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FQH-Based Topological Quantum Computer: Materials, Devices & Algorithms

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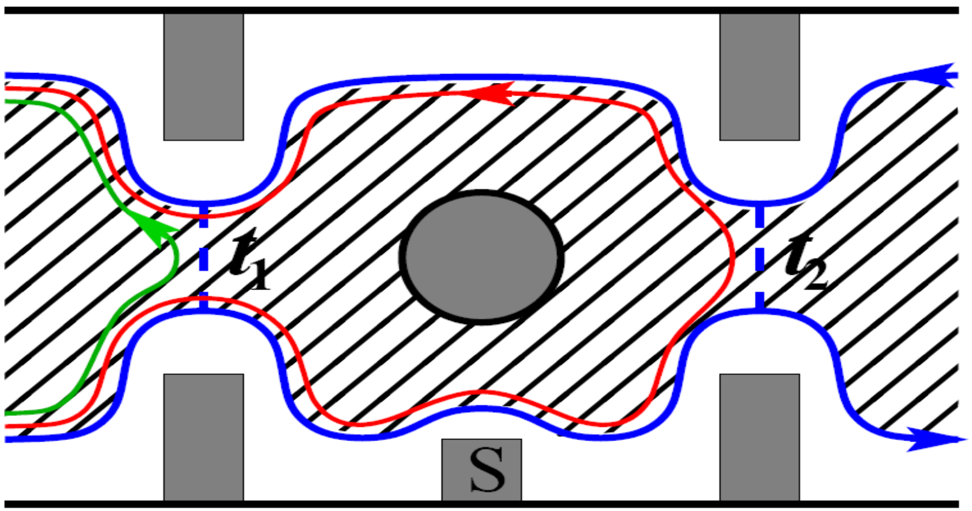
Motivation

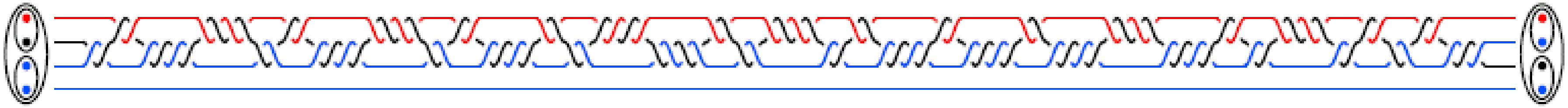
- Goal: “Engineering the topological quantum processor”.

- Materials
  - 2DEG in GaAs/AlGaAs
  - Graphene



- Devices
  - Quantum point contact
  - Interferometer (double quantum point contact)

- Engineering quantum gates – algorithms only



Nobel Laureates Said …

- Technology evokes new physics

  “It is frequently said that having a more or less specific practical goal in mind will degrade the quality of research. I do not believe that this is necessarily the case and to make my point in this lecture I have chosen my examples of the new physics of semiconductors from research projects which were very definitely motivated by practical considerations.”

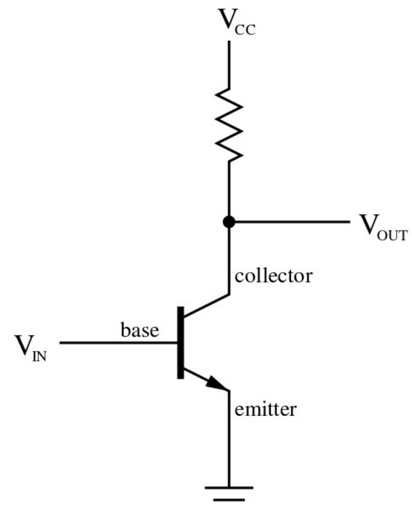
  -- William Shockley, Nobel Lecture, Dec. 11, 1956

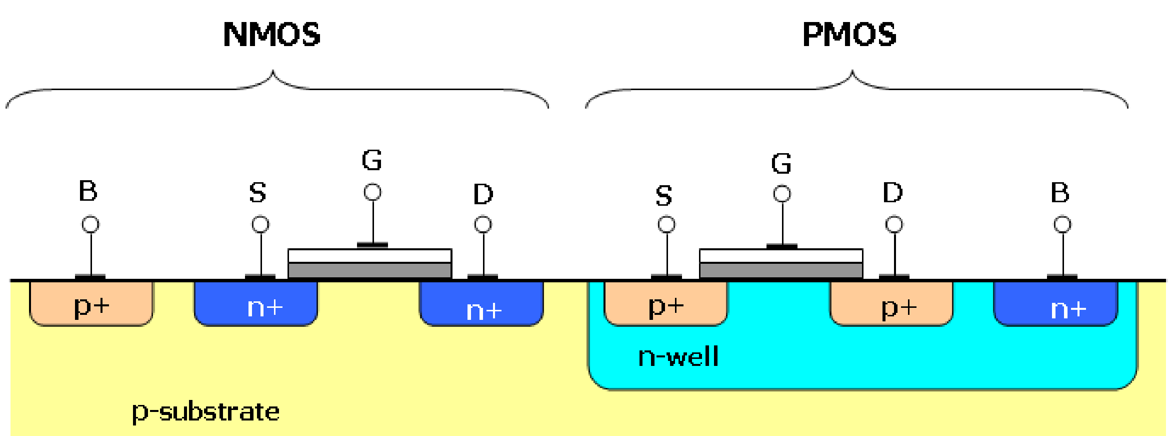
- Futuristic, but not crazy

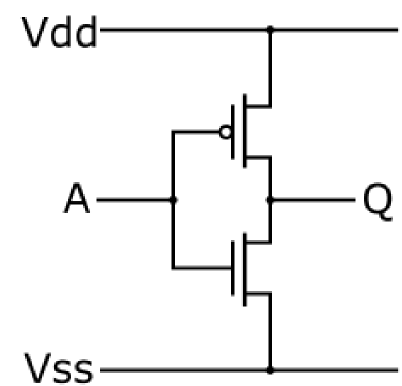
  [Frank] Wilczek also notes a number of new proposals to look for more exotic anyon states of FQH systems that could form the basis for quantum computers. Such ideas are “futuristic,” he says, "but not as crazy as they used to be.“

  -- Phys. Rev. Focus 16, 14 (2005)

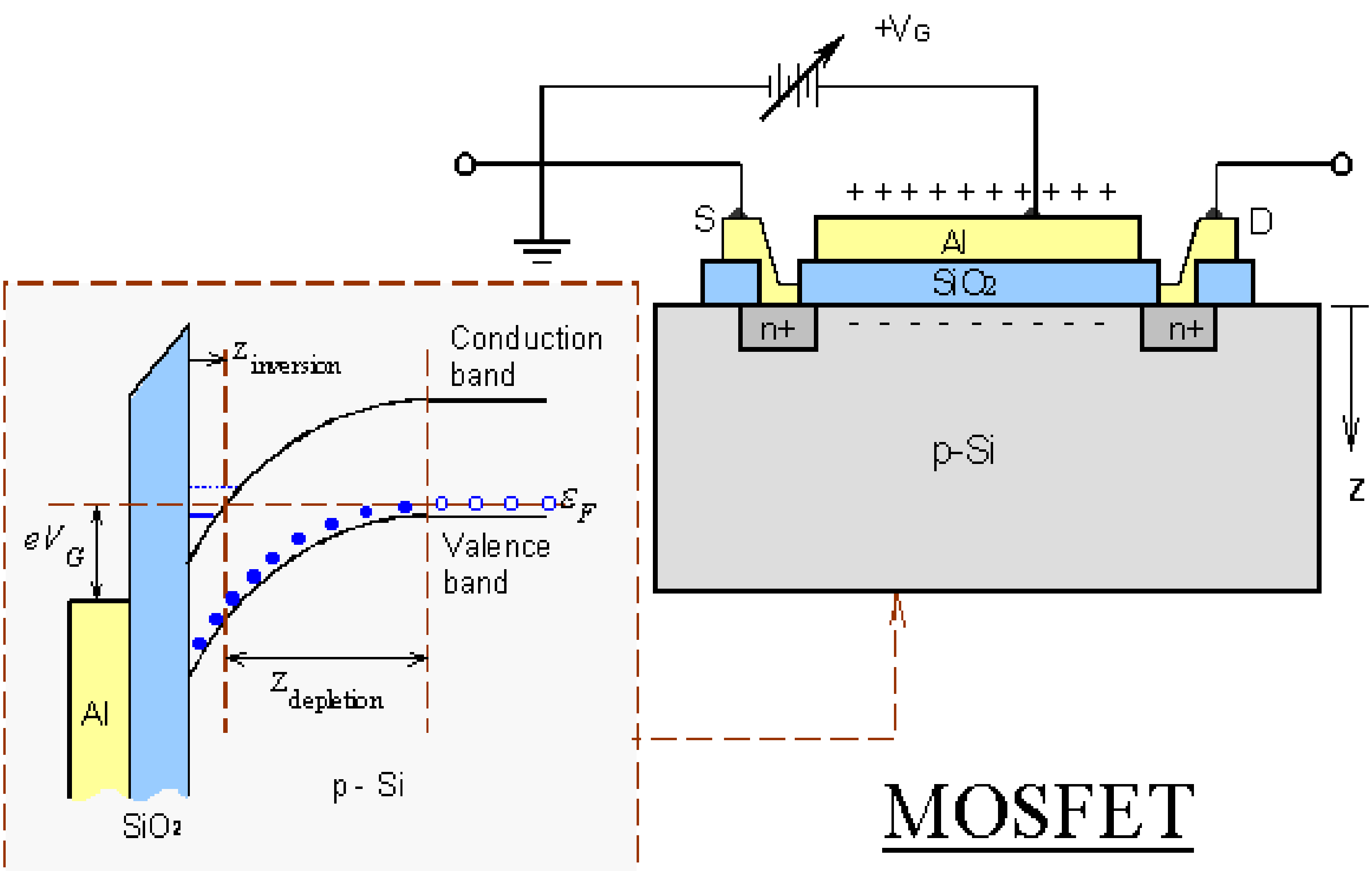
From Ge Transistor to Si CMOS





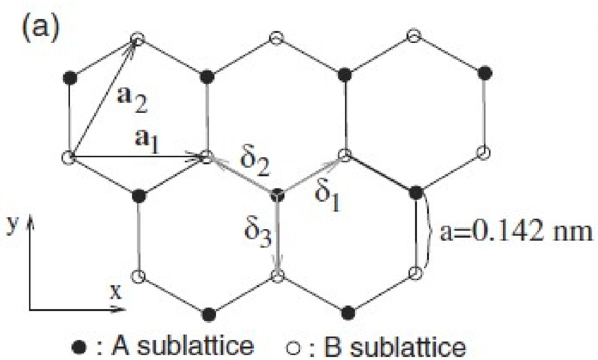


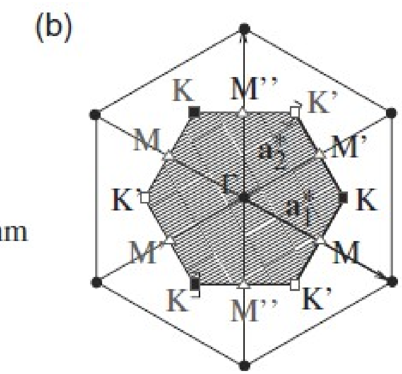
Metal-Oxide-Semiconductor Field Effect Transistor

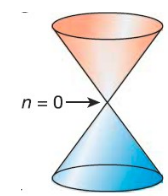


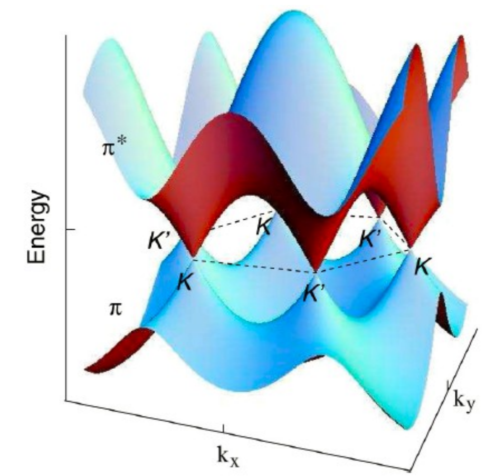
MOSFET

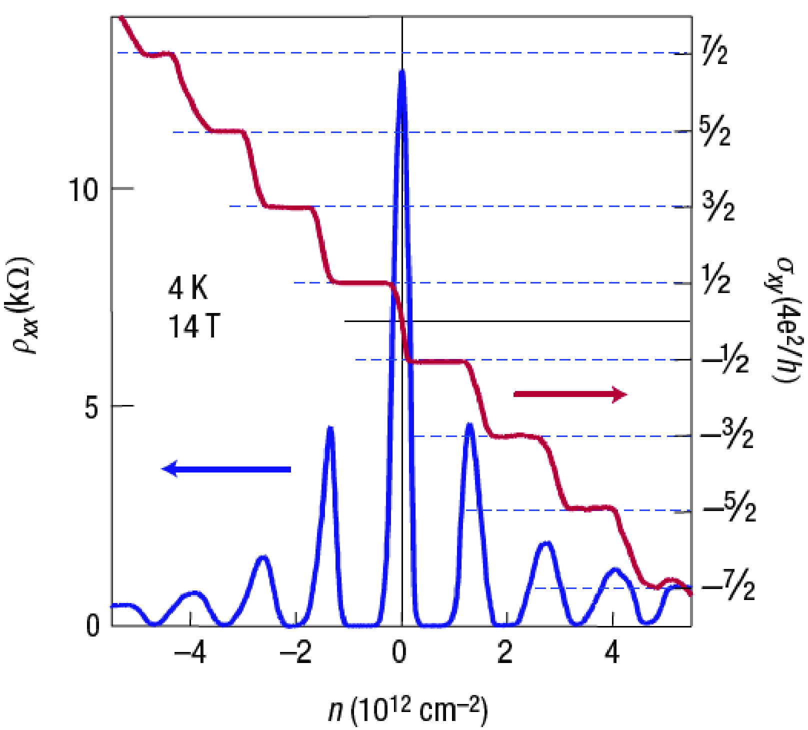
QHE in Graphene

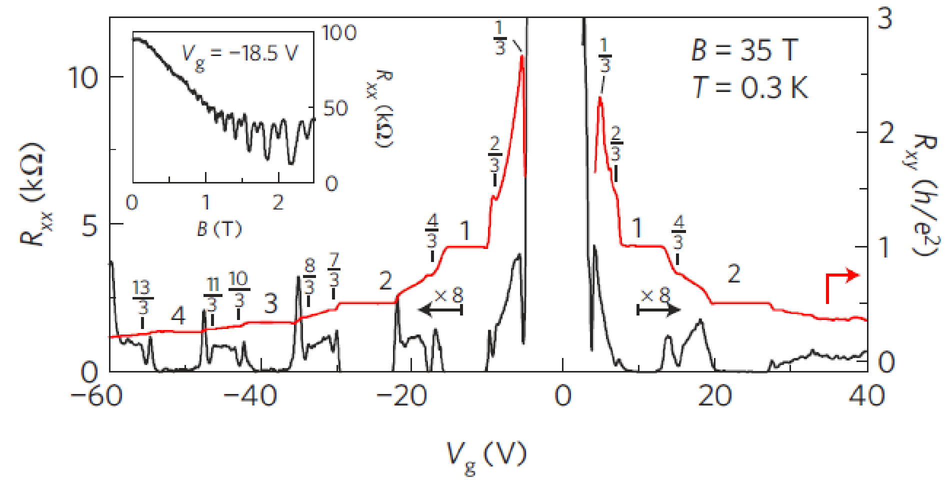








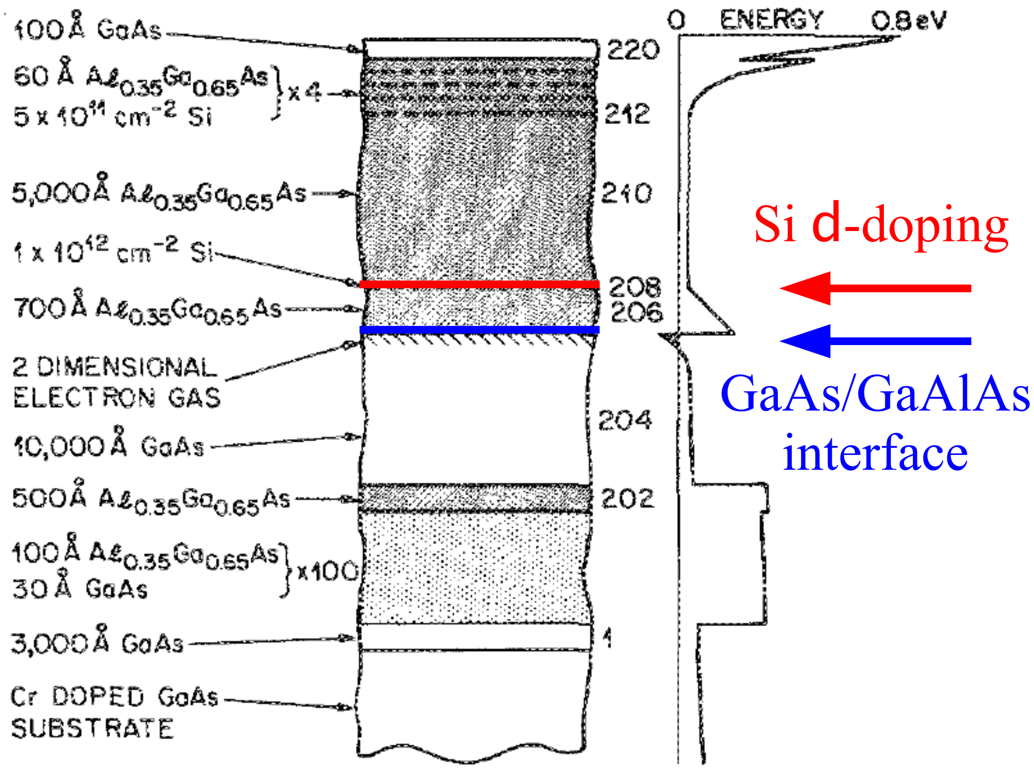


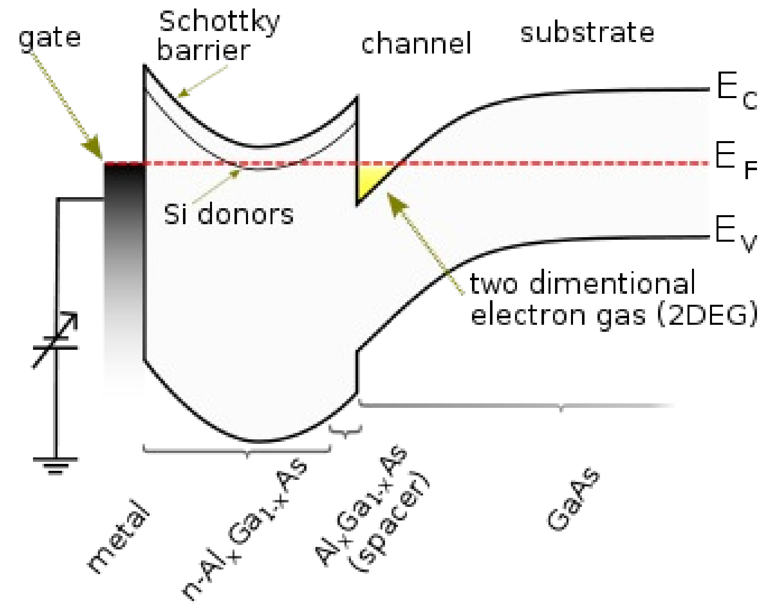


Novoselov *et al*., Nature (2005);
Zhang *et al*., Nature (2005)

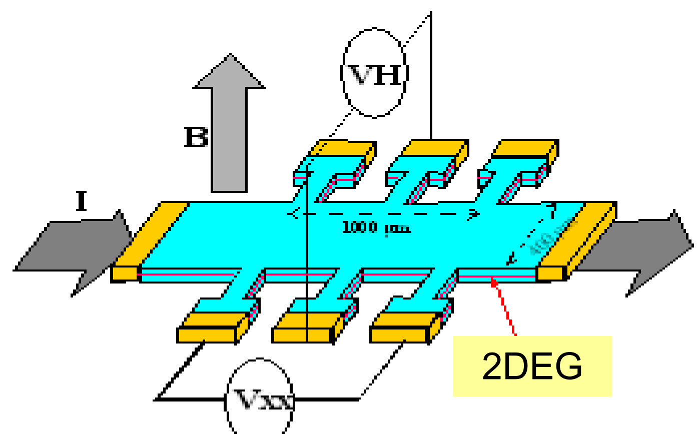
Dean et al., Nature Physics (2011);
Du et al. Nature (2009); Bolotin et al., ibid. (2009)

Two-Dimensional Electron Gas





Pfeiffer et al., Appl. Phys. Lett. **55**, 18 (1989)



Fractional Quantum Hall Effect (1982)

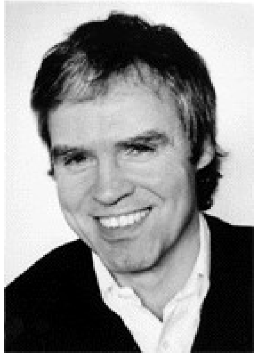
- High quality sample
- Low temperature
- High magnetic field

$$R_H = \frac{V_H}{I} = \frac{h}{\nu e^2}$$

$$R = \frac{V_{xx}}{I}$$

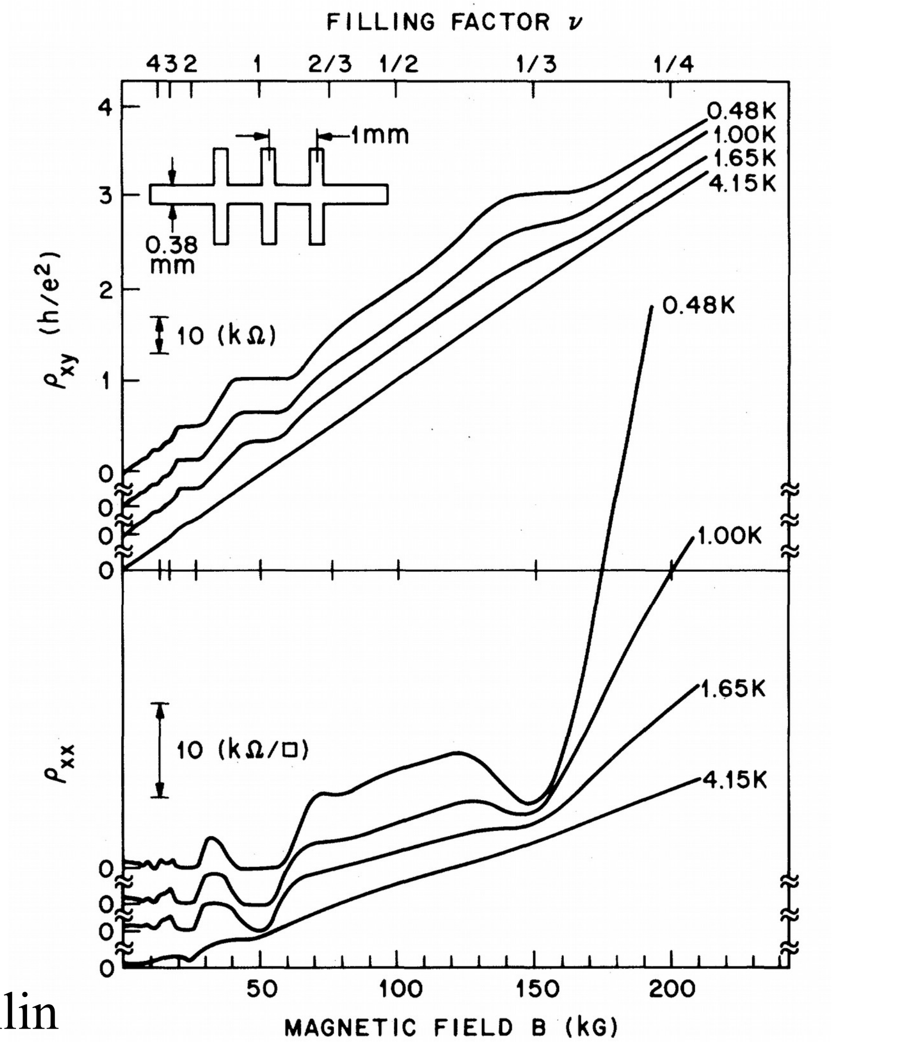
Fractional filling factor:
interaction important!

