dagger \hat{U} = \mathbb{1}$. Unitary operators are used to represent symmetry transformations in Quantum Mechanics. Consider a state $|\psi\rangle$, solution of the Schrödinger equation
$$i\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle \,,$$
where $\hat{H}$ is the Hamiltonian for the system. Show that $|\psi'\rangle \equiv U|\psi\rangle$ is also a solution of the Schrödinger equation if and only if $\hat{U}\hat{H}\hat{U}^\dagger = \hat{H}$.

   Note that, if $\hat{U}$ represents a continuous symmetry operation, then $\hat{U} = \exp[-i\alpha\hat{T}]$, where $\hat{T}$ is a hermitian operator. The condition $\hat{U}\hat{H}\hat{U}^\dagger = \hat{H}$ implies $[\hat{T}, \hat{H}] = 0$, i.e. the observable associated with $\hat{T}$ is conserved.

2. Consider the unitary parity operator $\hat{P}$, defined in such a way that $\hat{P}\hat{\vec{x}}\hat{P}^\dagger = -\hat{\vec{x}}$.

   (a) Show that $\hat{P}^2 = \mathbb{1}$ (up to a phase). What can we say about $\hat{P}^\dagger$?

   (b) What are the eigenvalues of $\hat{P}$?

3. Consider the Pauli matrices
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

   (a) Using the conventions $[A, B] = AB - BA$, $\{A, B\} = AB + BA$, show that the matrices $\frac{\sigma_i}{2}$ satisfy the commutation relations
$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\,\epsilon_{ijk}\,\frac{\sigma_k}{2} \,,$$
   where ϵ_{ijk} is totally antisymmetric and $\epsilon_{123} = +1$. Show also that
$$\left\{\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right\} = \frac{\mathbb{1}}{2}\,\delta_{ij} \,,$$
   with $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ denoting the two-dimensional identity matrix.

   (b) Consider the matrix $G \equiv \exp(-\frac{i}{2}\theta\sigma_3)$, dependent on the real parameter θ. Show by explicit computation that
$$\exp\left(-\frac{i\theta}{2}\sigma_3\right) = \mathbb{1}\cdot\cos(\theta/2) - i\,\sigma_3\,\cdot\sin(\theta/2) \,.$$

   Perform the above transformation using $\theta = 2\pi$ and $\theta = 4\pi$, respectively, and compare your result to ordinary three-dimensional rotations by an angle θ about the 3-axis.

(c) Consider the matrix $U(\vec{\theta}) \equiv \exp(-\frac{i}{2}\theta_k\sigma_k)$ with real parameters $\vec{\theta} \equiv (\theta_1, \theta_2, \theta_3)$. Show, either by explicit computation or with the help of part (b) and symmetry arguments, that

$$\exp\left(-\frac{i\theta_k}{2}\sigma_k\right) = \mathbb{1} \cdot \cos(\theta/2) - i\,(\hat{\theta}_k\sigma_k) \cdot \sin(\theta/2)\,.$$

Here, $\hat{\theta} = \vec{\theta}/\theta$ is the unit vector in the direction of $\vec{\theta}$, and $\theta \equiv |\vec{\theta}|$.

4. Consider the set of matrices of the form $U(\vec{\theta}) \equiv \exp(-\frac{i}{2}\theta_k\sigma_k)$.

   (a) Using the known relation $\det(\exp(A)) = \exp(\mathrm{Tr}A)$, show that any $U(\vec{\theta})$ has unit determinant. What can we say about the determinant of the product $U(\vec{\theta}_1)U(\vec{\theta}_2)$?

   (b) From question 2, we have seen that $U(\vec{\theta}) = a_0\mathbb{1} + ia_k\sigma_k$, where a_0, a_1, a_2, a_3 are real numbers. Compute $|a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2$. Imagine that (a_0, a_1, a_2, a_3) are points in a four-dimensional space. What surface do they describe while $\vec{\theta}$ varies?

   (c) Show that $[U(\vec{\theta})]^{-1} = [U(\vec{\theta})]^\dagger = U(-\vec{\theta})$.
   These properties, and the fact that $U(\vec{0}) = \mathbb{1}$ imply that the matrices $U(\theta)$ form the *group* of 2×2 unitary matrices with unit determinant, a.k.a. $SU(2)$.

5. Given a three-dimensional vector $\vec{v} = (v_1, v_2, v_3)$, we construct the 2×2 matrix $\bar{v} = v_i\sigma_i$, with $\sigma_i, i = 1, 2, 3$ the three Pauli matrices, as follows

$$\bar{v} = \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix}$$

   (a) Show that $\vec{v}^{\,2} = -\det(\bar{v})$. Then show that, for any two vectors $\vec{v}$ and $\vec{w}$,

$$\vec{v} \cdot \vec{w} = \frac{1}{4}\left[\det(\bar{v} - \bar{w}) - \det(\bar{v} + \bar{w})\right]\,.$$

   (b) Using the properties of Pauli matrices, show that, for any matrix $U \in SU(2)$, the matrix

$$\bar{v}' = U\,\bar{v}\,U^\dagger\,,$$

   can be written in the form $\bar{v}' = v'_i\sigma_i$, where

$$v'_i = \Omega_{ij}\,v_j\,, \qquad \Omega_{ij} = \frac{1}{2}\mathrm{Tr}\left[\sigma_i U \sigma_j U^\dagger\right]\,.$$

   Hint. Any 2×2 *complex matrix* M *can be written as* $M = M_0\mathbb{1} + M_i\sigma_i$.

   (c) Show that Ω is an orthogonal transformation, i.e. if $\vec{v}' = \Omega\vec{v}$ and $\vec{w}' = \Omega\vec{w}$, then $\vec{v}' \cdot \vec{w}' = \vec{v} \cdot \vec{w}$. What kind of ortogonal transformation is Ω?