Symmetry in Physics, Problem Sheet 3

For simplicity, we work in the natural system of units where $\hbar = c = 1$.

1. It is customary to solve Maxwell's equations using the vector potential $\vec{A}$ and the scalar potential ϕ, i.e. setting

$$\vec{B} = \vec{\nabla} \times \vec{A}\,, \qquad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\,.$$

Show that the following *gauge* transformations of $\vec{A}$ and $\vec{\phi}$

$$A \to \vec{A}' = \vec{A} + \vec{\nabla}\alpha\,, \qquad \phi \to \phi' = \phi - \frac{\partial \alpha}{\partial t}\,,$$

with $\alpha = \alpha(\vec{x}, t)$, leave $\vec{E}$ and $\vec{B}$ unchanged.

2. (a) Show that the (full, time-dependent) Schrödinger equation for a free particle is invariant under a "global" transformation $\psi \to \psi'$, where $\psi'(\vec{x}, t) = e^{i\alpha}\psi(\vec{x}, t)$, of the phase of the wave function, where α is an arbitrary real number. That is, show that ψ' satisfies the Schrödinger equation if and only if ψ does.

   (b) Now consider a "local" transformation $\psi' = e^{iq\alpha}\psi$, where $\alpha(\vec{r}, t)$ is an arbitrary function. Find the Schrödinger equation for the transformed wave funtion ψ' – it is no longer the free Schrödinger equation.

   (c) Show that if one modifies the Schrödinger equation by the substitution

$$\hat{\vec{p}} \to \hat{\vec{p}} - q\vec{A}, \qquad i\frac{\partial}{\partial t}\psi \to \left(i\frac{\partial}{\partial t} - q\phi\right)\psi\,, \tag{1}$$

   then the S.E. is invariant under the combined transformations

$$\psi \to \psi' = e^{iq\alpha}\psi\,, \qquad \vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\alpha\,, \qquad \phi \to \phi' = \phi - \frac{\partial \alpha}{\partial t}\,, \tag{2}$$

   i.e. a local change of phase accompanied by a gauge transformation on the vector and scalar potential. The modified Schrödinger equation is precisely the one that is postulated for a particle in an electromagnetic field. Introducing interactions in this way is called the *gauge principle*.