Daniel C. Tsui



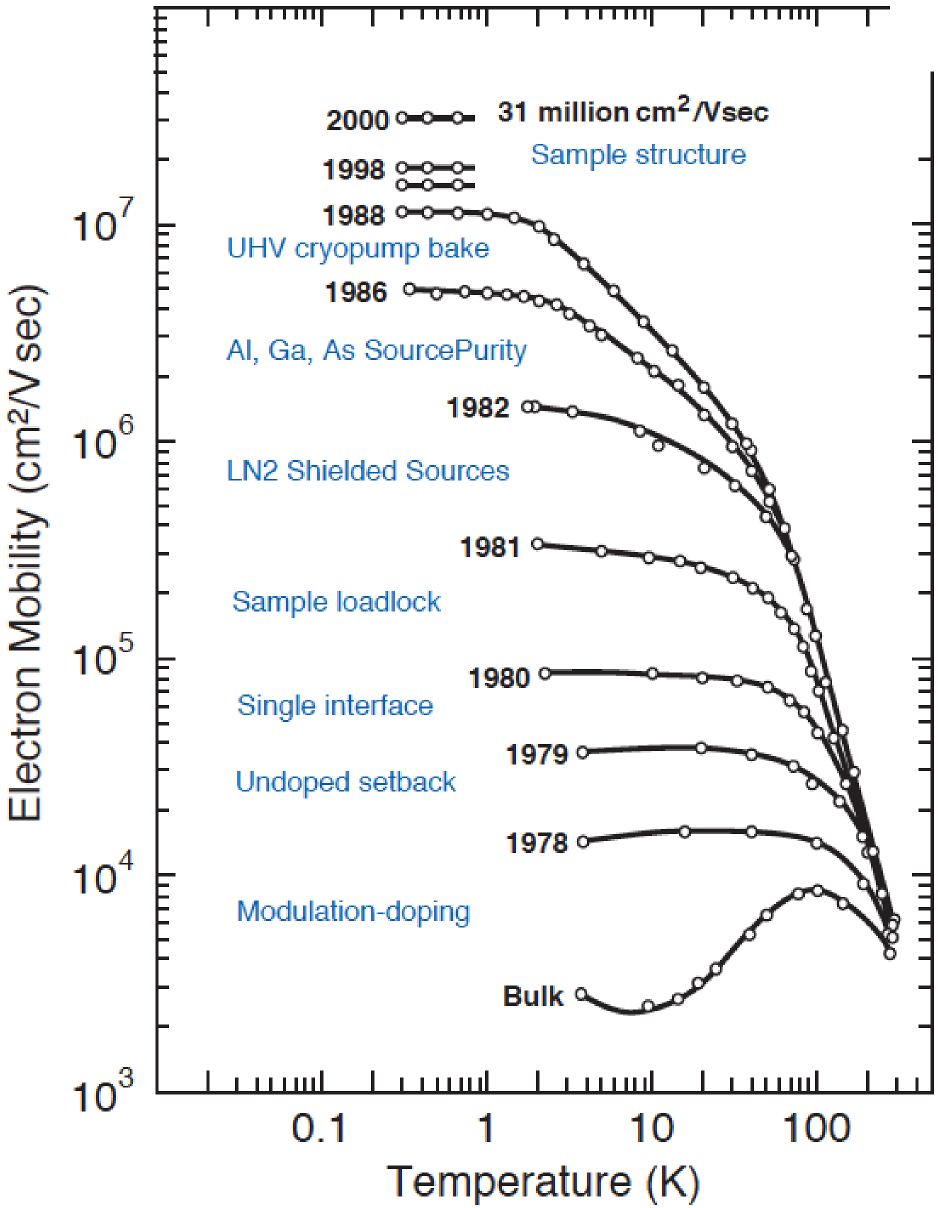
Horst L. Störmer

Robert B. Laughlin

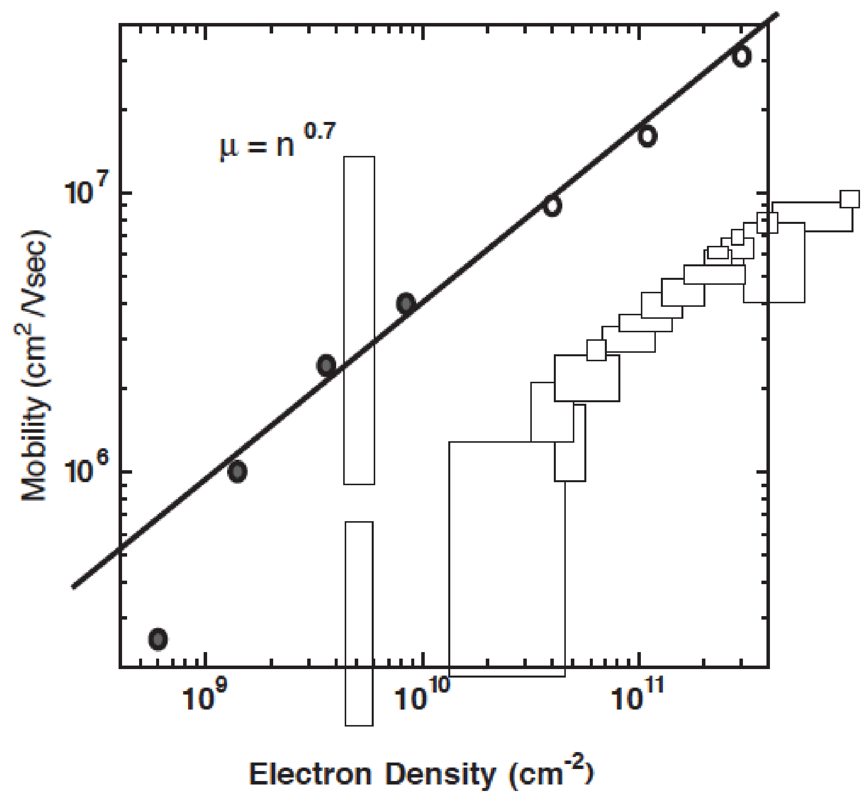


Nobel Prize 1998: "for their discovery of a new form of quantum fluid with fractionally charged excitations."

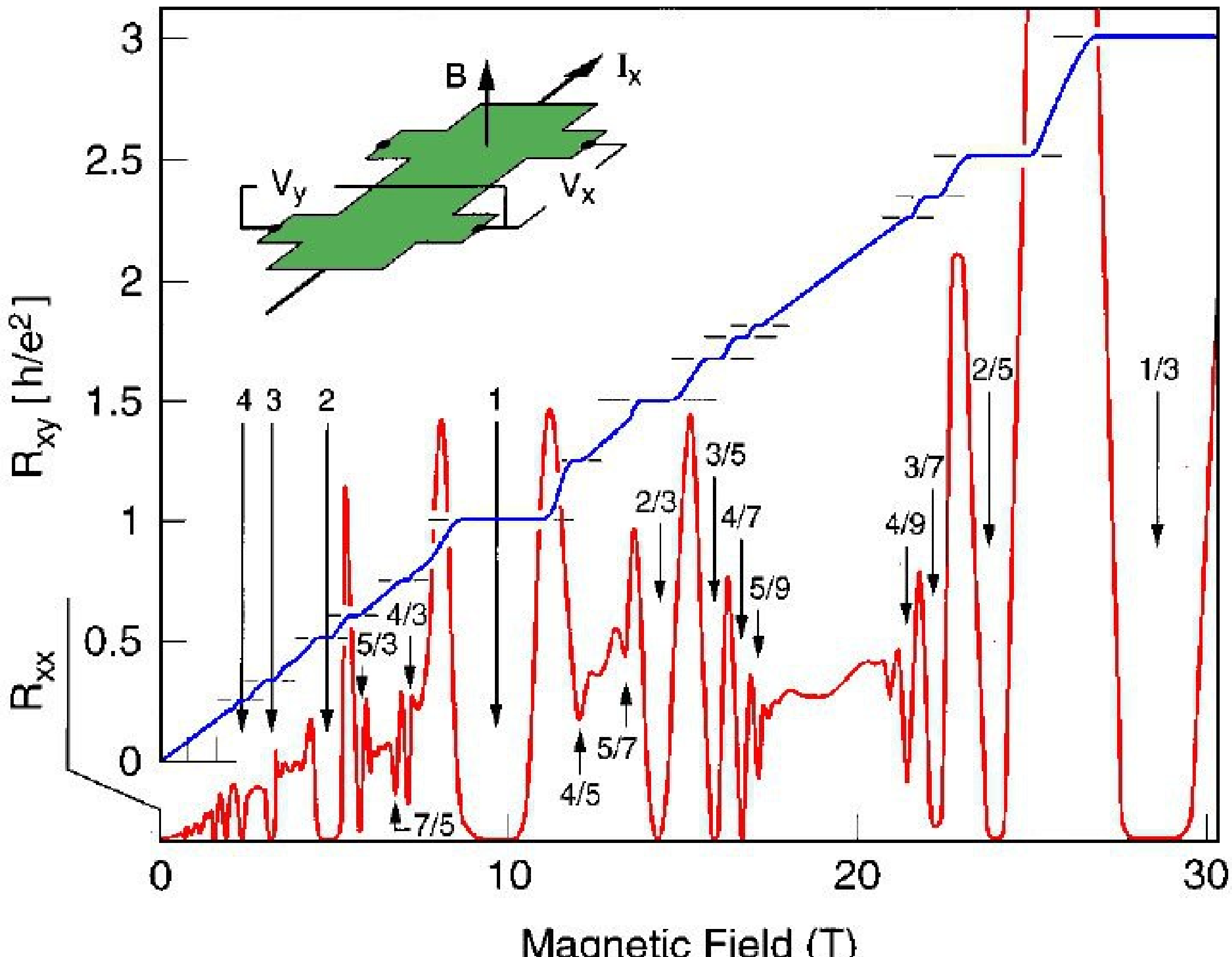
On Samples

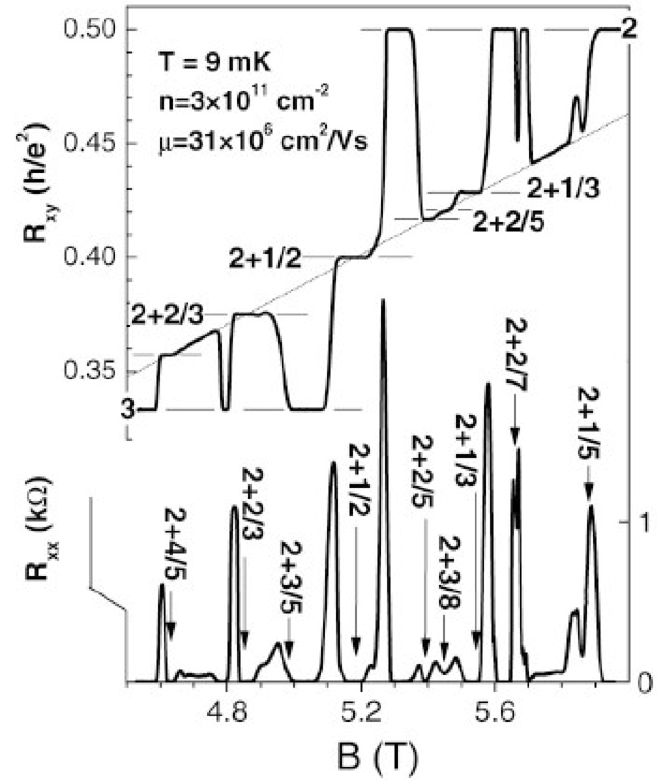


L. Pfeiffer, K.W. West / Physica E 20 (2003) 57–64



FQHE: Distinct Topological Phases



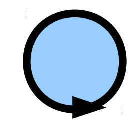


- Dominated by odd denominators, with notable exception at (5/2)
- Condensate of charge and flux composites

2DEGs: Algebraic Approach

- Coordination of electrons in a plane described by a complex $z = x + iy$
- Perpendicular magnetic field, choose symmetric gauge
- Hamiltonian (free spin-polarized electrons)

$$H_0 = \frac{1}{2\mathrm{m}}(\vec{p} - e\vec{A})^2 \longrightarrow H_0 = \hbar\omega_c\left(a^+ a + \frac{1}{2}\right)$$

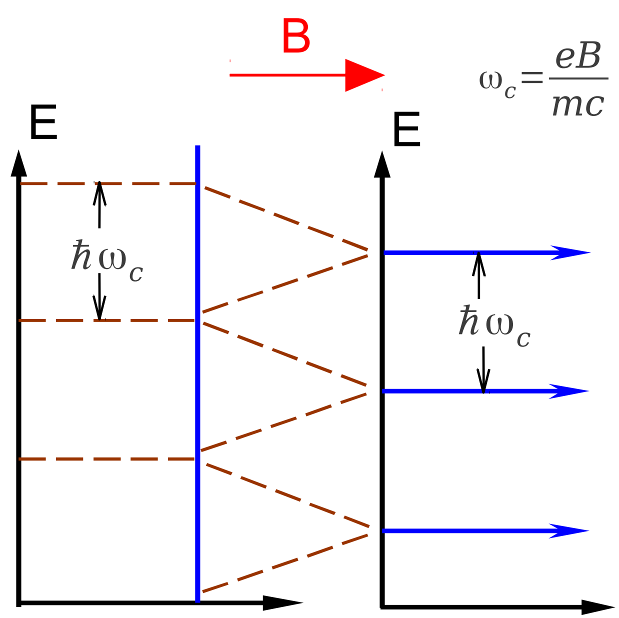


- Two sets of ladder operators

Inter-LL $\quad a = \sqrt{2}\left(l_B \partial_{\bar{z}} + \frac{1}{4 l_B} z\right)$ $\quad$ cyclotron motion

Intra-LL $\quad b = \sqrt{2}\left(l_B \partial_z + \frac{1}{4 l_B} \bar{z}\right)$ $\quad$ guiding center motion

$$l_B = \sqrt{\frac{\hbar c}{eB}} \qquad E_n = \hbar\omega_c\left(n + \frac{1}{2}\right)$$

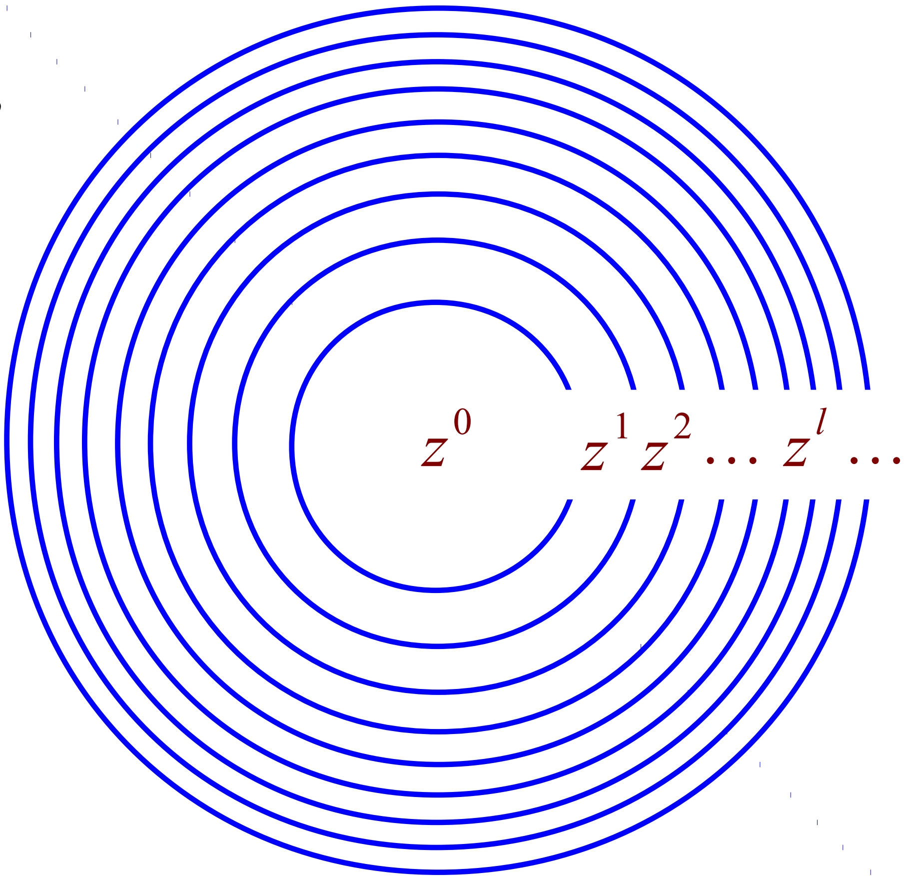


Density of States (DOS)

Laughlin State – Disk Geometry

- In the LLL, electron-electron interaction is not a perturbation. Nevertheless,

$$\phi_l(z) \sim z^l e^{-|z|^2/4} \qquad z = x + iy$$



$$R_l = \sqrt{\langle l | r^2 | l \rangle} = \sqrt{2(l+1)}$$

- Basic requirement for an electron wave function in the LLL:
  - antisymmetric function
  - analytic function
  - a universal Gaussian factor

- Laughlin state

$$\Psi_L = \prod_{i<j} (z_i - z_j)^m e^{-\sum_i |z_i|^2/4}$$

Model Hamiltonian for the Laughlin State

- Laughlin wavefunction is the ground state of

$$H_{hardcore} = \sum_{i<j}^{N} \partial_i^2 \delta^2 (z_i - z_j)$$

$$\Psi_L = \prod_{i<j} (z_i - z_j)^3 e^{-\sum_i |z_i|^2/4}$$

- Its LLL projection has a simple pseudopotential form
  - Two-particle wavefunction

$$(z_1 + z_2)^M (z_1 - z_2)^m$$

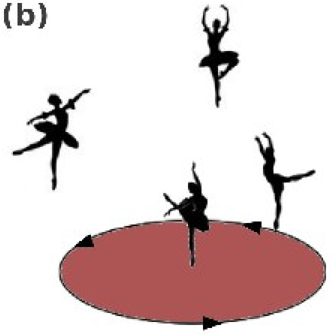
  - Interaction can be written, in general, as

$$H_i = \sum_m V_m P_m(1,2)$$

  - One produces the 1/3 Laughlin factor by $V_1 > 0$ only

When N = 2 particles approach the same point, the wavefunction vanishes as $q = 3$ powers.

(b)

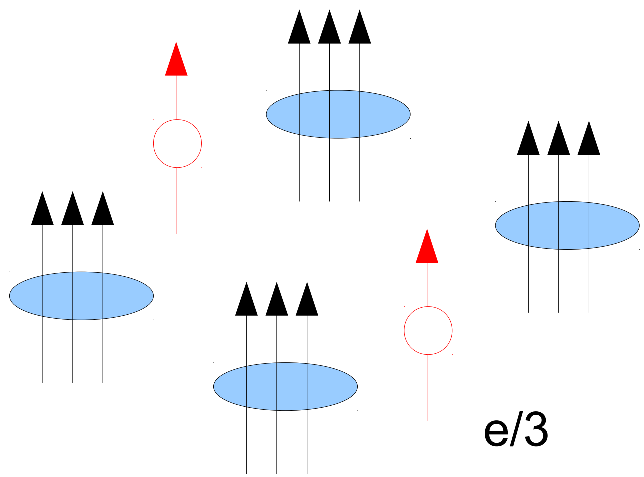


- In general, the Laughlin state is the zero-energy ground state of

$$H = \sum_{m=0}^{q-1} V_m \sum_{i<j} P_m(i,j)$$

Abelian Laughlin Quasiholes

- FQHE for electrons ($\nu = 1/3, 1/5, \ldots$)
  - Condensate of composite bosons



qps created away from 1/3 filling

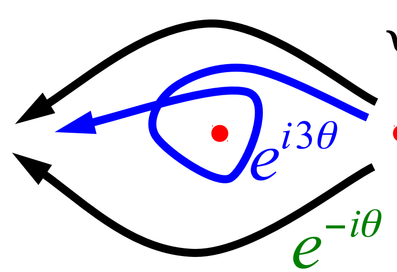
$$\Psi_L = \prod_{i<j} (z_i - z_j)^3 \, e^{-\sum_i |z_i|^2/4}$$

$$\Psi_{\xi}^{1\mathrm{qh}} = \prod_j (z_j - \xi) \prod_{i<j} (z_i - z_j)^3 \, e^{-\sum_i |z_i|^2/4}$$

$$\Psi_{\xi_1,\xi_2}^{2\mathrm{qh}} = \prod_j (z_j - \xi_1)(z_j - \xi_2) \prod_{i<j} (z_i - z_j)^3 \, e^{-\sum_i |z_i|^2/4}$$

Path equiv. in 3D; NOT equiv. in 2D:

Abelian anyons (i.e., different by a phase)



$$\Psi \rightarrow \Psi \, e^{i\theta}$$

Exercises on the Laughlin State

- Why is the filling fraction for the following Laughlin state?

$$\Psi_{Laughlin} = \prod_{1 \leqslant i < j \leqslant N} (z_i - z_j)^m e^{-\sum_i |z_i|^2/4}$$

  - m = 2: bosonic
  - m = 3: fermionic

- What is its total angular momentum?

- What is the fractional charge of the m = 2 state?

Hint: Two-Electron Laughlin State

- Laughlin state for electrons ($\nu = 1/3$)

$$\Psi_L = \prod_{i<j} (z_i - z_j)^3 e^{-\sum_i z_i^2/4}$$

$$(z_1 - z_2)^3 \;=\; 1\cdot(z_1^3 - z_2^3) + (-3)\cdot(z_1^2 z_2 - z_1 z_2^2)$$

1 ●□□● + (- 3) □●●□ Orbitals: 0,1,2,3

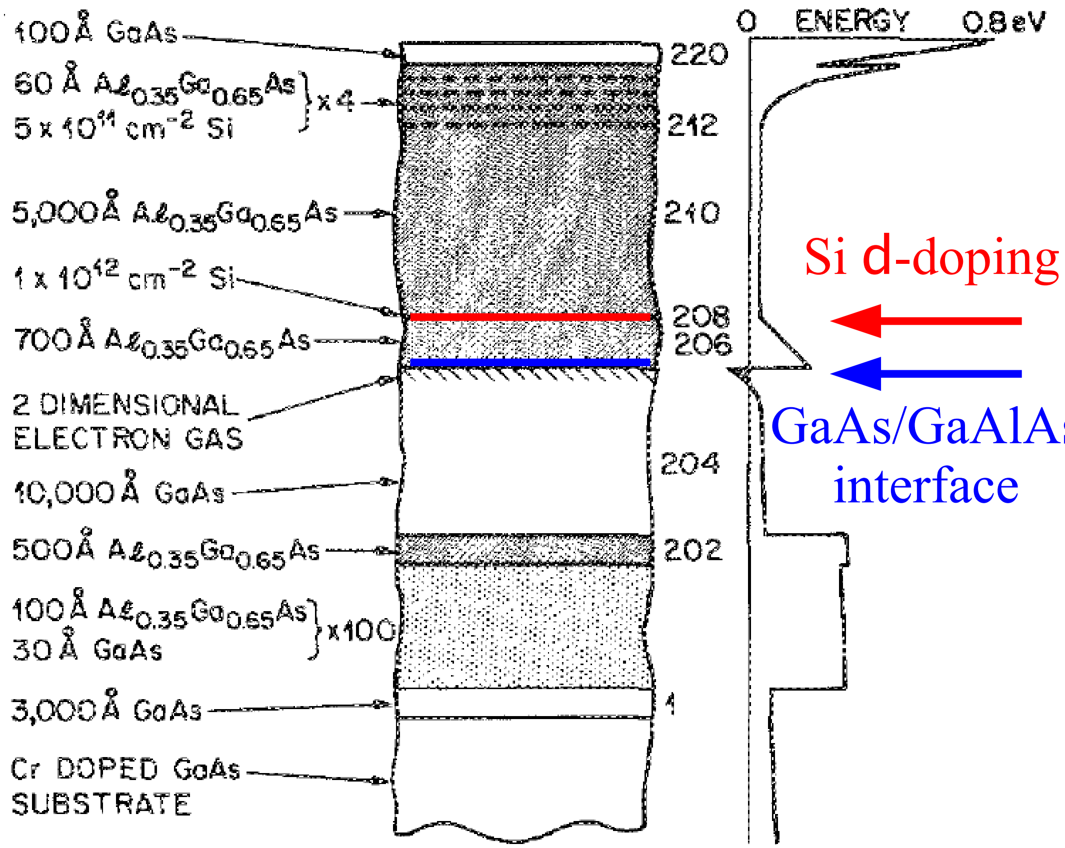
Generalize it for 3 electrons. Use Mathematica for N electrons.

- For N electrons,

$$\Psi_L = Sym\left(z_1^{3(N-1)} z_2^{3(N-2)} \cdots z_N^0 + \cdots\right) e^{-\sum_i |z_i|^2/4}$$

$$\nu \;=\; \lim_{N\to\infty} \frac{N}{3(N-1)+1} \;=\; \frac{1}{3}$$

Realistic Model

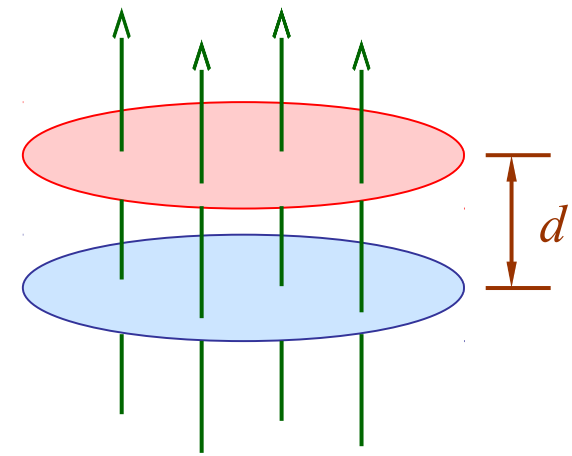


Si d-doping

GaAs/GaAlAs interface

Background charge (+Ne)

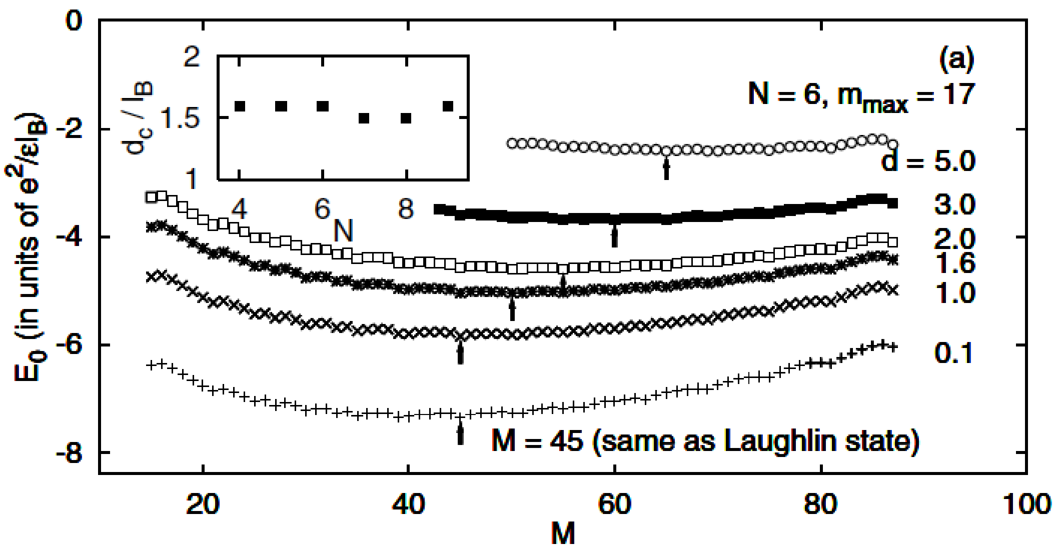
Electron layer (-Ne)

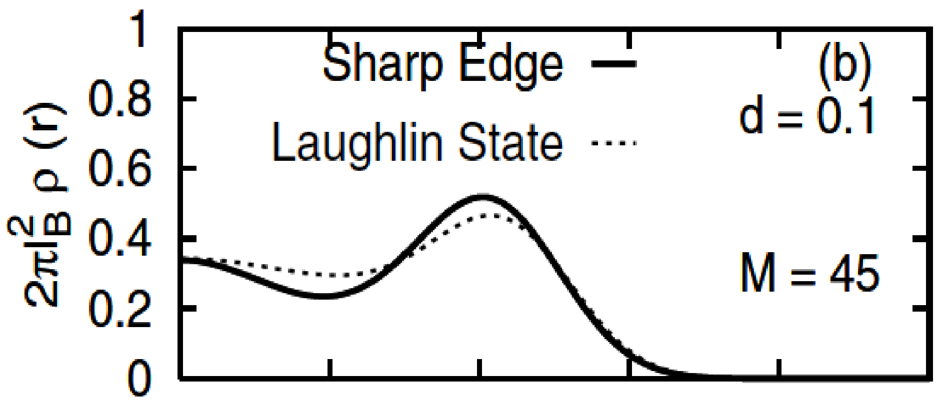


$$H = \frac{1}{2}\sum_{mnl} V^{l}_{mn} c^{+}_{m+l} c^{+}_{n} c_{n+l} c_{m} + \sum_{m} U_{m} c^{+}_{m} c_{m}$$

Coulomb interaction Confining potential

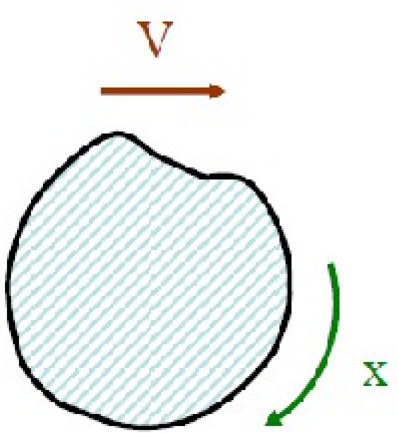
Ground State and Edge Spectrum



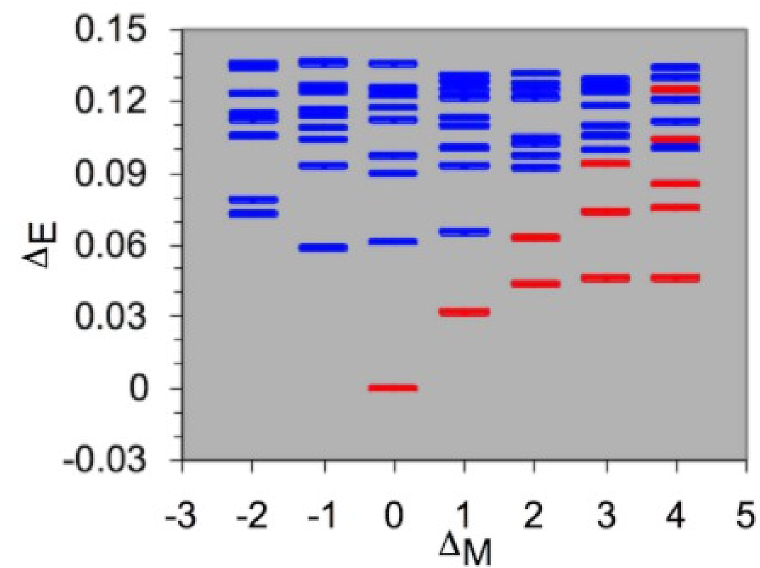


- Edge excitations generated by symmetric polynomials

$$P(\{z_i\})\prod_{i<i}(z_i - z_j)^m e^{-\sum_i |z_i|^2/4}$$

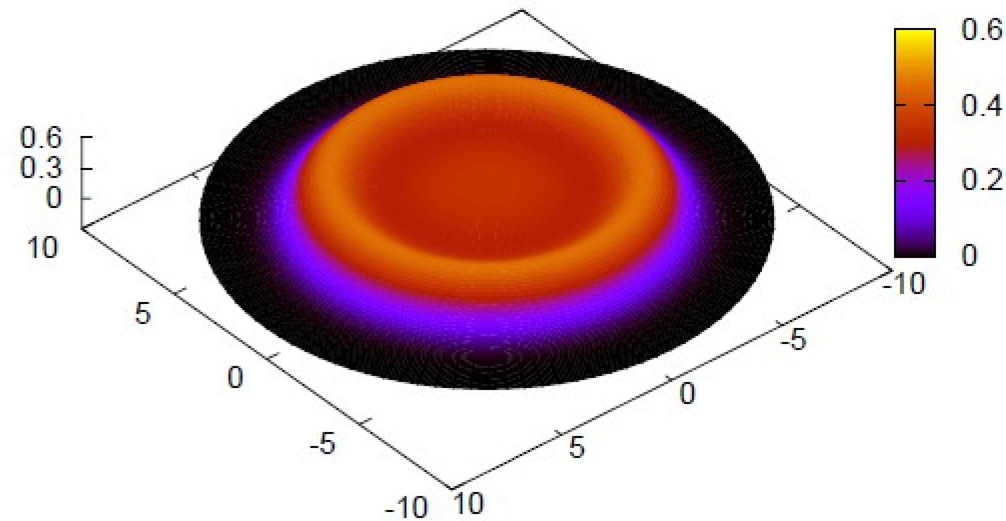


Gapless chiral bosonic charge mode

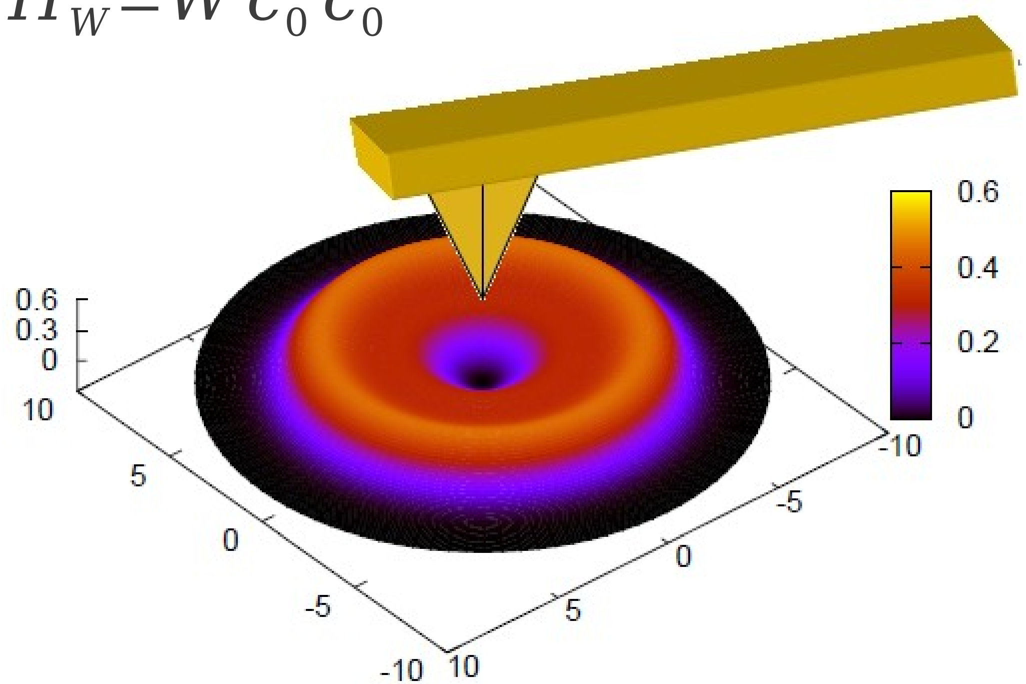


Introducing Quasihole

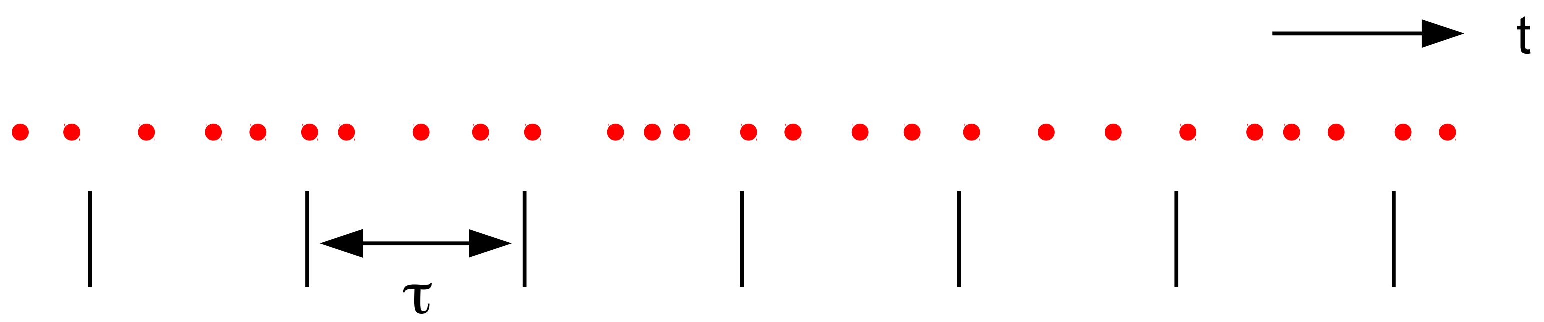
9-electron
Laughlin state



$$H_W = W c_0^+ c_0$$



How to Measure the Charge Experimentally?

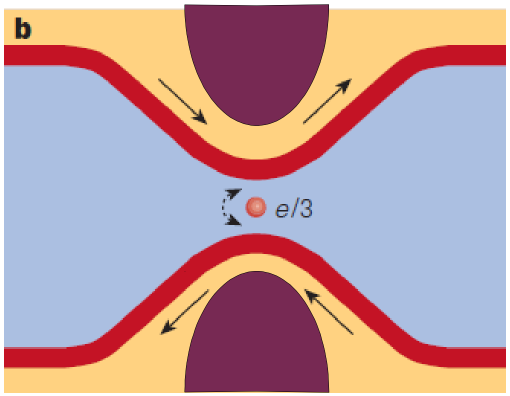


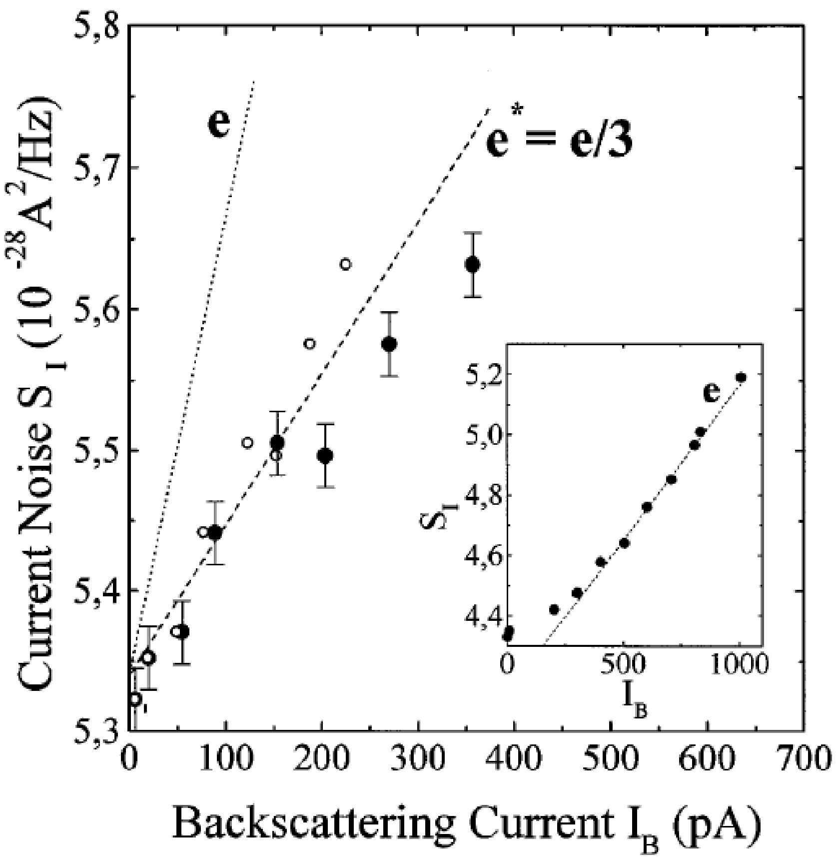
$$I \;=\; \frac{\text{average \# of events in } \tau}{\tau} \times \text{charge contribution per event}$$

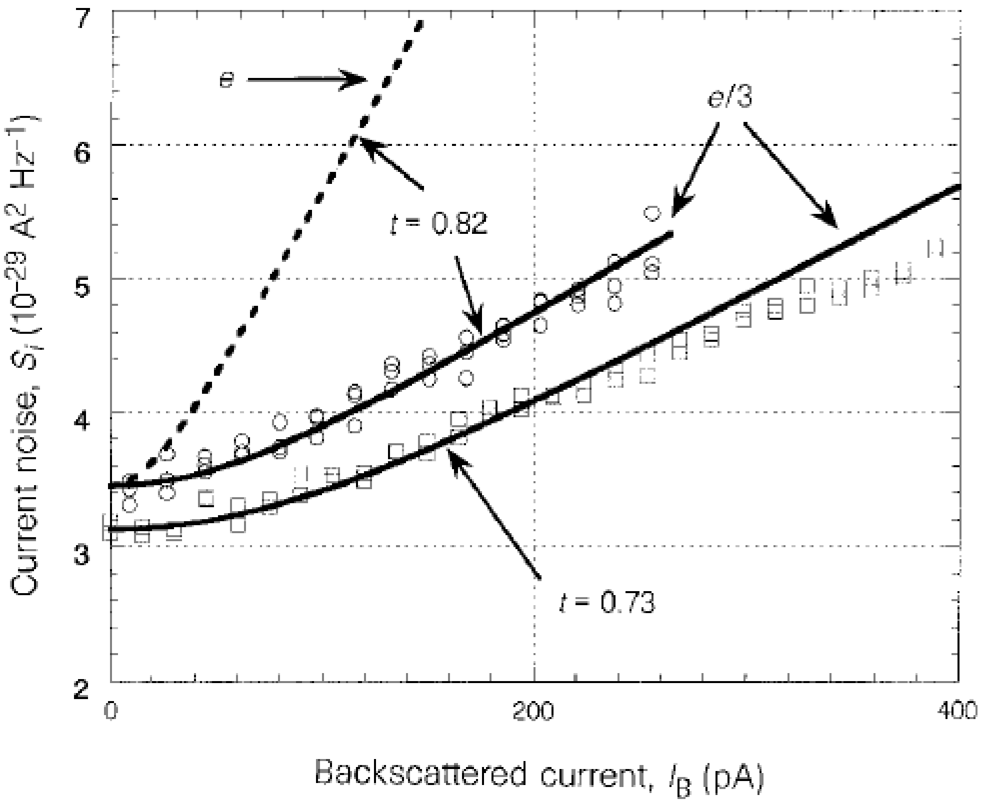
$$I_n \;=\; \frac{\text{rms fluctuation of \# in } \tau}{\tau} \times \text{charge contribution per event}$$

$$S_I \;\propto\; I_n^2 \;\propto\; 2e^* I$$

Fractional Charge in Shot Noise





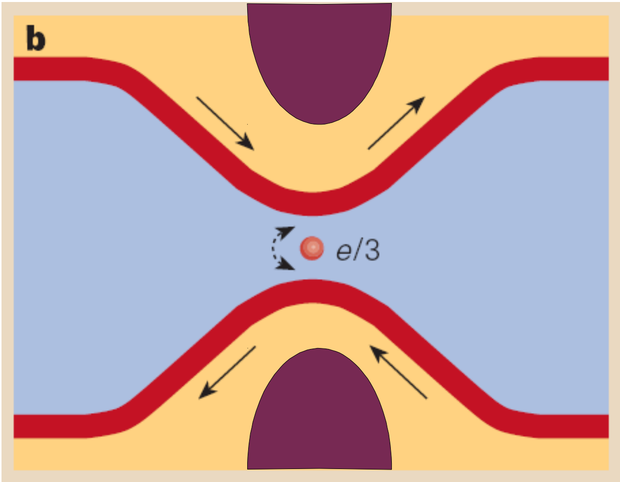


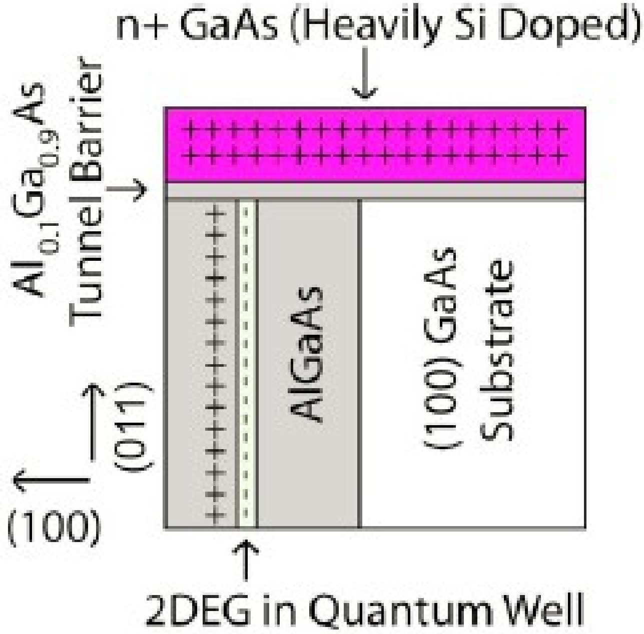
▲ De-Picciotto et al., Nature 389, 162 (1997)

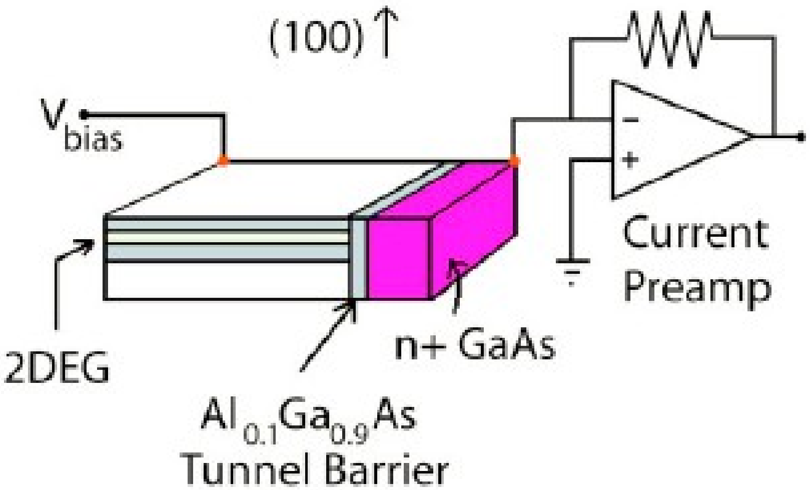
◀ Saminadayar et al., PRL 79, 2526 (1997)

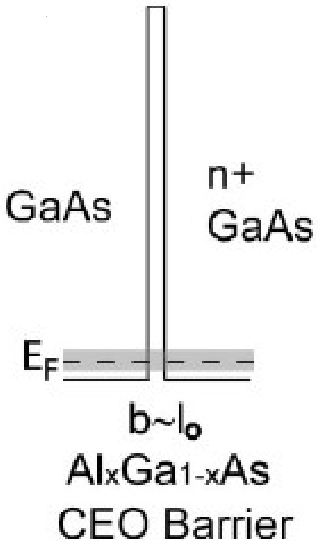
Devices for Edge Physics

- Quantum point contact
  - Smooth potential, tunable
- Cleaved-edge overgrowth
  - Broad energy range







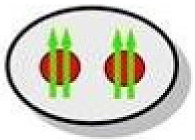


#1: Messages So Far

- FQH effect can be routinely observed in two-dimensional electron systems in GaAs quantum wells or in high-mobility graphene.
- Mobility is an important quantity to determine which fractions can be observed. Higher mobility means smaller disorder.
- A model wave function can be thought of as the fixed point for the corresponding topological phase, which is stable under long-range interaction and disorder.
- Laughlin states support (gapped) Abelian quasiparticle excitations which carry a fraction of an electron charge. The fractional charge has been detected by shot noise measurement.
- Laughlin states support gapless chiral edge excitations. Quasiparticles can propagate along the edge.
- Other odd-denominator FQH states can be thought of as the descendants of the Laughlin states.

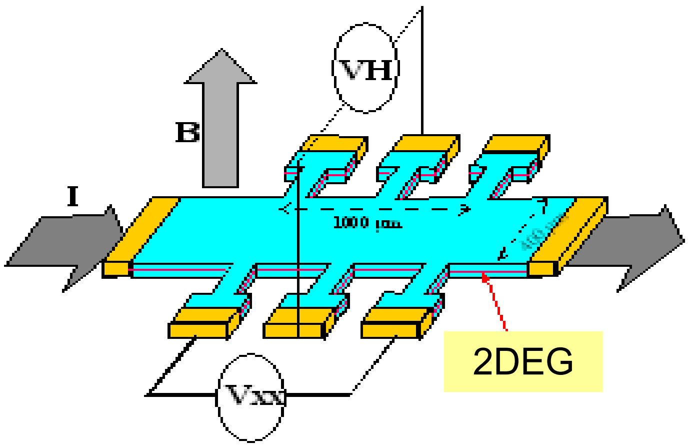
Next: non-Abelian state at $\nu = 5/2$

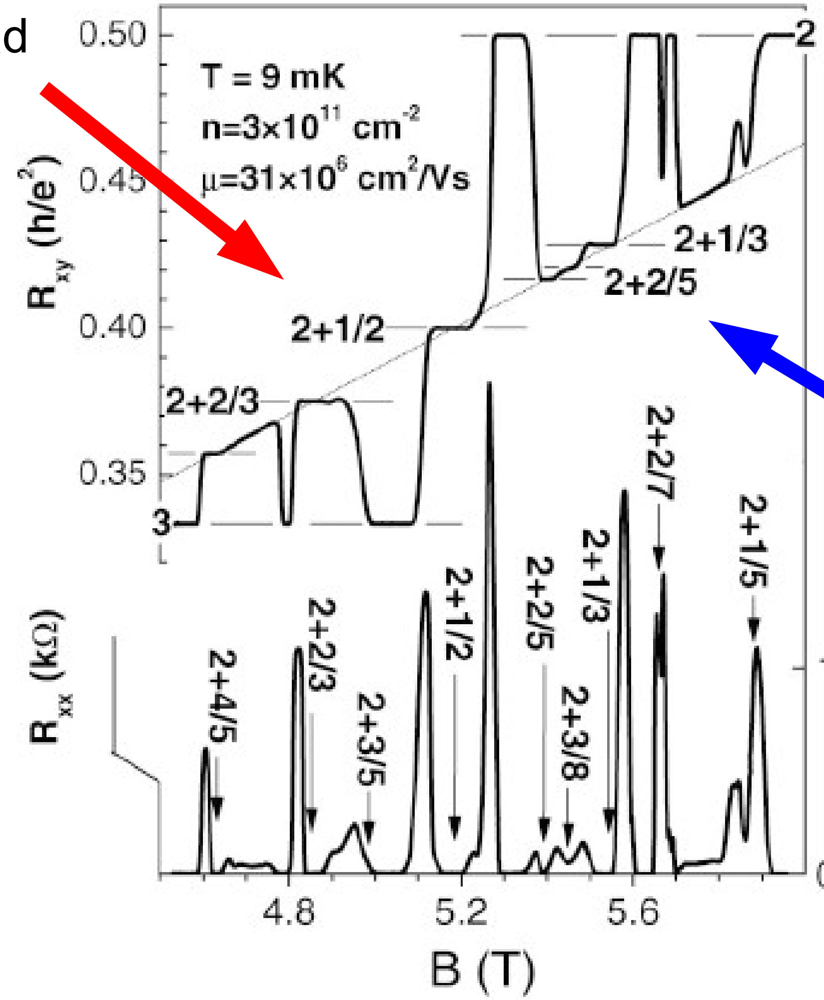
FQH at the First Excited Landau Level



Ising anyon / **Majorana** fermion mode

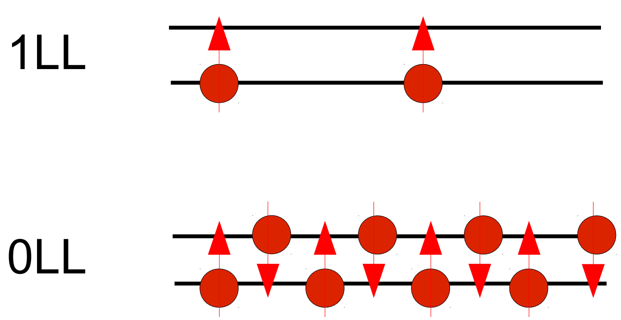
Moore-Read





Read-Rezayi?

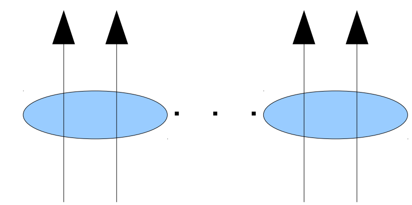
Fibonacci anyon



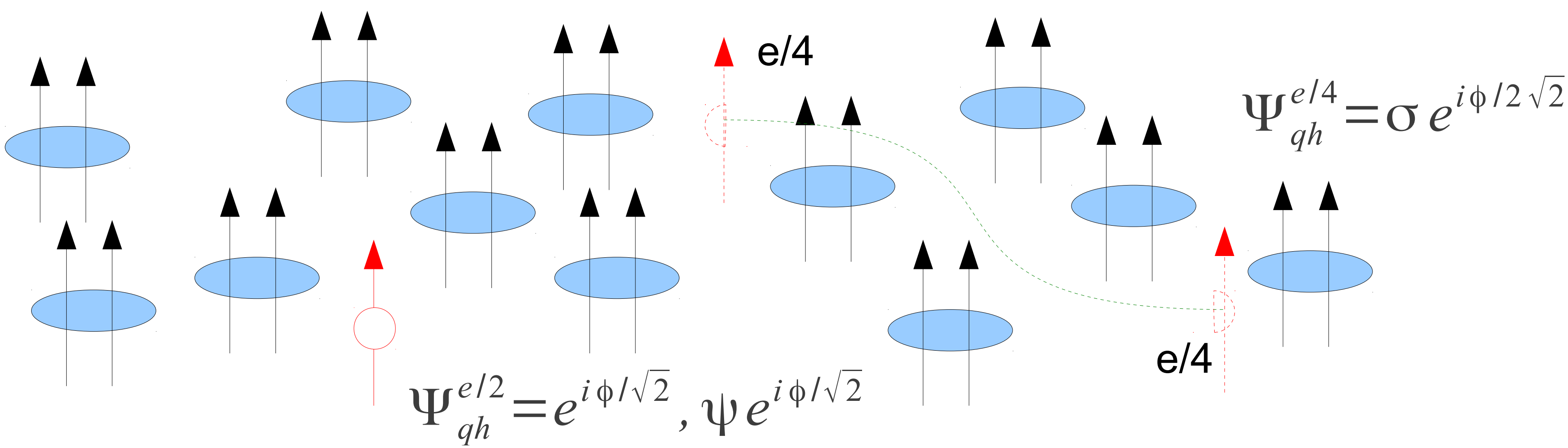
Xia et al., PRL (04)

A Cartoon of the Moore-Read State

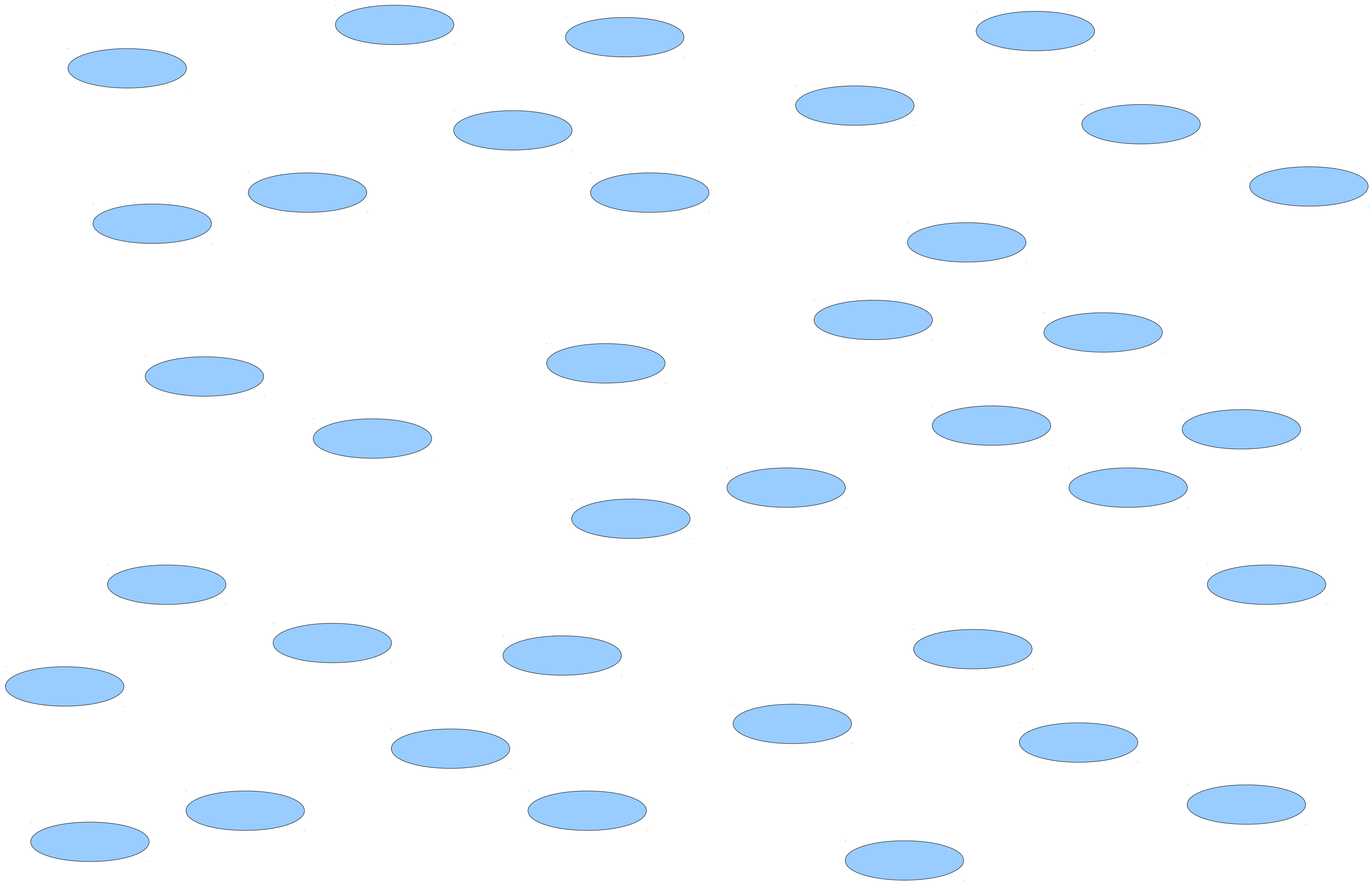
- Half-filling ν = 1/2: CF at zero effective field (B* = 0)
  - 0LL (or LLL): Fermi sea of composite fermions
  - 1LL: Superfluid of Cooper pairs of composite fermions
  - 2+LL: Charge density wave

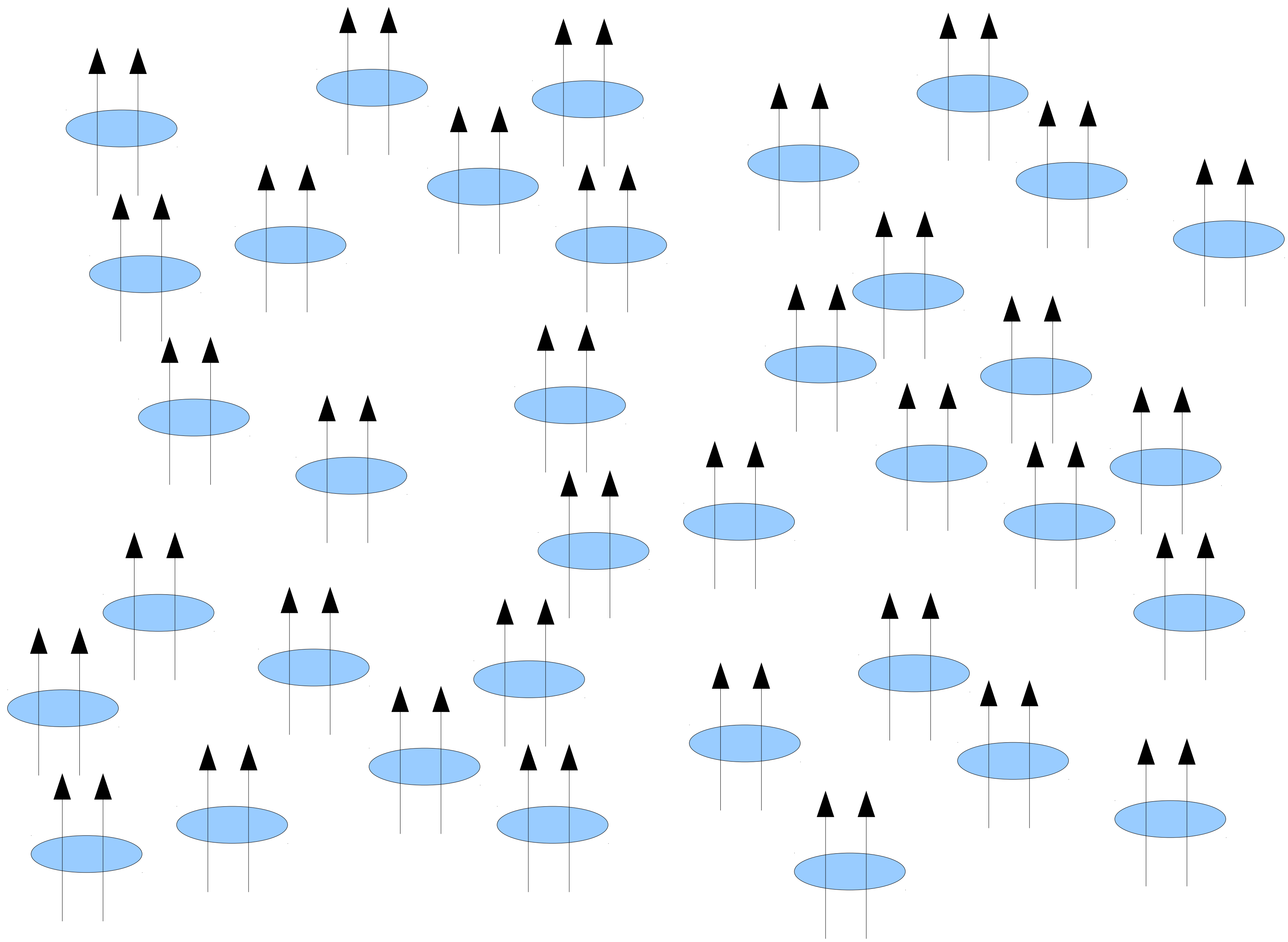


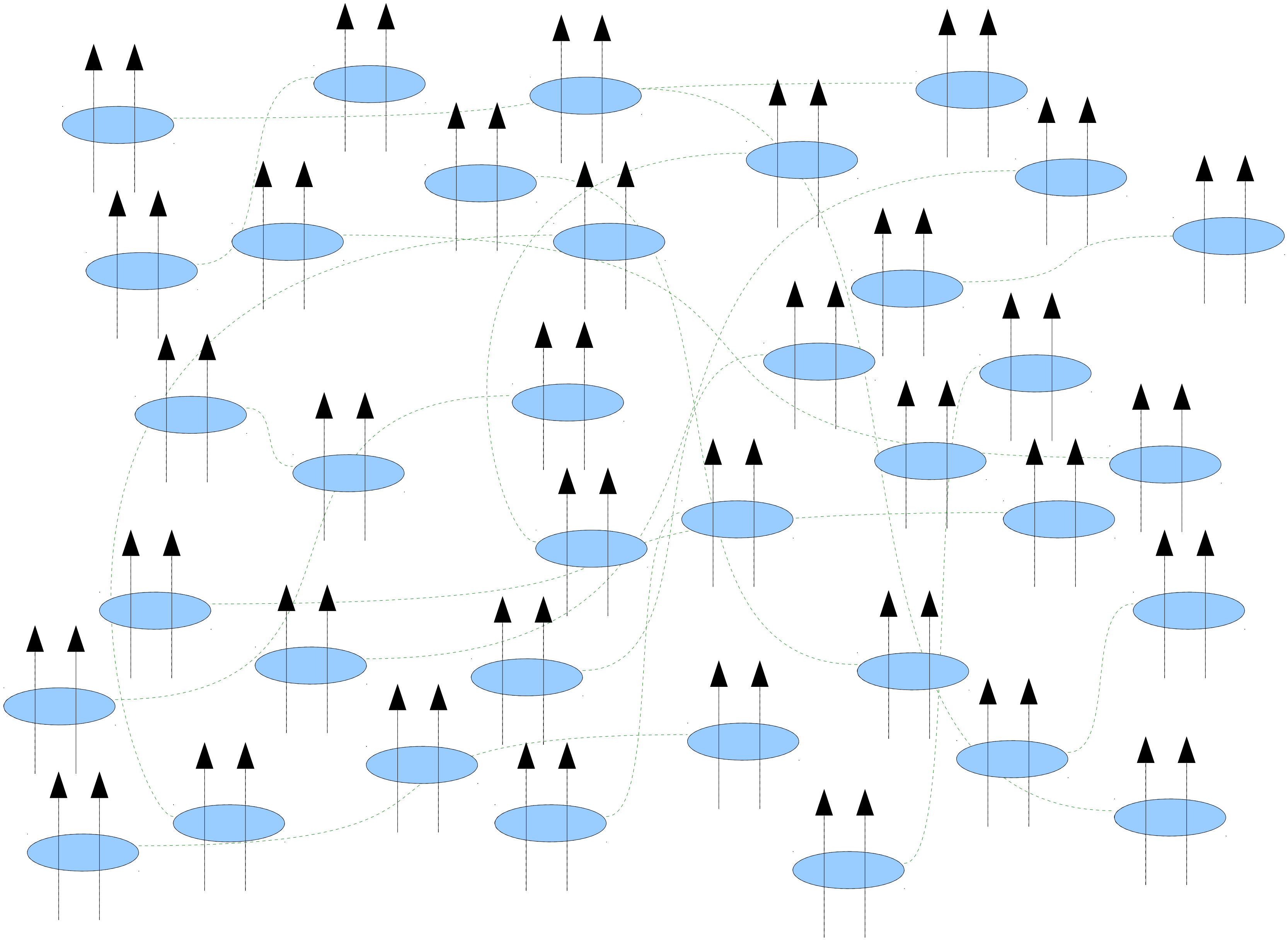
- Condensate of composite fermions (ν = 5/2 = 2 + 1/2)

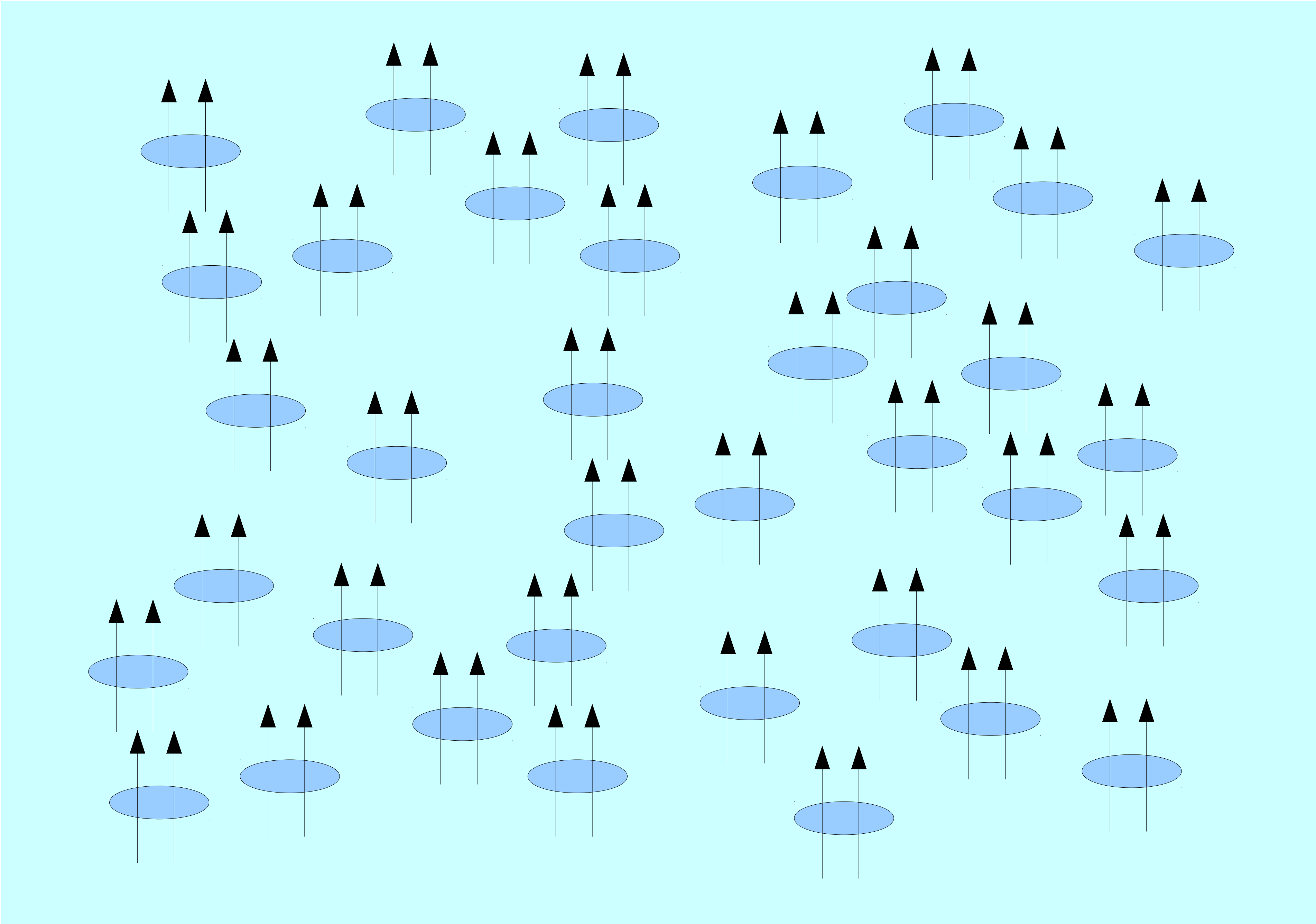


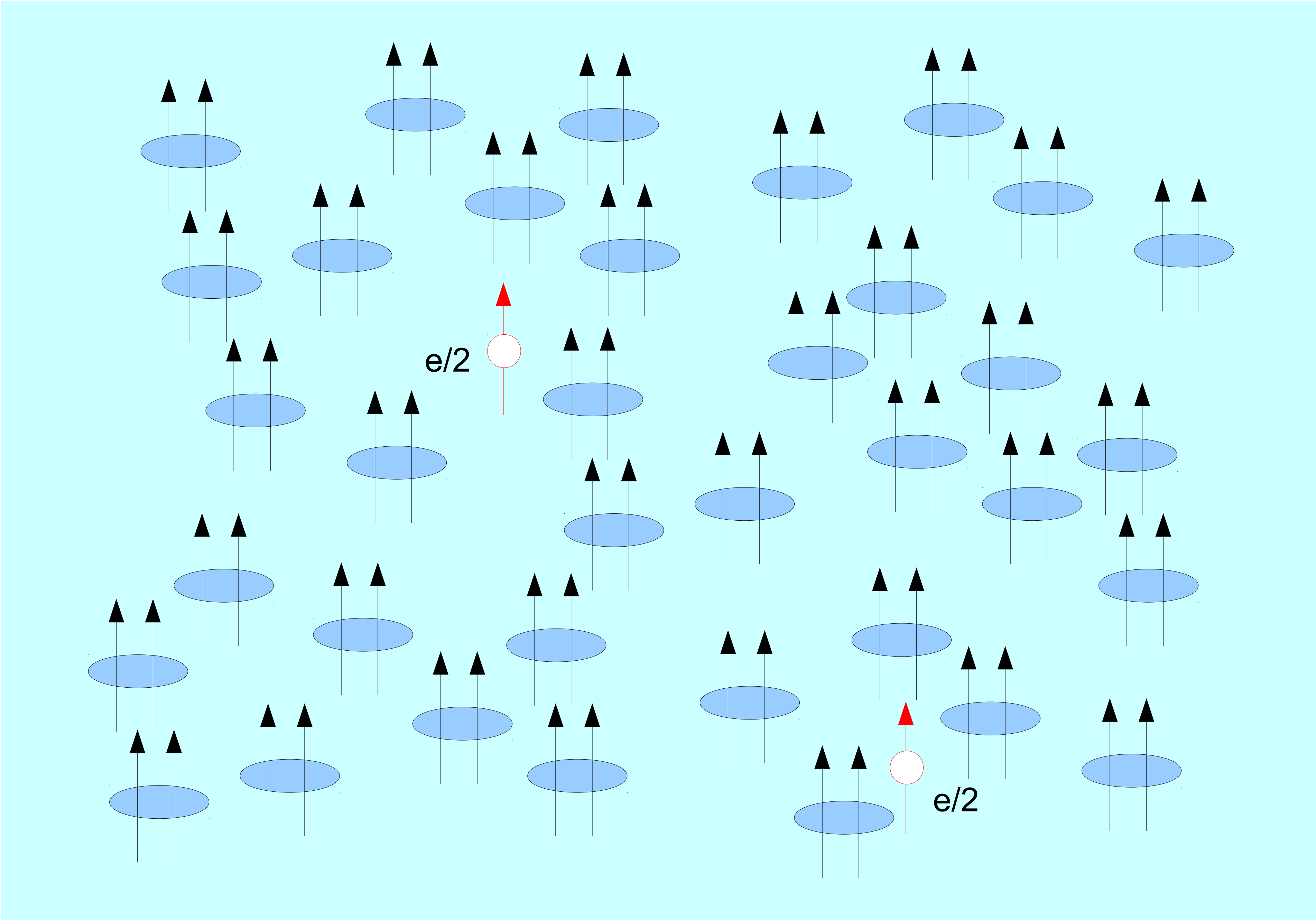
e/4 quasihole = charge-e/4 boson + neutral Majorana fermion mode

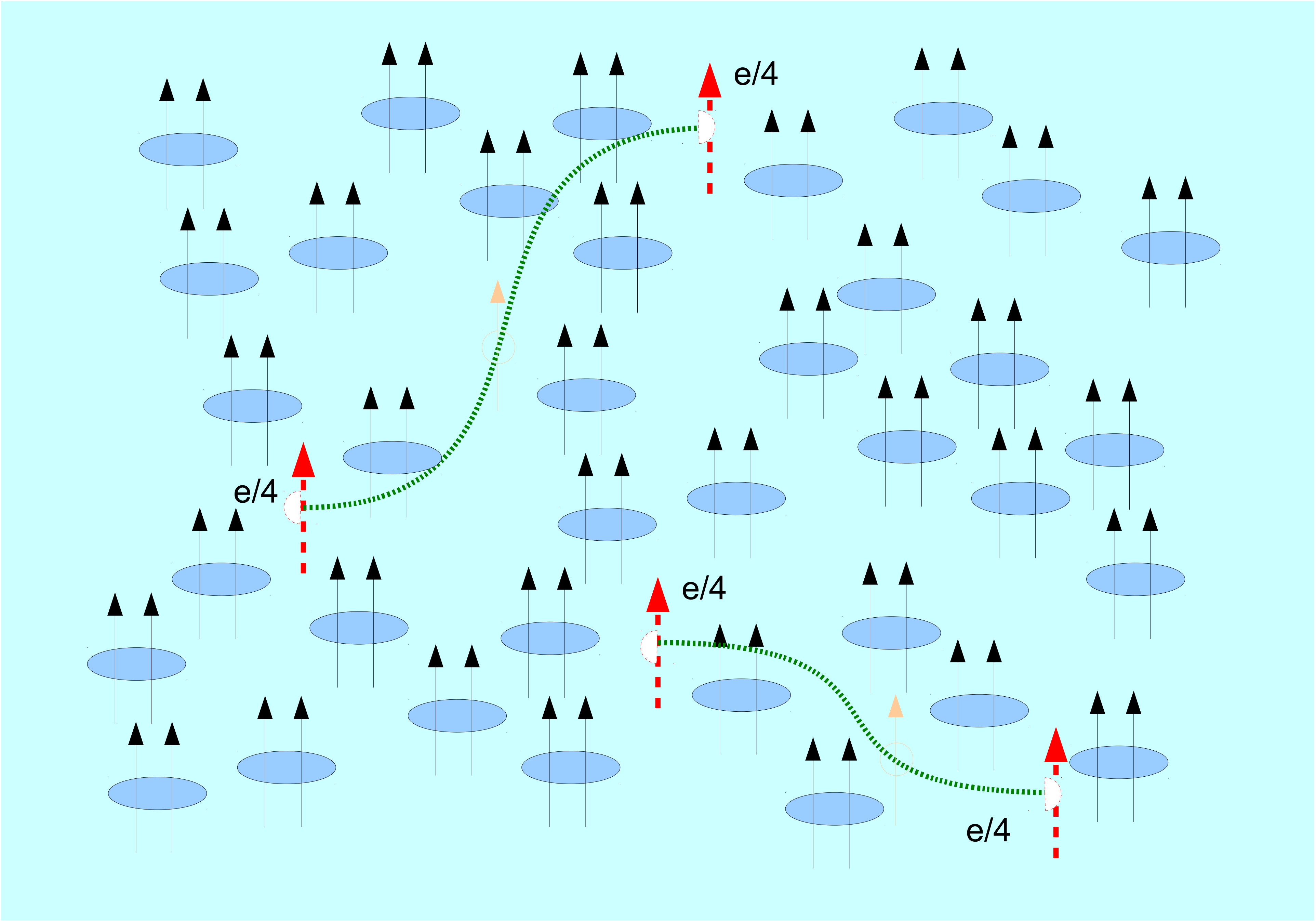




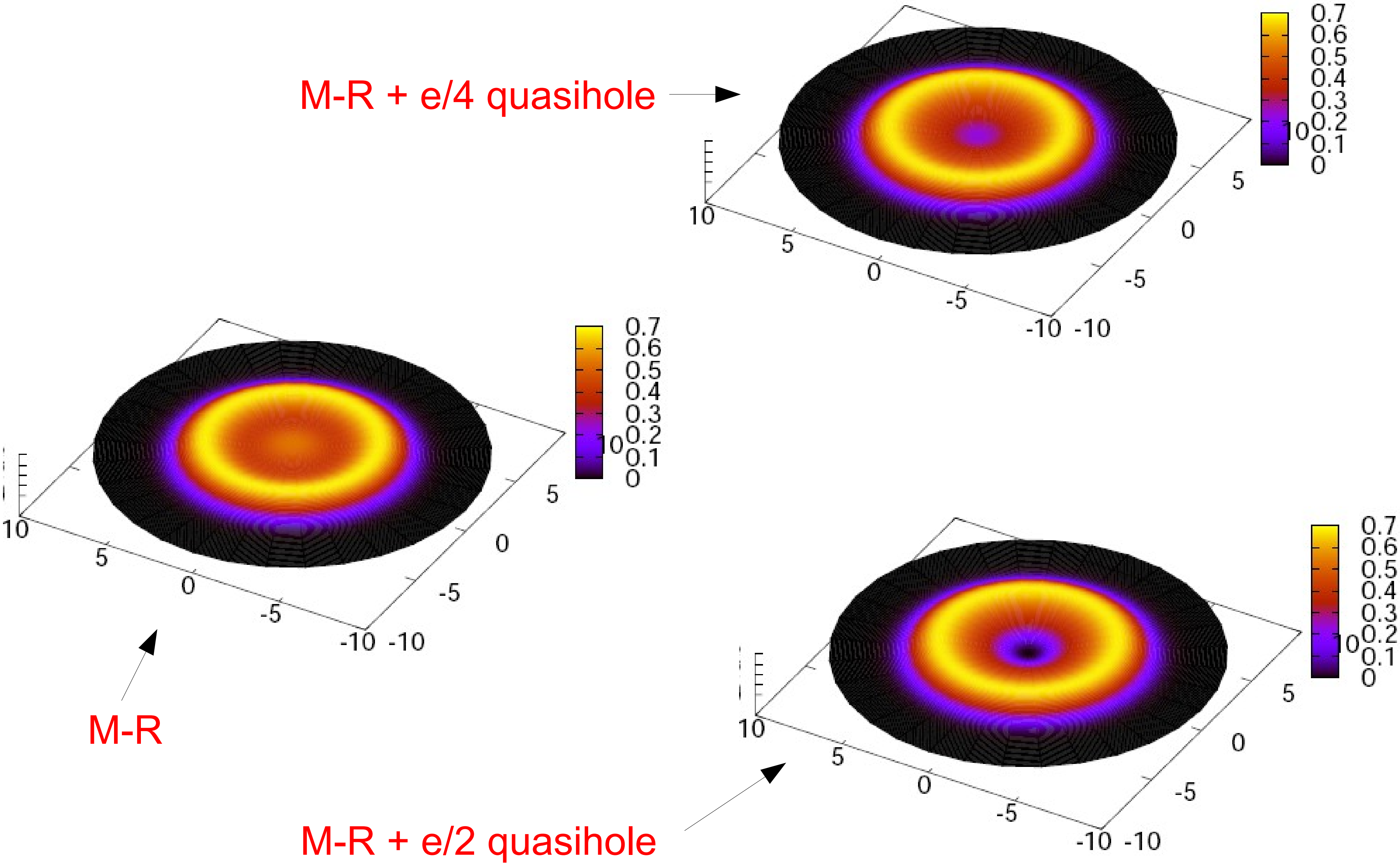





e/2
e/2


e/4
e/4
e/4
e/4

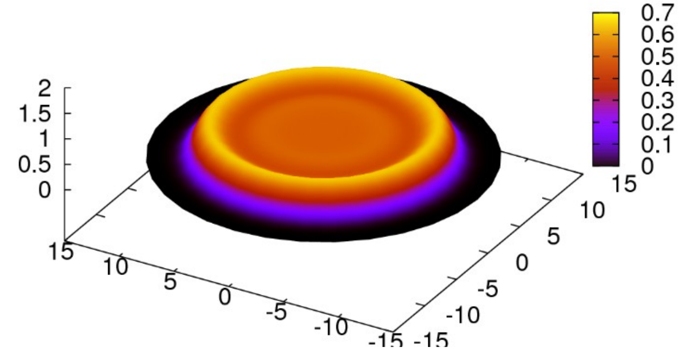
Density Profiles for a 12-electron Droplet (ED)



Quasiholes Wavefunctions in the Moore-Read State

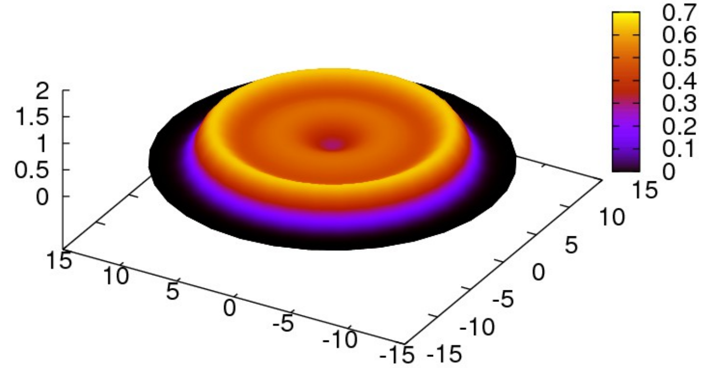
- Moore-Read state (Moore & Read, 1991)

$$\Psi_{Pf} = Pf\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)^m$$



- Quasiholes in Moore-Read condensate
  - Charge e/2, Abelian (Laughlin type)

$$\prod_i (z_i - \xi_1)(z_i - \xi_2) Pf\left(\frac{1}{z_i - z_j}\right)\prod_{1\leqslant i<j\leqslant N}(z_i - z_j)^2$$



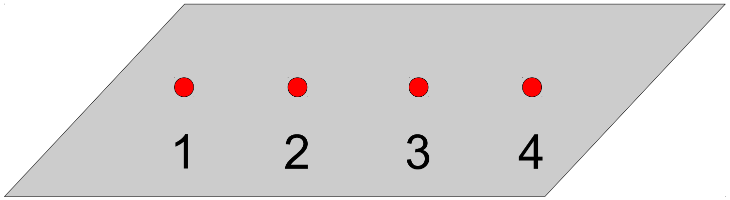
  - Charge e/4, non-Abelian

$$\Psi_{(12)(34)} = Pf\left(\frac{(z_i-\xi_1)(z_i-\xi_2)(z_j-\xi_3)(z_j-\xi_4) + i \Leftrightarrow j}{z_i - z_j}\right)\prod_{1\leqslant i<j\leqslant N}(z_i - z_j)^2$$

Quasiparticles cannot be generated by local operators, but can be moved around by local operators adiabatically.

Four-Anyon States

- Even when one fixes the location of all quasiholes, there are more than one states



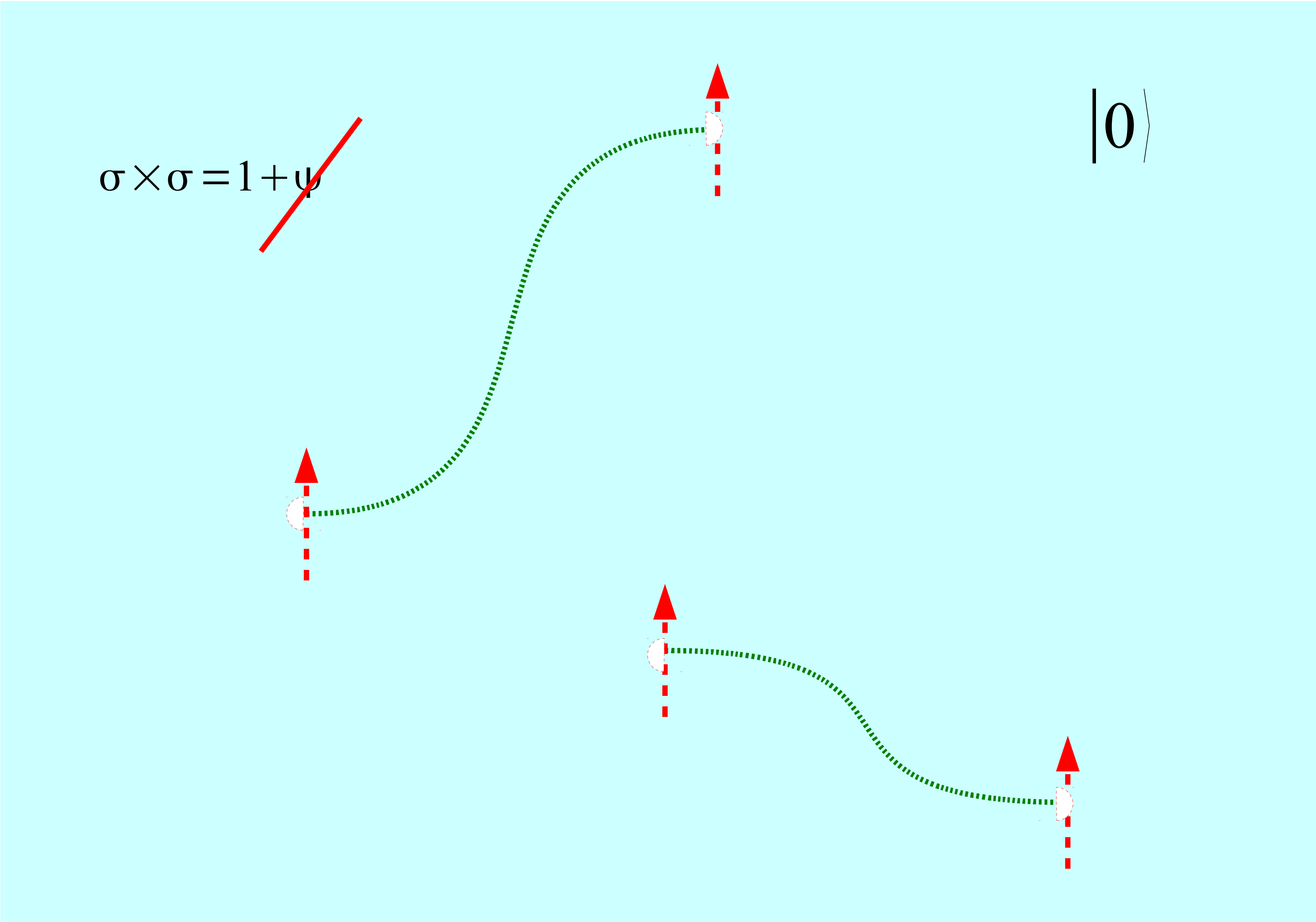
$$\Psi_{(12)(34)} = Pf\left(\frac{(z_i-\xi_1)(z_i-\xi_2)(z_j-\xi_3)(z_j-\xi_4)+i\Leftrightarrow j}{z_i-z_j}\right)\prod_{1\le i<j\le N}(z_i-z_j)^2$$

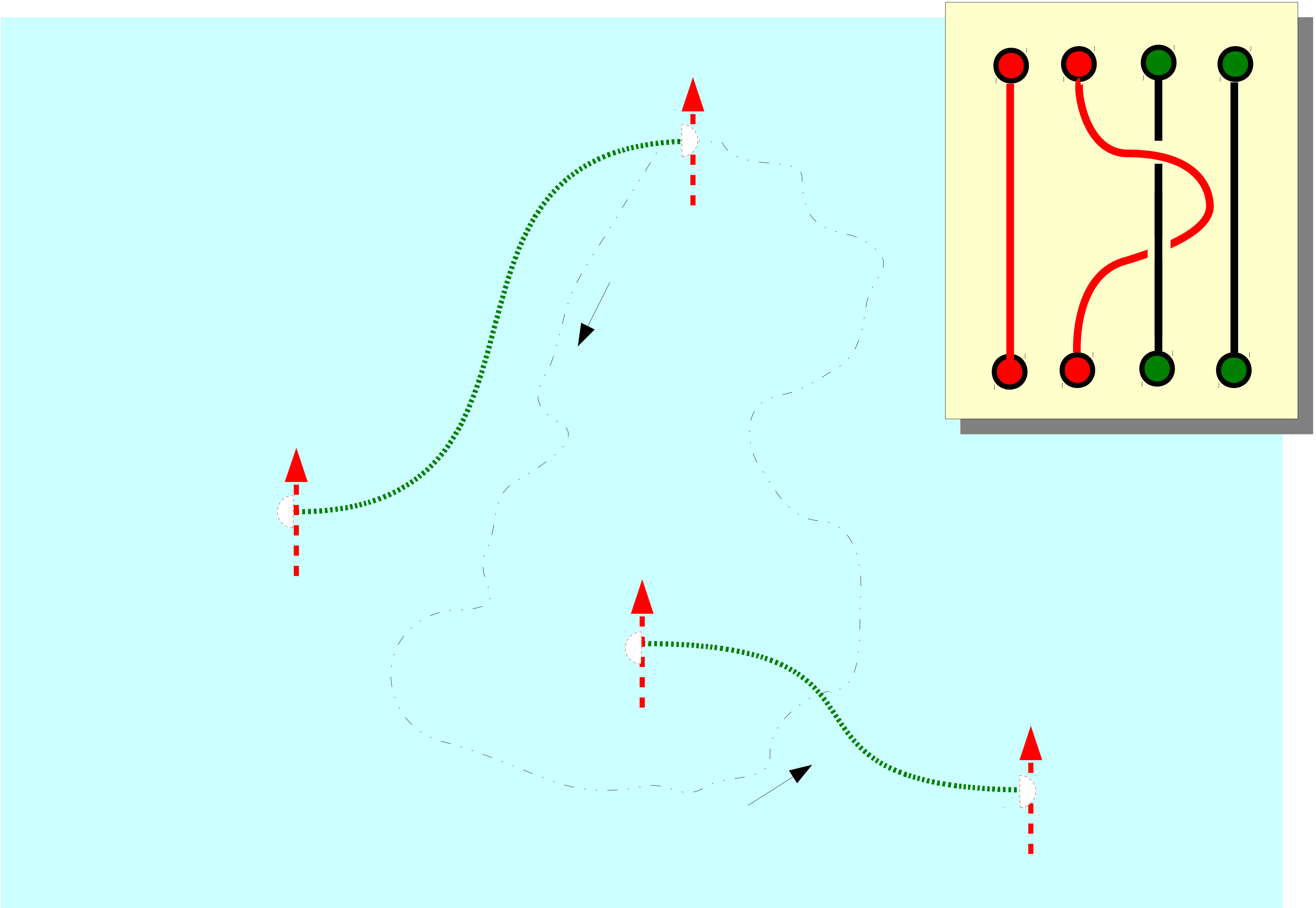
$$\Psi_{(13)(24)} = Pf\left(\frac{(z_i-\xi_1)(z_i-\xi_3)(z_j-\xi_2)(z_j-\xi_4)+i\Leftrightarrow j}{z_i-z_j}\right)\prod_{1\le i<j\le N}(z_i-z_j)^2$$

$$\Psi_{(14)(23)} = Pf\left(\frac{(z_i-\xi_1)(z_i-\xi_4)(z_j-\xi_2)(z_j-\xi_3)+i\Leftrightarrow j}{z_i-z_j}\right)\prod_{1\le i<j\le N}(z_i-z_j)^2$$

- But they are not linearly independent!

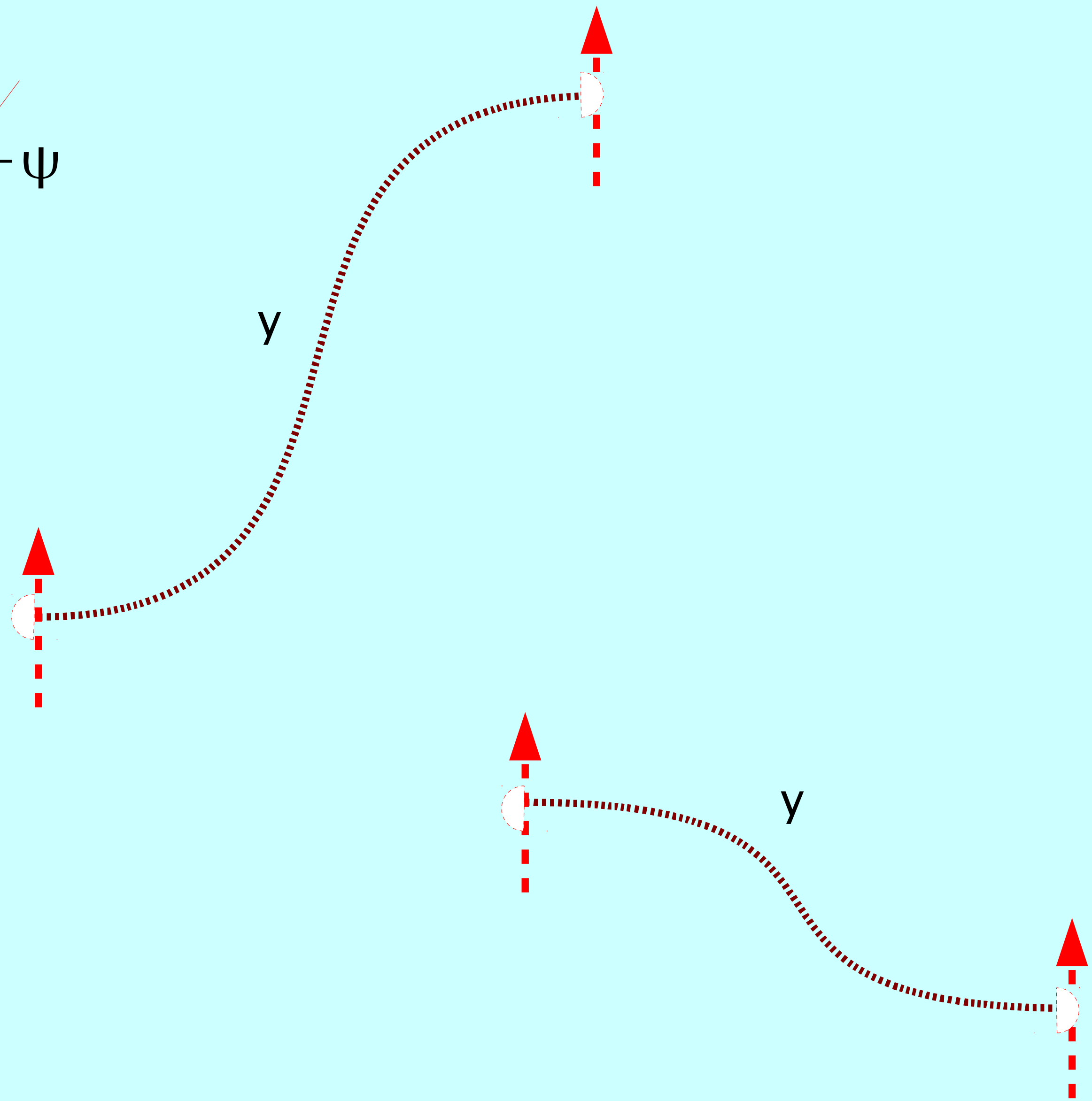
$$\Psi_{(12)(34)}-\Psi_{(13)(24)}=(1-x)\left(\Psi_{(12)(34)}-\Psi_{(14)(23)}\right) \qquad x=\frac{(\xi_1-\xi_2)(\xi_3-\xi_4)}{(\xi_1-\xi_3)(\xi_2-\xi_4)}$$


$\sigma \times \sigma = 1 + \cancel{\psi}$
$|0\rangle$



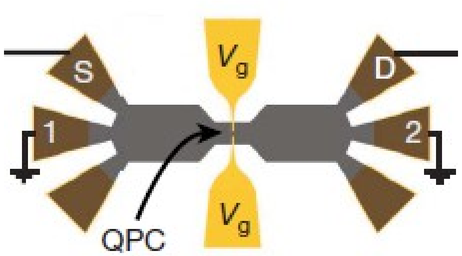
$$\sigma \times \sigma = \cancel{1} + \psi$$

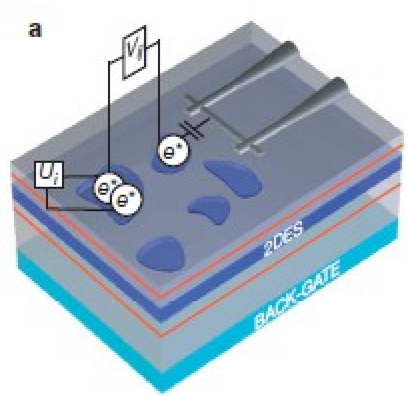
$$\left|1\right\rangle$$

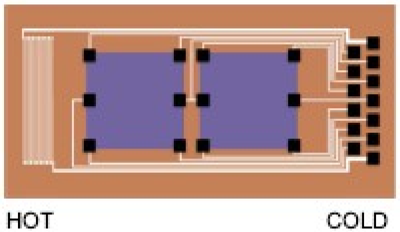


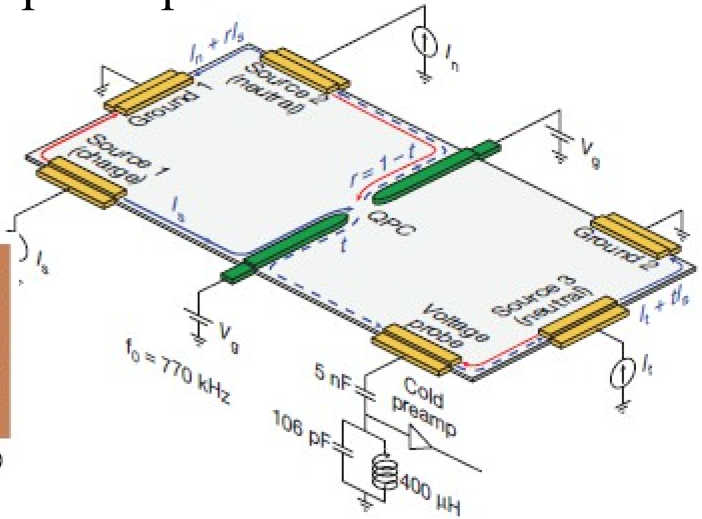
Experimental Progress

- e/4 charge probing (indirect)
  - Noise in current across quantum point contact (Heiblum group, 2008)
  - Tunneling conductance across a QPC (MIT-Harvard group, 2008; Lin 2015)
  - Local charge via coupling with single electron transistor (Yacoby group, 2011)
- Quasiparticle statistics (direct)
  - Interference (Willett, 2009, 2010, 2013; Kang group, 2011)
  - Ground state degeneracy via thermopower (Eisenstein group, 2012)
- Other consistent results
  - Spin polarization (Muraki group, 2012), ruled out spin-unpolarized states
  - Neutral current noise (Heiblum group, 2010)

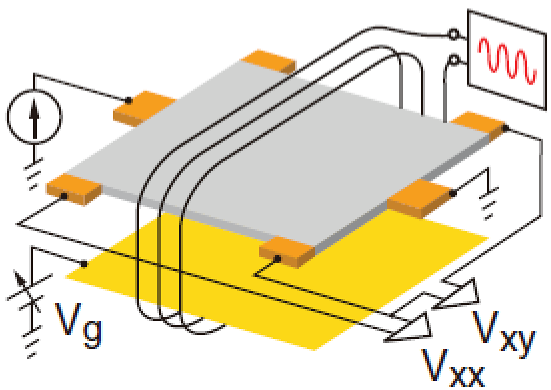




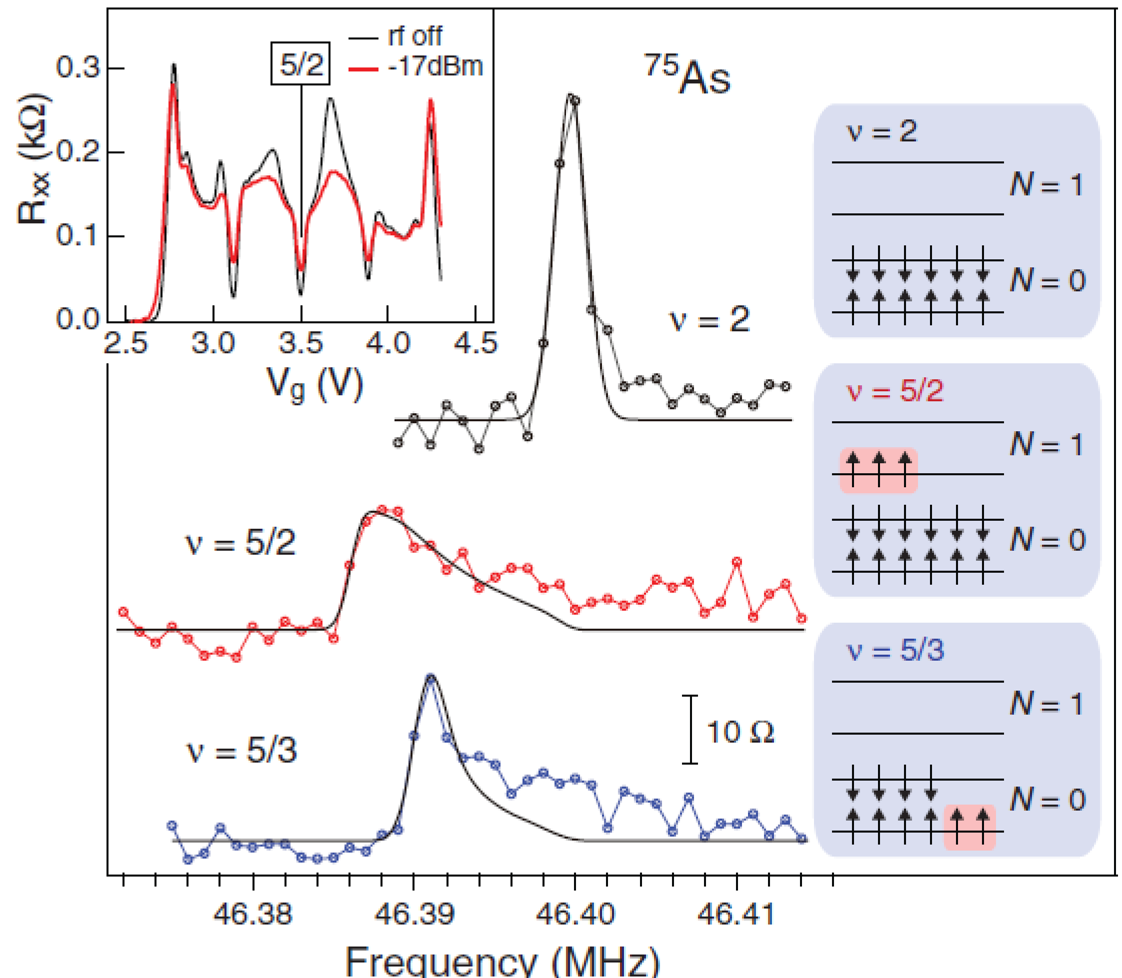


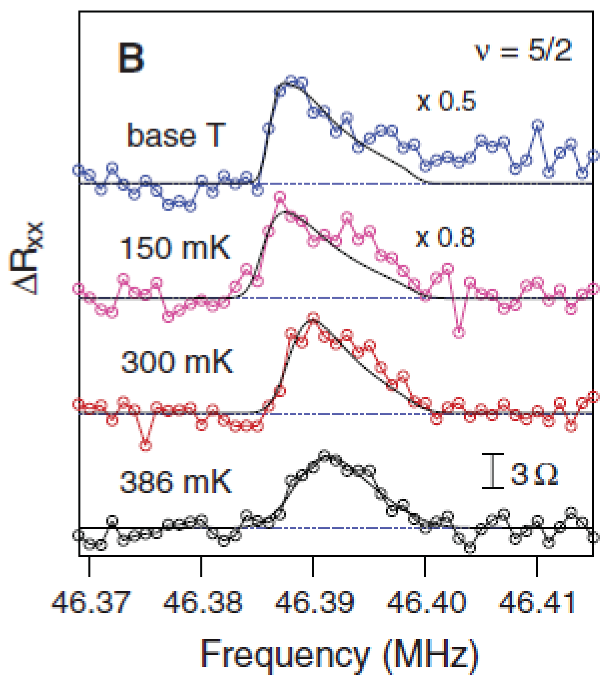


Spin Polarization: Tiemann et al., Science (2012)



Resistively detected NMR (RD-NMR)

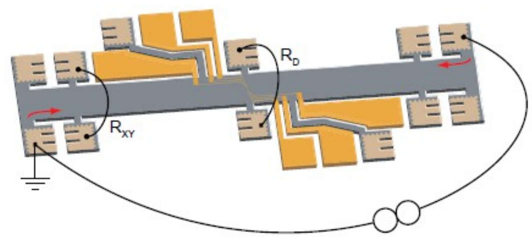


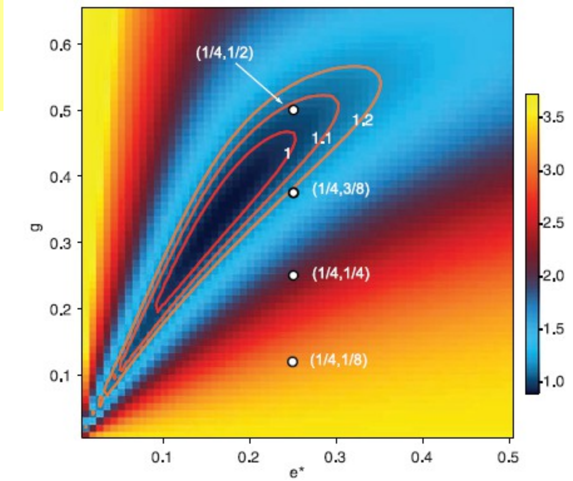


5/2 plateau vanishes ~ 150 mK
maximum polarization up to 200 mK

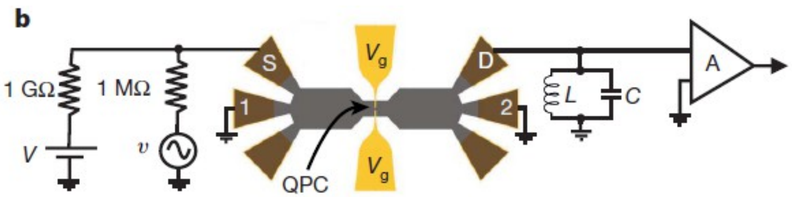
Anyons Anywhere?

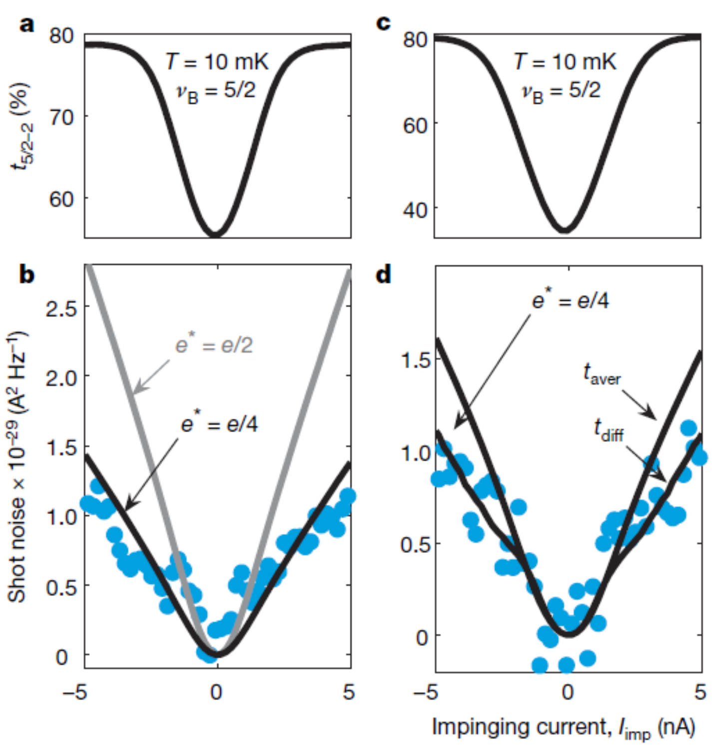
Radu et al., Science 320, 899 (2008)



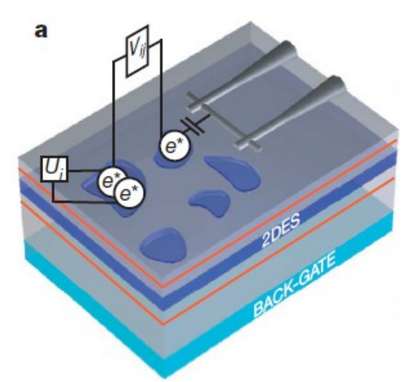


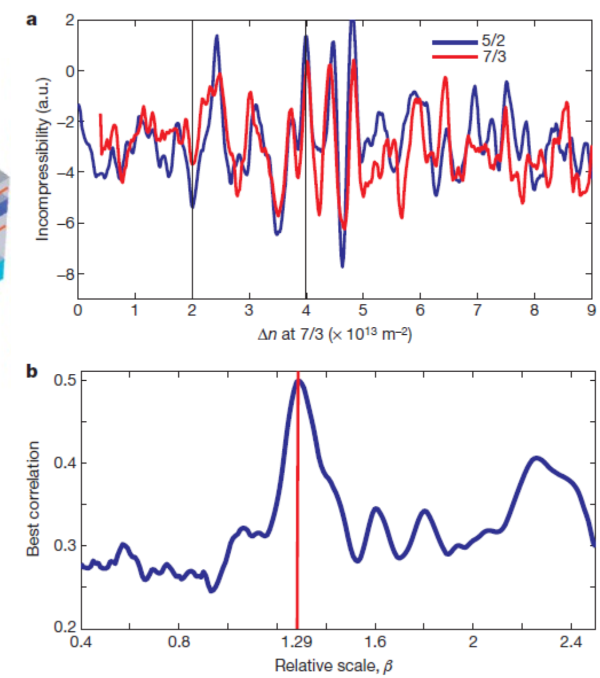
Dolev et al., Nature 452, 829 (2008)





Venkatachalam et al., Nature 469, 285 (2011)



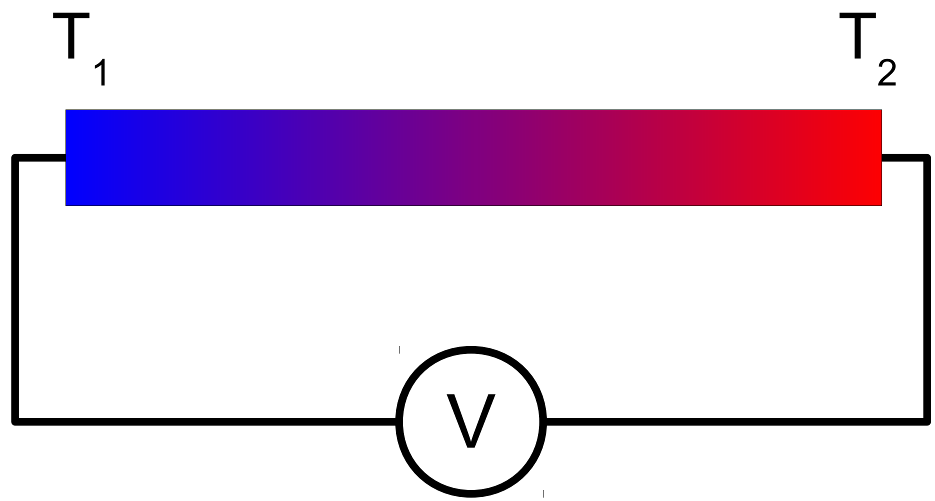


Noise, tunneling conductance, and local incompressibility support the existence of e/4 anyons. But what about their statistics?

How to Measure Ground State Degeneracy?

- Seebeck effect

$$Q = \frac{E_{emf}}{\nabla T} = -\frac{\nabla V}{\nabla T}$$



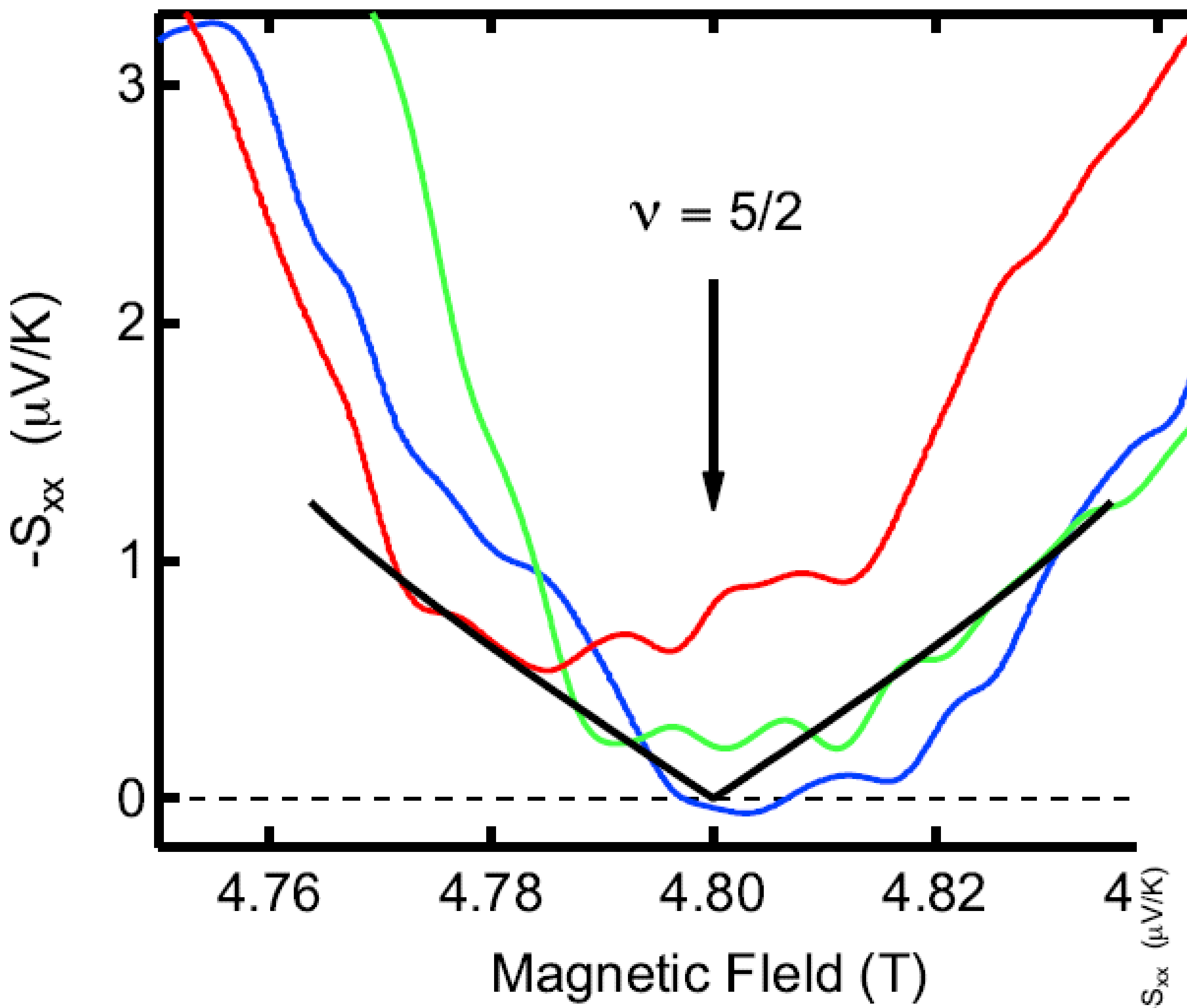
- Number of quasiparticles

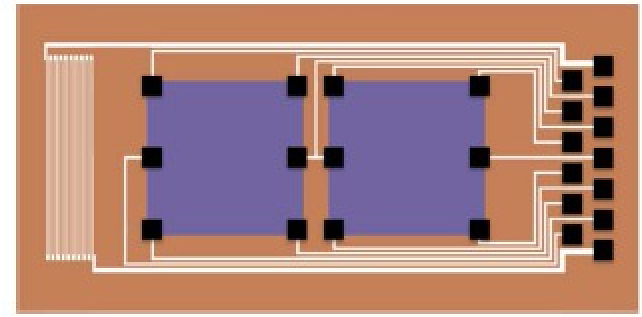
$$n = N_e \left| \frac{e}{e^*} \frac{(B - B_0)}{B_0} \right|$$

- Ground state degeneracy $\qquad d^n \quad \rightarrow \quad S = k\, n \ln d$
- Thermopower measures "entropy per charge carrier"

$$Q = -\frac{S}{N_e e} = -\left| \frac{(B - B_0)}{B_0} \right| \frac{k}{|e^*|} \ln d$$

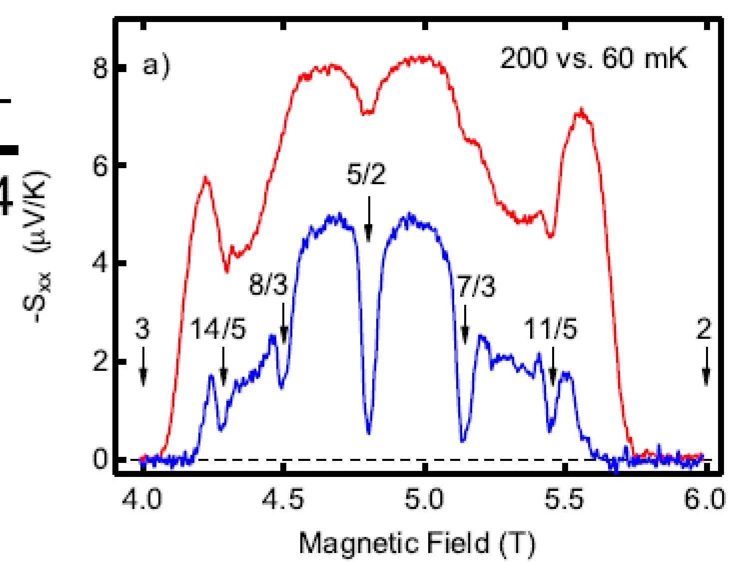
Thermopower (Caltech group)





Chickering et al.,
arXiv:1211.3672

T = 20 mK (B), 28 mK (G) and 41 mK (R)



$$Q = -\left|(B-B_0)/B_0\right|\left(k/|e^*|\right)\ln d$$

Yang & Halperin, 2009

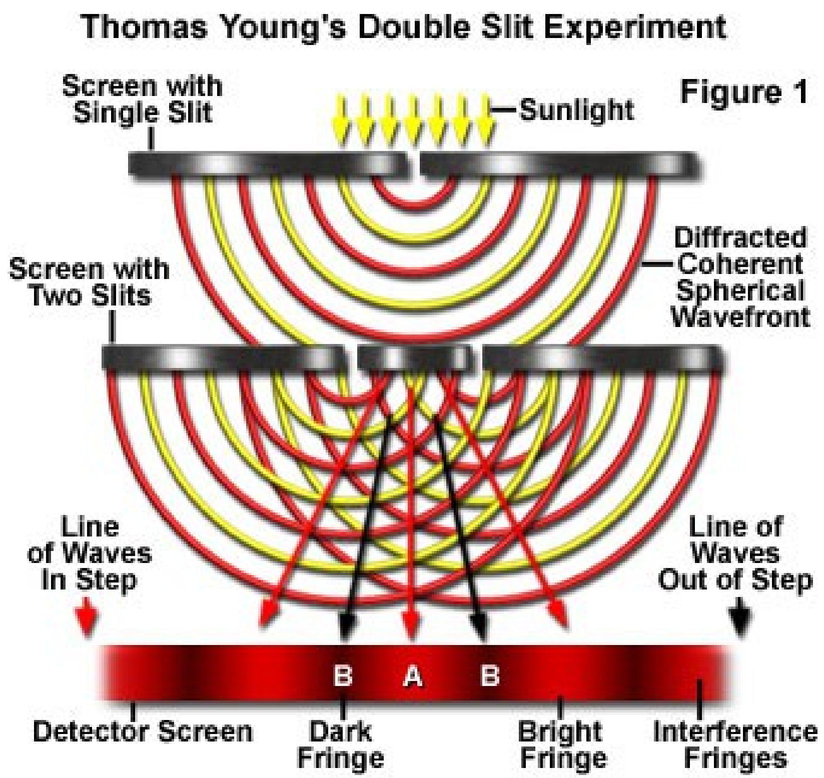
#2: More Messages

- Fractional quantum Hall effect at filling fraction 5/2 (and 7/2) is distinct from odd-denominator states. It can be thought of as the reincarnation of a p+ip superfluid in quantum Hall regime.
- The 5/2 state support both Abelian charge e/2 quasiparticles and non-Abelian charge e/4 quasiparticles.
- The charge of the elementary quasiparticles has been proved by shot noise and local charge measurements to be e/4.
- There are tangible evidences in tunneling conductance and thermal power experiments that the 5/2 state may be of non-Abelian nature.

Next: quantum Hall interferometer

Young and Double Slit Interference

- Double-slit experiment (1801)



Thomas Young (1773-1829)

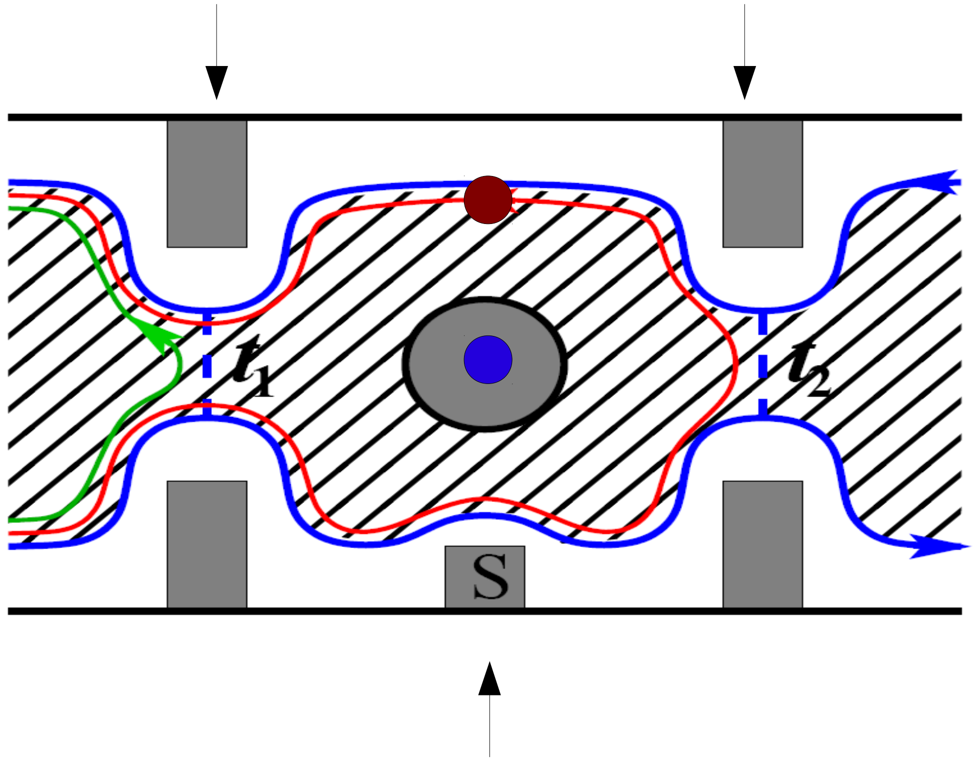
Double-slit interference demonstrates the wave nature of light and, later, other quantum particles.

Detecting Quasiparticle Statistics by Interference

Gates controlling the strength of tunneling

path via point contact 1

path via point contact 2

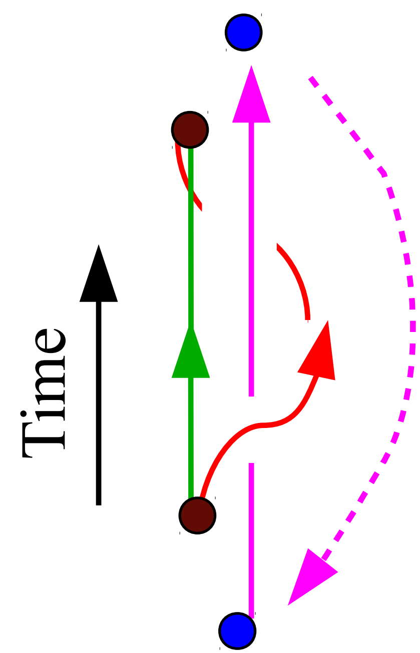


edge of the ½ droplet

edges of the filled Landau levels not included

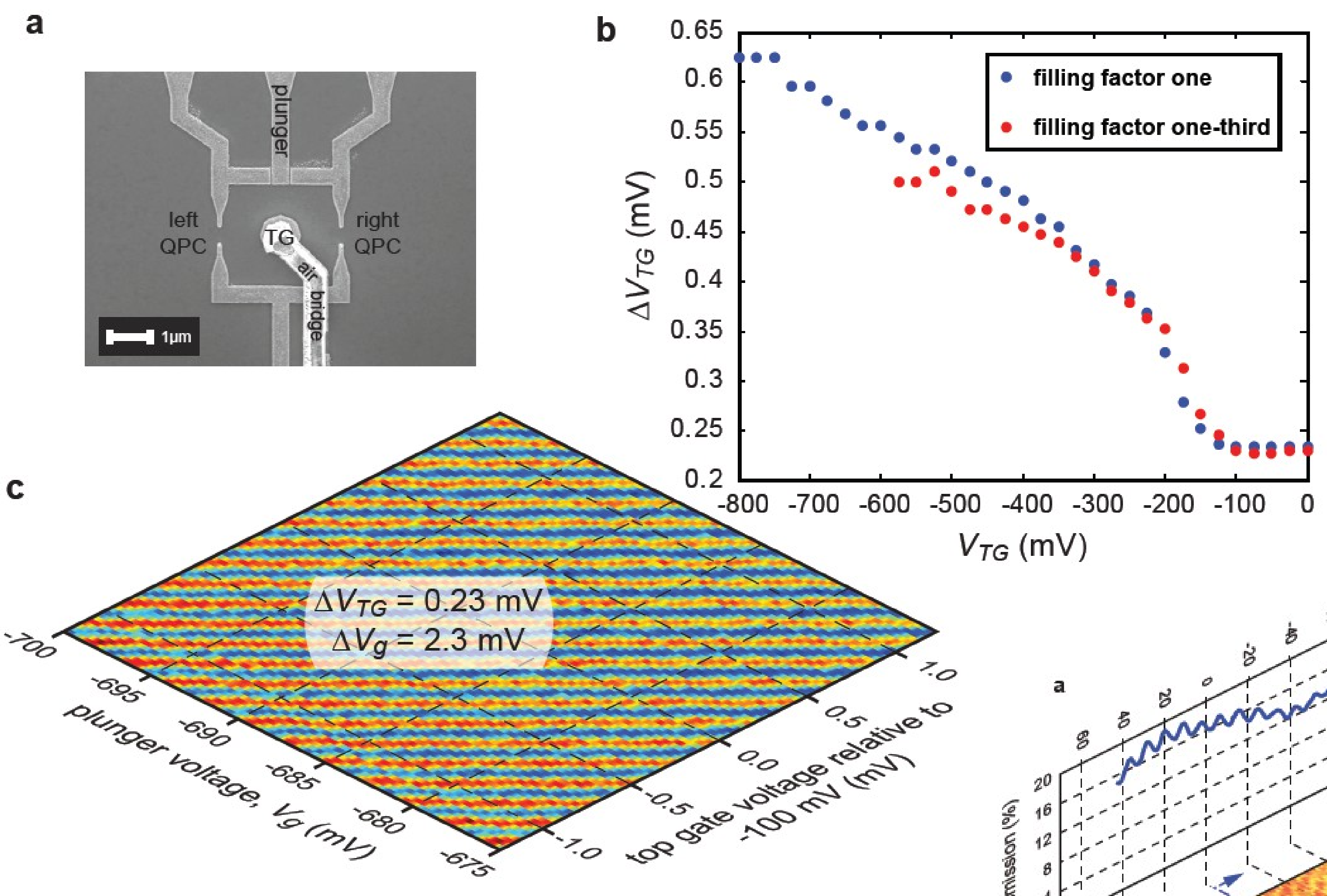
Side gate controlling the number of quasiparticles on the central antidot

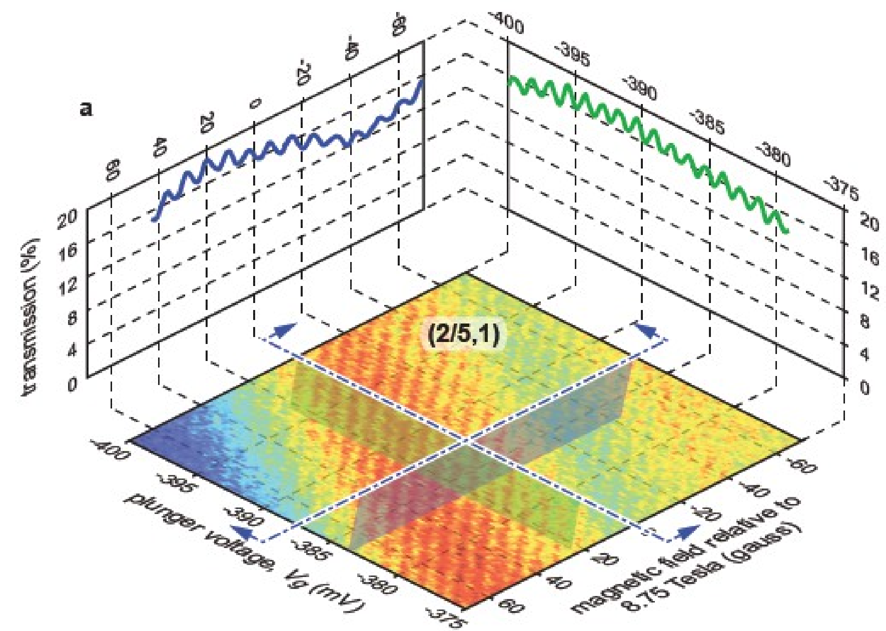
$$G \propto \left| t_1 U_1 + t_2 U_2 \middle| \Psi \right\rangle \Big|^2 = |t_1|^2 + |t_2|^2 + 2\,\Re\left\{ t_1^* t_2 e^{i\Phi} \langle \Psi | M_n | \Psi \rangle \right\}$$



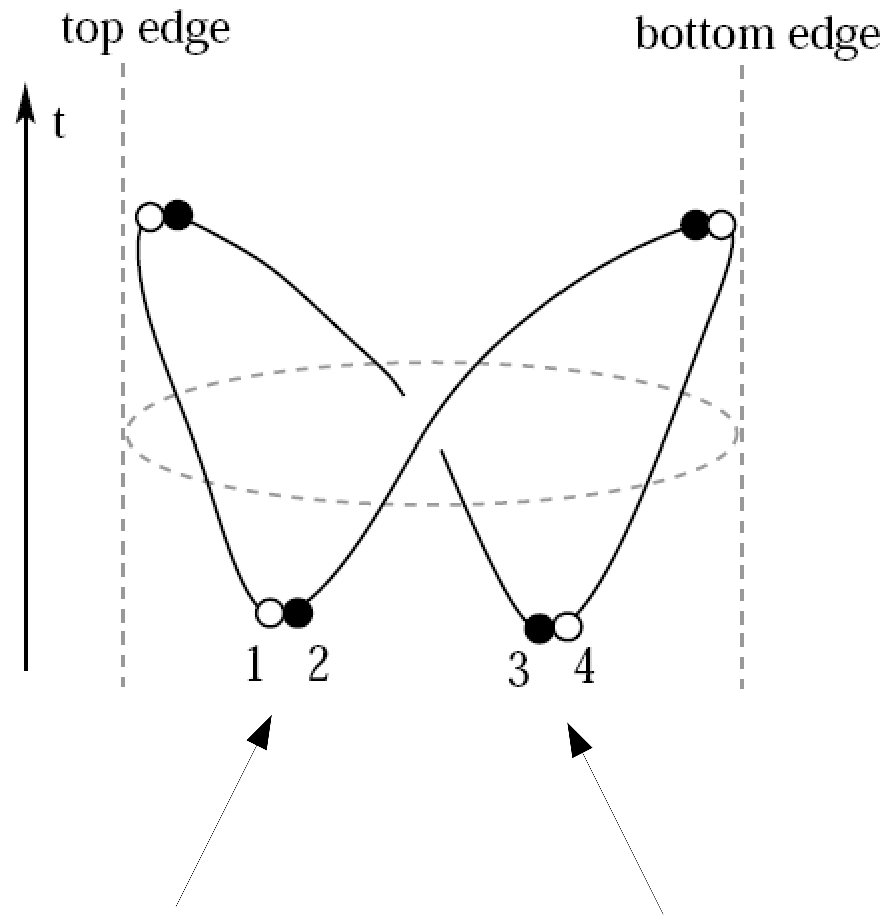
Chamon et al., 1997; Fradkin et al., 1998

Heiblem Group, PNAS (2010)



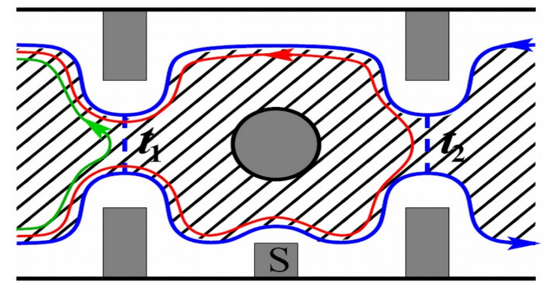


Relevant Process and Diagram

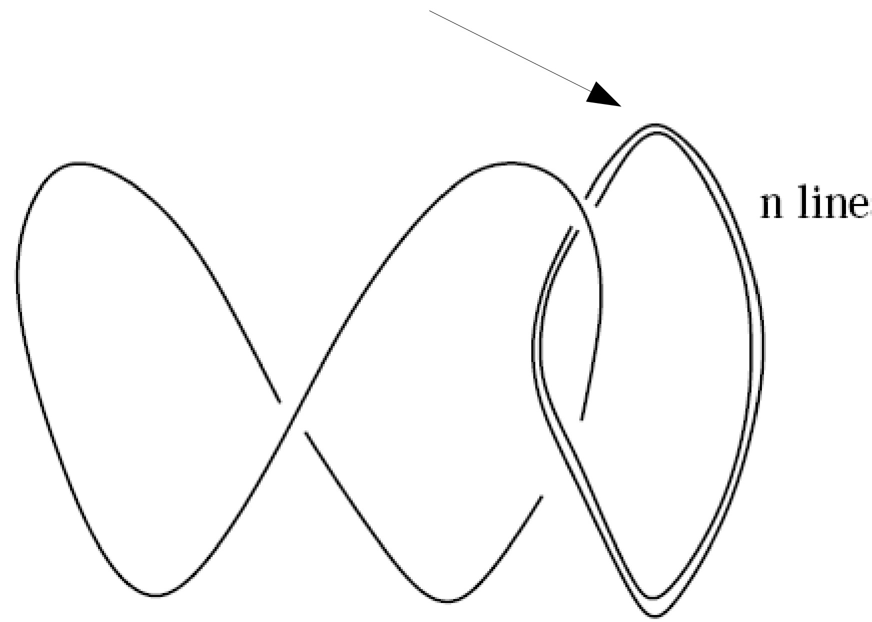


point contact 1

point contact 2

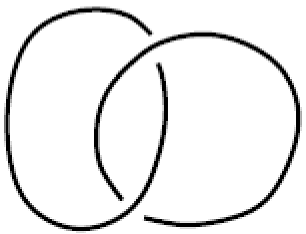
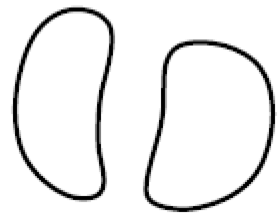


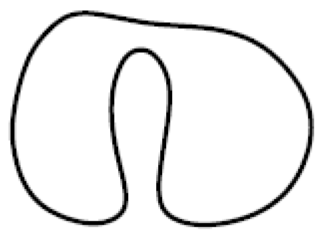
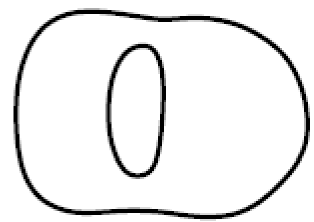
central antidot (n non-Abelian qps)



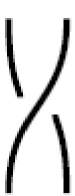
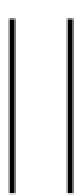
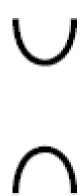
evaluate corresponding Jones polynomials

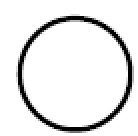
Evaluation of the Jone Polynomial

 $=$ q  $+$ 

$+$  $+\ q^{-1}$ 

$= (q + q^{-1})\, d^2 + 2d$ $= 0$!!

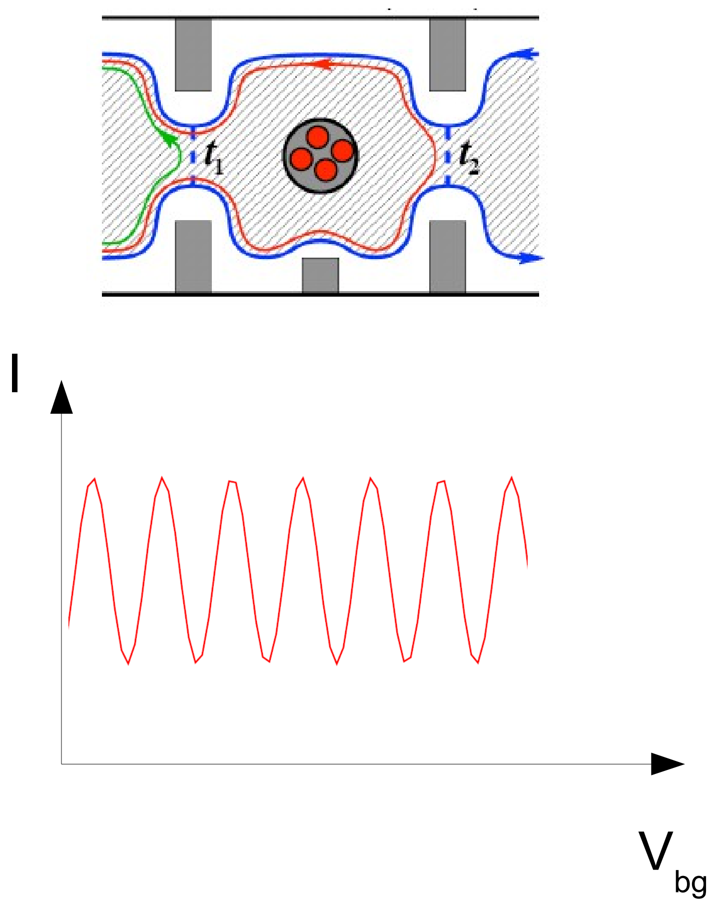
 $= q^{1/2}$  $+\ q^{-1/2}$ 

 $= d = -q - q^{-1} = \sqrt{2}$

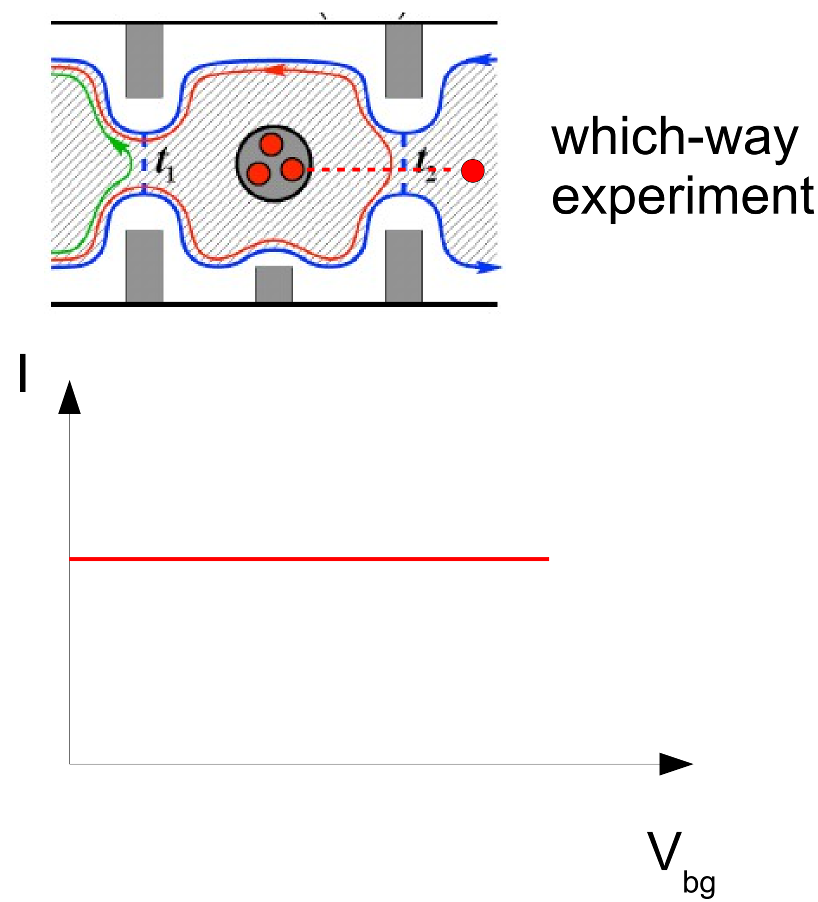
$q = -e^{i\pi/4}$

Expected Experimental Signature

Odd-even effect: Stern & Halperin (06); Bonderson, Kitaev & Shtengel (06)



Even number of non-Abelian quasiparticles inside the interference loop



Odd number of non-Abelian quasiparticles inside the interference loop

Willett et al., PNAS (2009); arXiv:0807.0221

PNAS

Measurement of filling factor 5/2 quasiparticle interference with observation of charge e/4 and e/2 period oscillations

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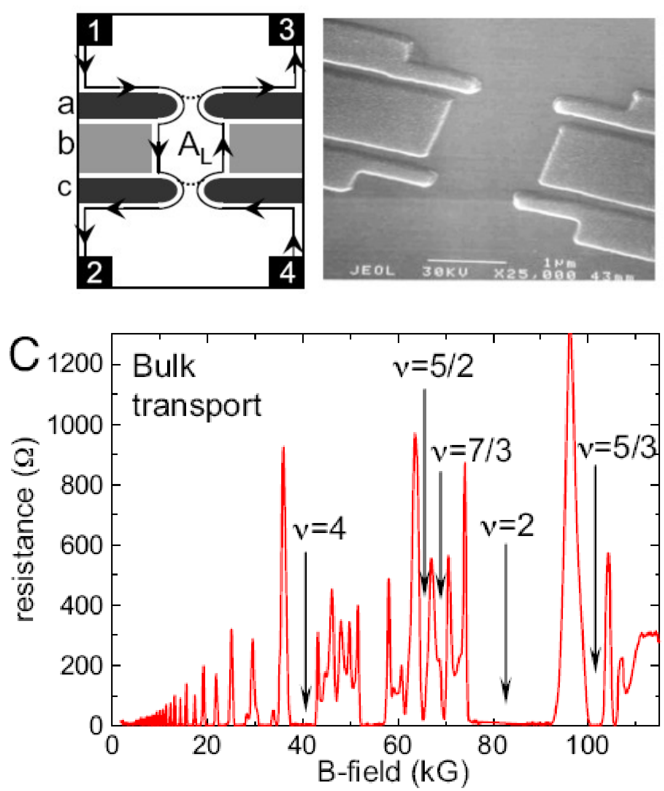
Edited by W. F. Brinkman, Princeton University, Princeton, NJ, and approved March 30, 2009 (received for review December 10, 2008)



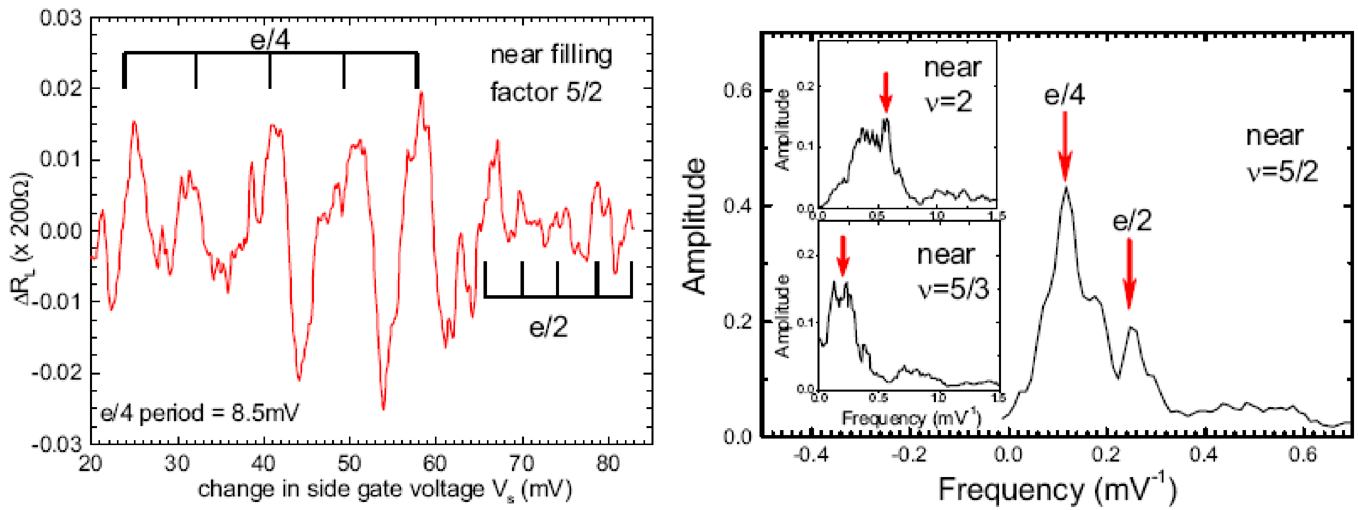


Fig. 3. A–B interferometric measurements at 5/2 filling factor displaying oscillations with periods corresponding to e/4 and e/2. R_L is measured for side-gate sweeps at filling factor $\nu = 5/2$ using similar measurements at $\nu = 2$ and 5/3 or 7/3 as metrics. Left and right

Charge-e/2 Quasiparticles?

$$\Psi_{qh}^{e/4} = \sigma\, e^{i\phi/2\sqrt{2}}$$

Most relevant.
Charge & neutral components.

$$\Psi_{qh}^{e/2} = e^{i\phi/\sqrt{2}},\ \cancel{\psi e^{i\phi/\sqrt{2}}}$$

Irrelevant to inter-edge tunneling in RG sense

Less relevant but relevant
Charge component only!

$\sigma \times \sigma = 1 + \psi$ (Ising/Majorana)

$$I_{12} \propto \sum_q s_q |\Gamma_1||\Gamma_2| e^{-|x_1-x_2|/L_\phi} \cos\left(2\pi \frac{q}{e}\frac{\Phi}{\Phi_0} + \phi_q + arg(\Gamma_1 \Gamma_2^*)\right)$$

tunneling amplitude

favors e/4 qps

coherence length due to thermal smearing

$$L_\phi = \frac{1}{2\pi k_B T}\left(\frac{g_c}{v_c} + \cancel{\frac{g_n}{v_n}}\right)^{-1}$$

favors e/2 qps

Coherence Length of e/4 Quasiparticles

XW, Hu, Rezayi & Yang, PRB (2008)

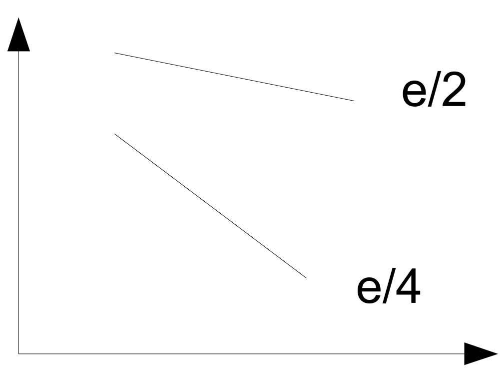
$$L_\phi = \frac{1}{2\pi k_B T}\left(\frac{g_c}{v_c}+\frac{g_n}{v_n}\right)^{-1}$$

Theory predicted at 25 mK:

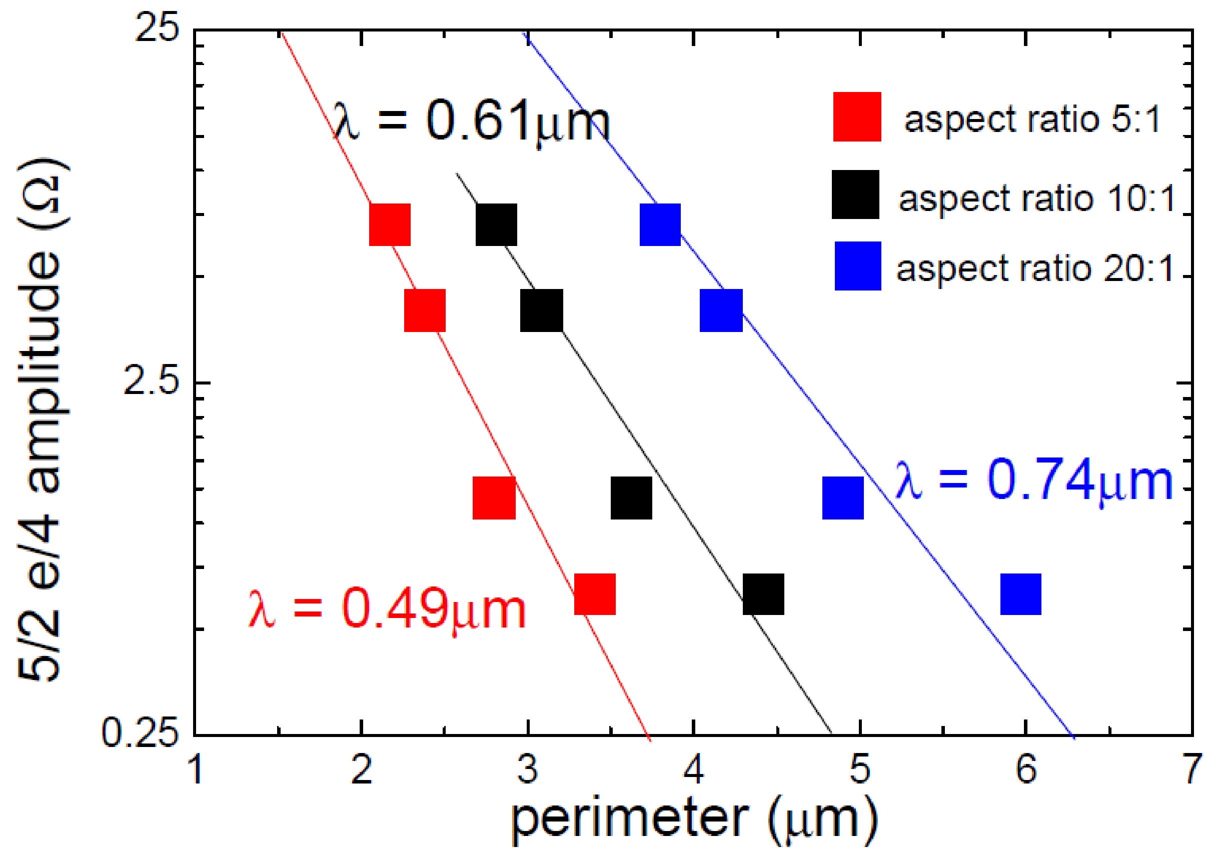
(e/4): L_ϕ ~ 1.5 mm

(e/2): L_ϕ ~ 5 mm

ln (exp{-L/Lϕ})



Willett et al., arXiv:1301.2594

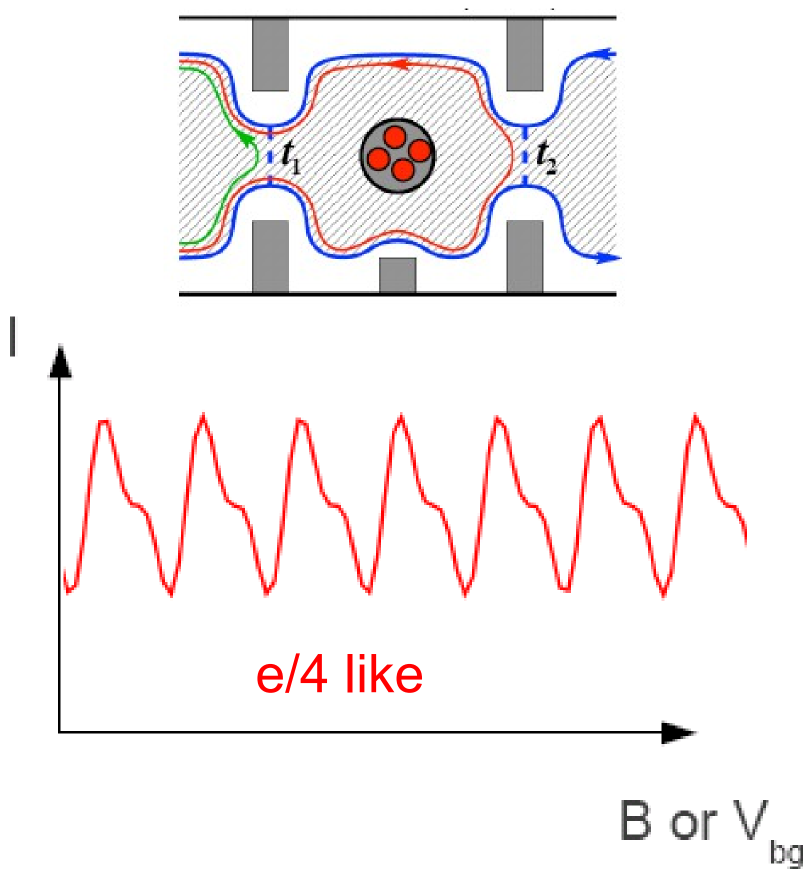


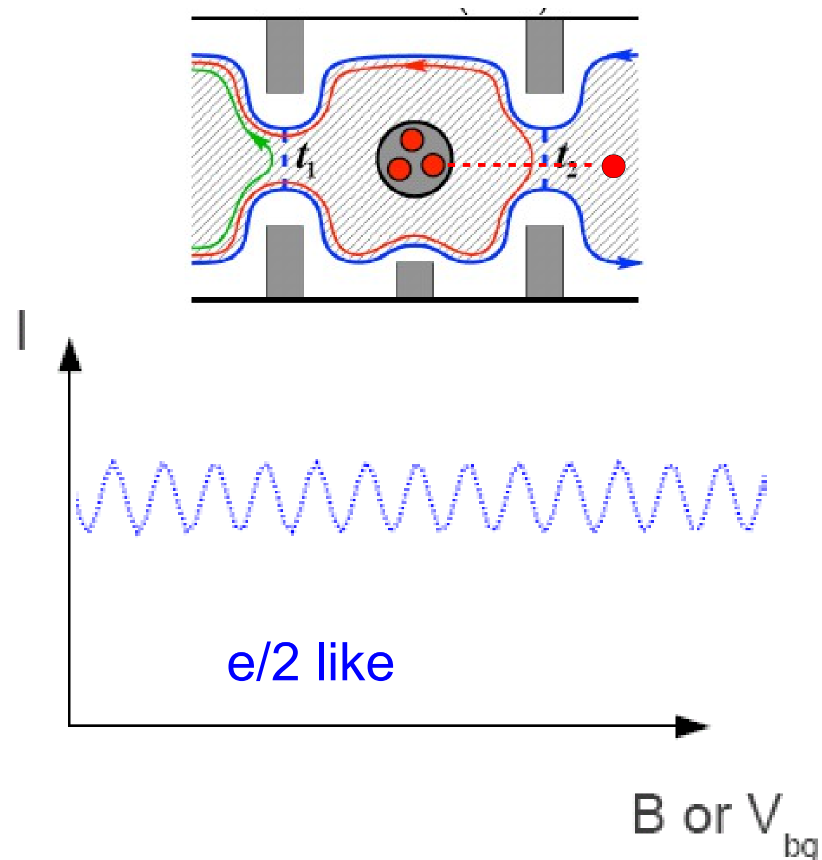
e/2: longer, because of the absence of the slow neutral mode. More visible at higher temperatures.

Signature for Non-Abelian Statistics

Background Abelian signal:

XW, Hu, Rezayi & Yang, PRB (2008)





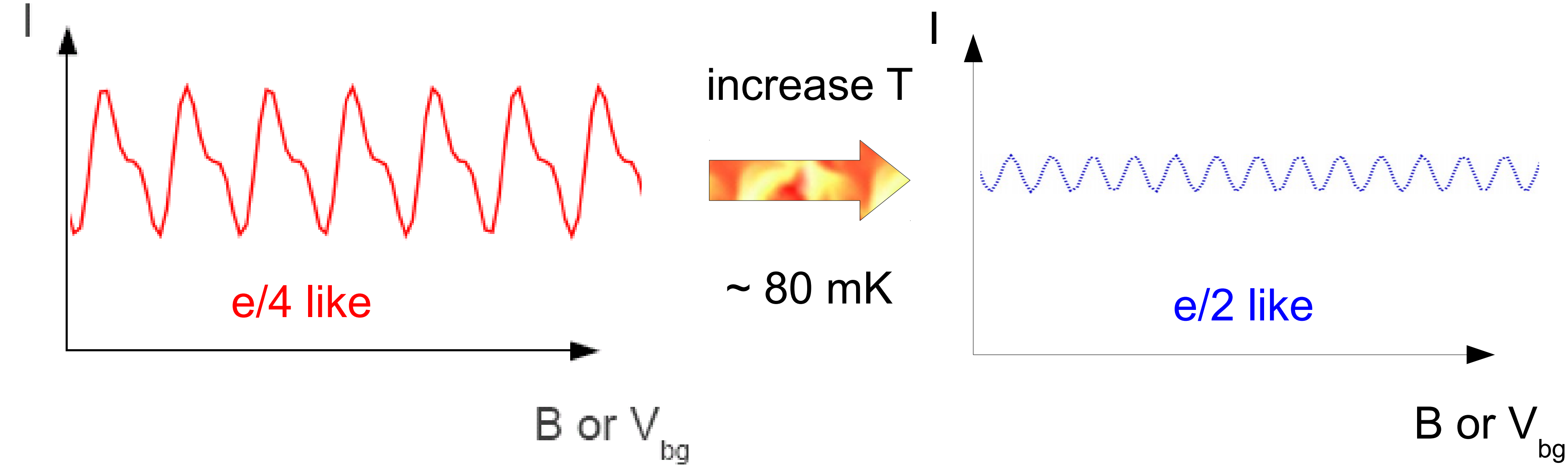
Even number of non-Abelian quasiparticles inside the interference loop

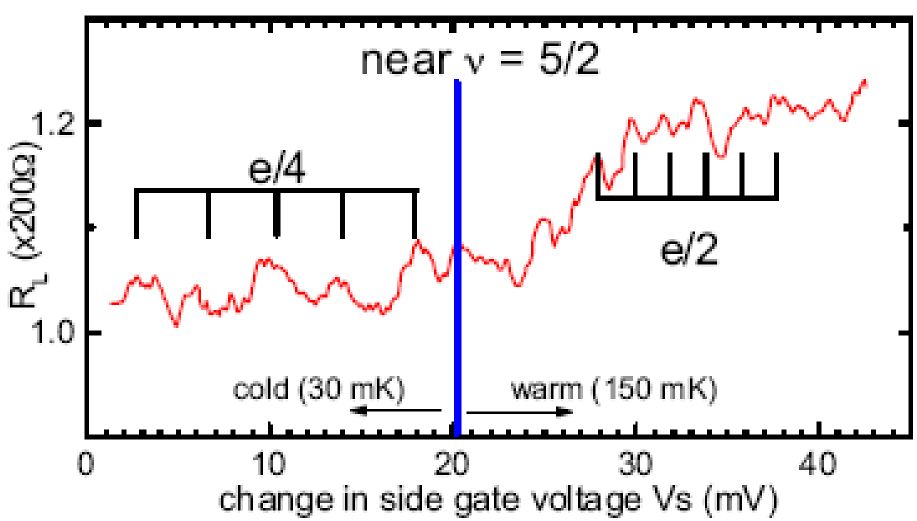
Odd number of non-Abelian quasiparticles inside the interference loop

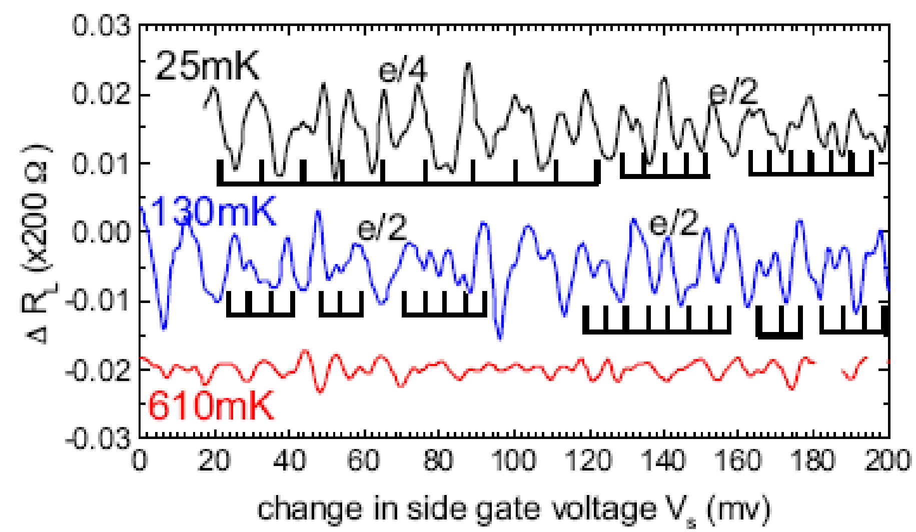
Coherence length ~ 1 μm

e/4 Pattern Suppressed at Higher T

XW, Hu, Rezayi & Yang, PRB (2008)





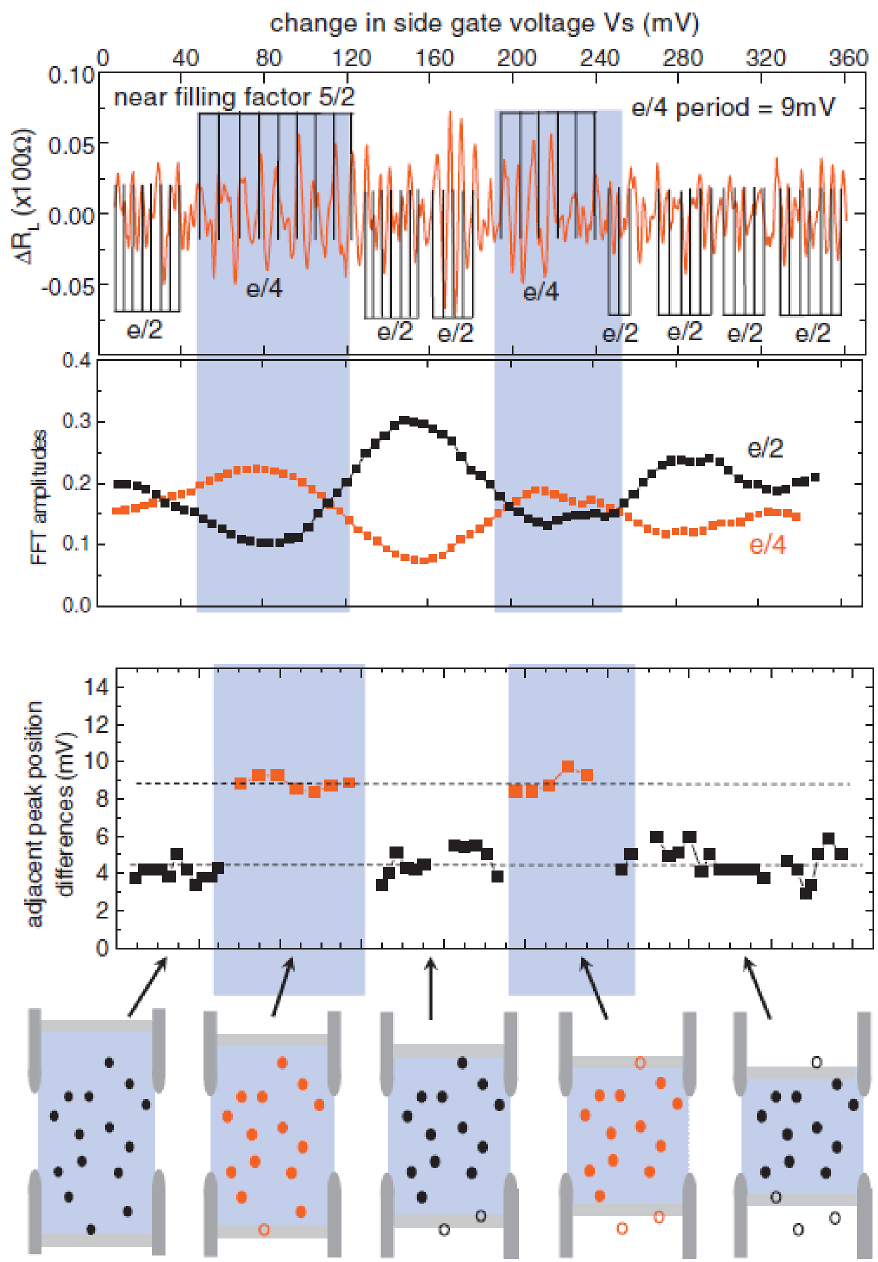


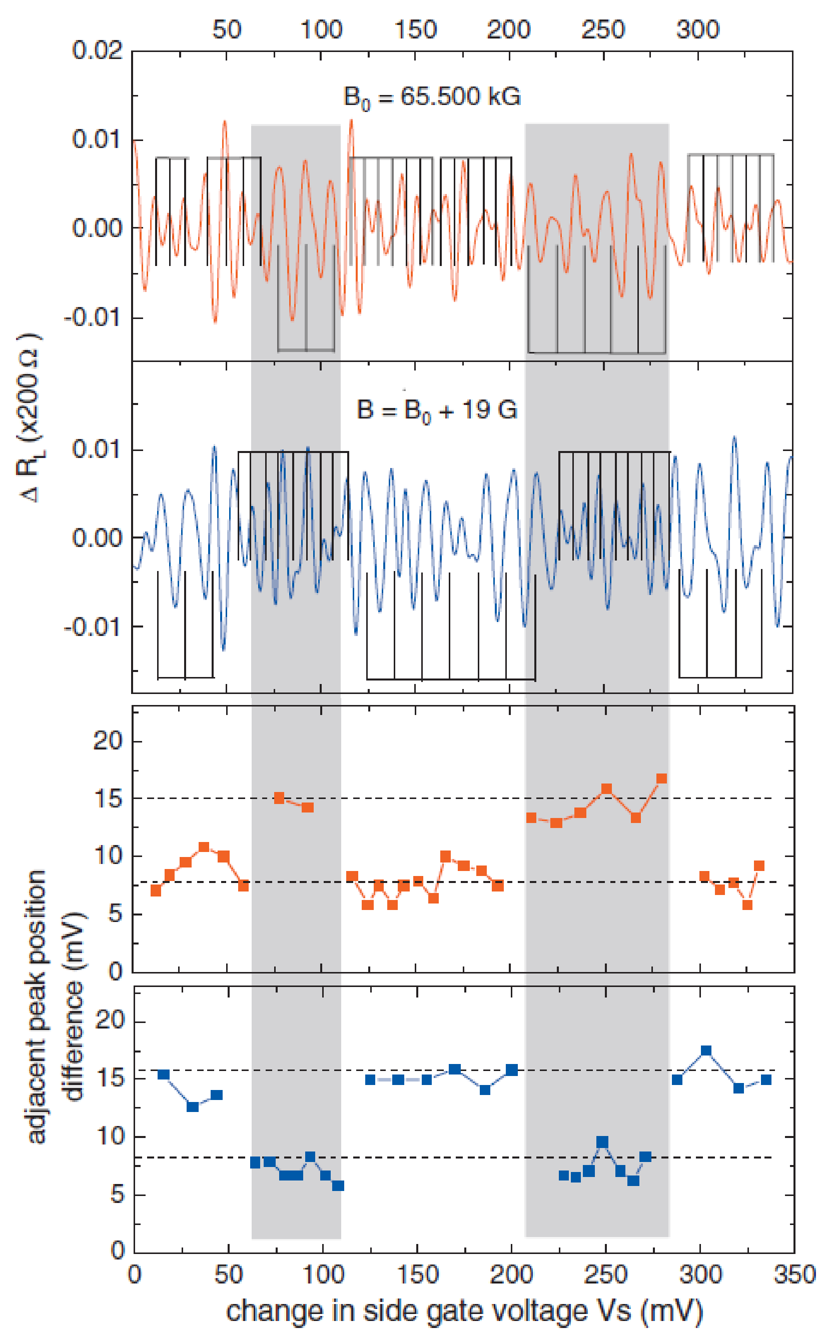
period lines in the swept side-gate data. (*C*) Data indicate temperature dependence of e/4 and e/2 oscillations: e/2 oscillations may be made more prevalent with an increase in temperature. The temperature of the sample was taken from

Willett et al., PNAS (2009)

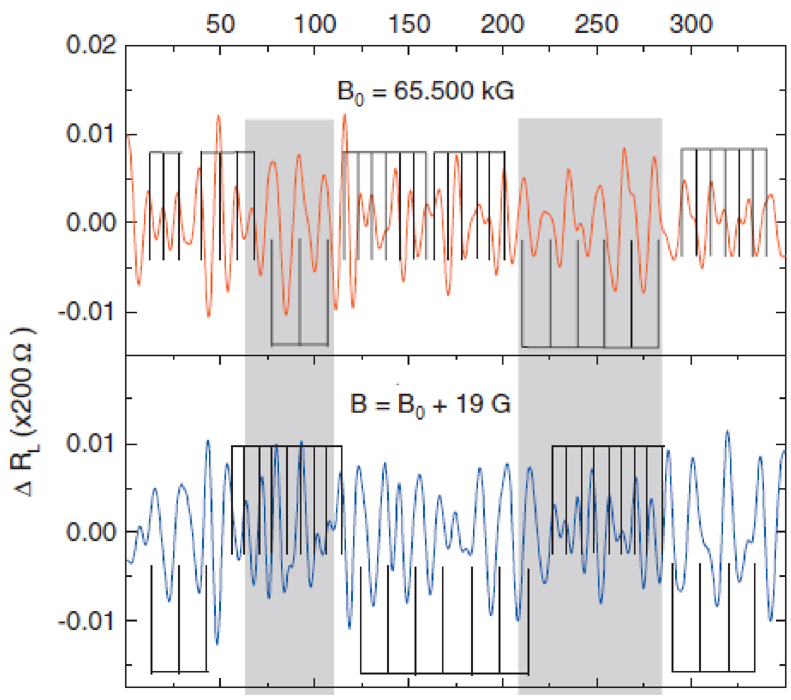
Alternative e/4 and e/2 Patterns

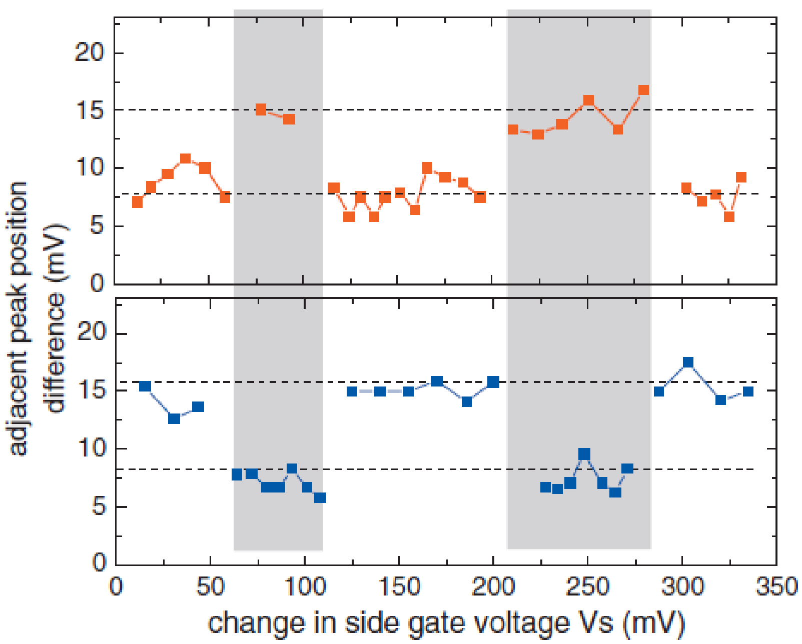
Willett et al., PRB (2010)





B-field Induced e/4 and e/2 Oscillation Swap





65 kG (upper panel) → 65 kG + 19 G (lower panel)

A suitable adjustment of the applied magnetic field is expected to change the parity in the encircled localized quasiparticle number, thus change the pattern of aperiodic e/4 and e/2 observed over the same side-gate sweep.

Willett et al., PRB (2010)

#3: Yet More Messages

- Willett's interference data agrees with the existence of both charge e/4 and e/2 quasiparticles.
- Experimental data does not violate the theoretical expectation that the ground state wave function of the 5/2 state is the Moore-Read state (or its particle-hole conjugate).
- Non-Abelian e/4 quasiparticles have short decoherence length, which limits the device size to 1 micron or so with today's technology.
- The interferometer experiment demonstrated that we have the technology to create anyons and to manipulate them to achieve braiding.
- Reproduction of data and significant improvements in experiments are desired.

Next: What to do with anyons?

Model of Anyons

- A model of anyons is a theory of a two-dimensional medium with a mass gap, where the particles carry locally conserved charges. One defines
  - A finite *label set* $\{a,b,c,\ldots\}$;
  - The *fusion rules* $a \times b = \Sigma_c N_{ab}^{c}\, c$;
  - The *F-matrix* (expressing associativity of fusion);
  - The *R-matrix* (braiding rules).

F & R satisfy self-consistency equations, known as the pentagon and hexagon equations.

Ising anyon model:

$\{1, \sigma, \psi\}$

$\sigma \times \sigma = 1 + \psi$
$\psi \times \psi = 1$
$\psi \times \sigma = \sigma \times \psi = \sigma$

$$F = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$R = \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{3i\pi/8} \end{pmatrix}$$

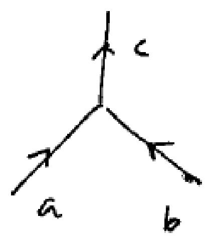
Diagrams

- Anyon a : 

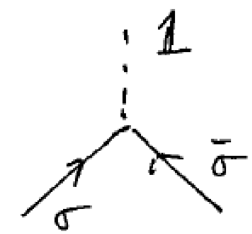
- Antiparticle $\bar{a}$: 

= 

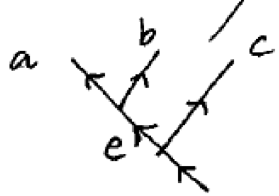
- Fusion $a \times b = c \; (+ \cdots)$

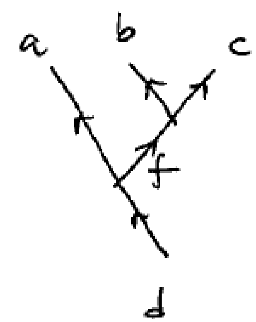
$\left(\frac{d_c}{d_a d_b}\right)^{1/4}$ 

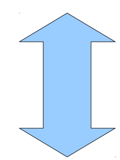
$= \langle ab; c|$



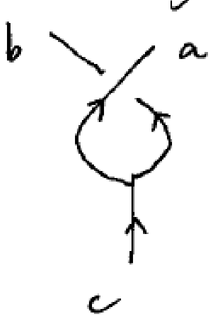
- Associativity

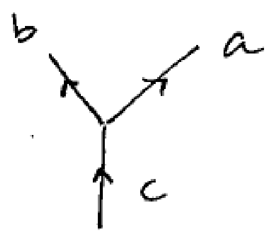


$= \sum_f \left[F^{abc}_d \right]_{ef}$ 

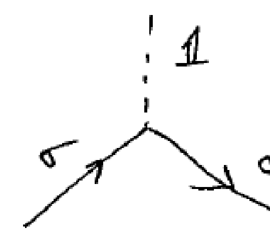


- Braiding



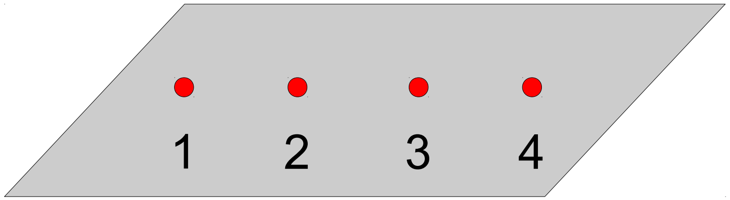
$= R^{ab}_c$ 

phase



Four Ising Anyons as a Qubit

- Even when one fixes the location of all quasiholes, there are more than one states



$$\Psi_{(12)(34)} = Pf\left(\frac{(z_i-\xi_1)(z_i-\xi_2)(z_j-\xi_3)(z_j-\xi_4)+i\Leftrightarrow j}{z_i-z_j}\right)\prod_{1\leqslant i<j\leqslant N}(z_i-z_j)^2$$

$$\Psi_{(13)(24)} = Pf\left(\frac{(z_i-\xi_1)(z_i-\xi_3)(z_j-\xi_2)(z_j-\xi_4)+i\Leftrightarrow j}{z_i-z_j}\right)\prod_{1\leqslant i<j\leqslant N}(z_i-z_j)^2$$

$$\Psi_{(14)(23)} = Pf\left(\frac{(z_i-\xi_1)(z_i-\xi_4)(z_j-\xi_2)(z_j-\xi_3)+i\Leftrightarrow j}{z_i-z_j}\right)\prod_{1\leqslant i<j\leqslant N}(z_i-z_j)^2$$

- But they are not linearly independent!

$$\Psi_{(12)(34)}-\Psi_{(13)(24)}=(1-x)\left(\Psi_{(12)(34)}-\Psi_{(14)(23)}\right) \qquad x=\frac{(\xi_1-\xi_2)(\xi_3-\xi_4)}{(\xi_1-\xi_3)(\xi_2-\xi_4)}$$

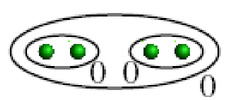
Four Ising Anyons as a Qubit

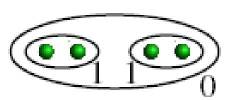
- Ansatz wavefunction (decomposition into two quasihole-paring wavefunctions)

$$\begin{aligned}\Psi^{(0,1)}(\xi_1,\xi_2,\xi_3,\xi_4;z_1,\ldots,z_N) &= A^{(0,1)}(\{\xi\})\,\Psi_{(12)(34)}(\{\xi\},\{z\}) \\ &+ B^{(0,1)}(\{\xi\})\,\Psi_{(13)(24)}(\{\xi\},\{z\})\end{aligned}$$

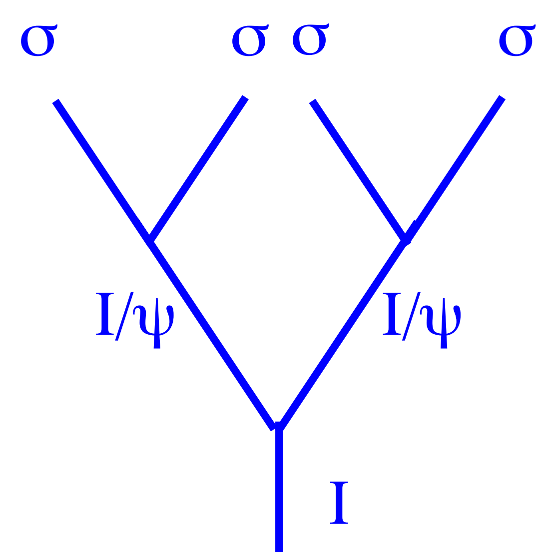
C. Nayak and F. Wilczek, Nucl. Phys. B 479 (1996) 529

E. Ardonne and K. Schoutens, Ann. Phys. 322 (2007) 201

$|0\rangle = |(\cdot\,\cdot)_0(\cdot\,\cdot)_0\rangle_0 =$ 

$|1\rangle = |(\cdot\,\cdot)_1(\cdot\,\cdot)_1\rangle_0 =$ 

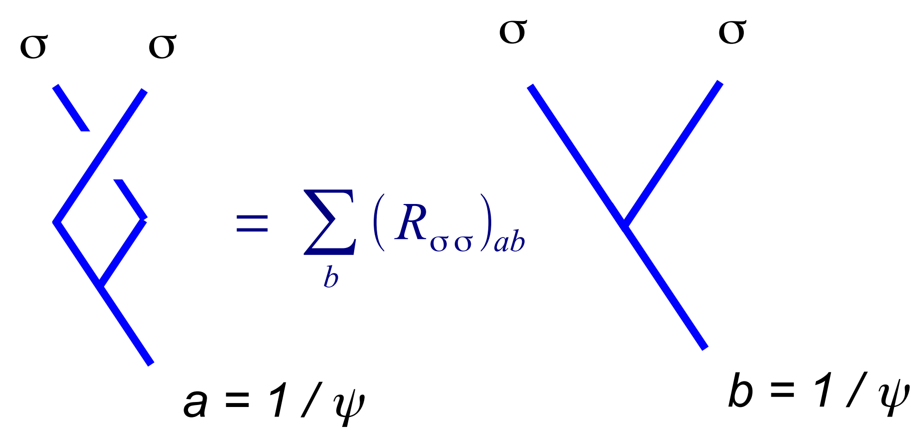
Ising: • = σ, 0 = 1, 1 = ψ



Identify the Two Fusion Channels

- The two linearly independent wave function can be written as

$$\Psi^{\pm}=\frac{[(\xi_1-\xi_3)(\xi_2-\xi_4)]^{1/4}}{(1\pm\sqrt{1-x})^{1/2}}\left(\Psi_{(13)(24)}\pm\sqrt{1-x}\,\Psi_{(14)(23)}\right) \qquad \Psi=a^{+}\,\Psi^{+}+a^{-}\,\Psi^{-}$$



- Exchanging ξ_1 and ξ_2, we have

$$1-x \;\rightarrow\; \frac{1}{1-x}$$

$$(\xi_1-\xi_3)(\xi_2-\xi_4) \;\rightarrow\; (\xi_2-\xi_3)(\xi_1-\xi_4) =(\xi_1-\xi_3)(\xi_2-\xi_4)(1-x)$$

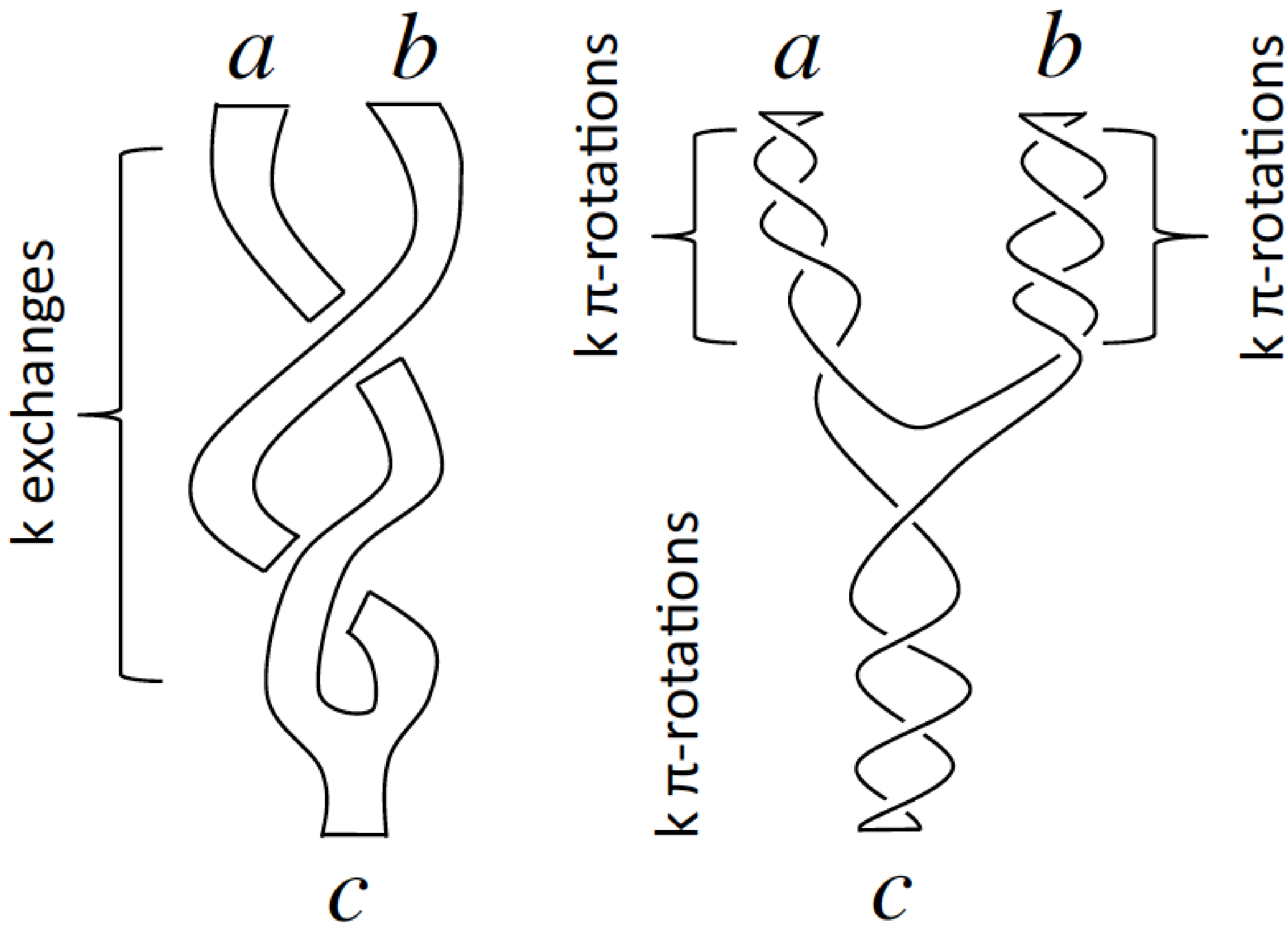
$$1-x=\frac{(\xi_1-\xi_4)(\xi_2-\xi_3)}{(\xi_1-\xi_3)(\xi_2-\xi_4)}$$

$$\Phi_{(13)(24)}\pm\sqrt{1-x}\,\Phi_{(14)(23)} \;\rightarrow\; \Phi_{(23)(14)}\pm\sqrt{\frac{1}{1-x}}\,\Phi_{(24)(13)} =\sqrt{\frac{1}{1-x}}\left[\pm\Phi_{(13)(24)}+\sqrt{1-x}\,\Phi_{(14)(23)}\right]$$

$$\boxed{\begin{pmatrix}\Psi^{+}\\ \Psi^{-}\end{pmatrix} \rightarrow \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\Psi^{+}\\ \Psi^{-}\end{pmatrix}}$$

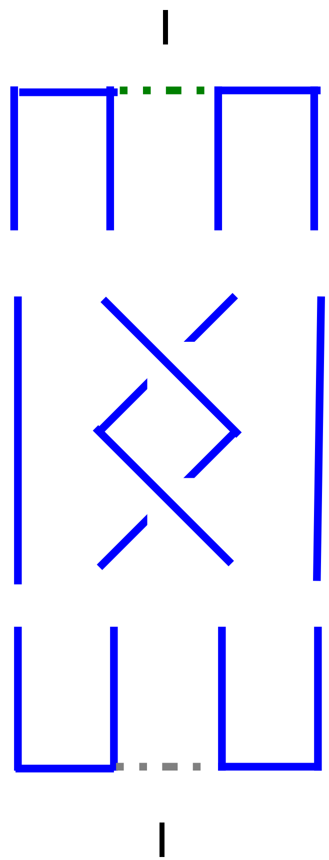
R-matrix (Ising x U(1))

Spin and Statistics



$$(R^c_{ab})^k = e^{-i\pi k s_a} e^{-i\pi k s_b} e^{i\pi k s_c}$$

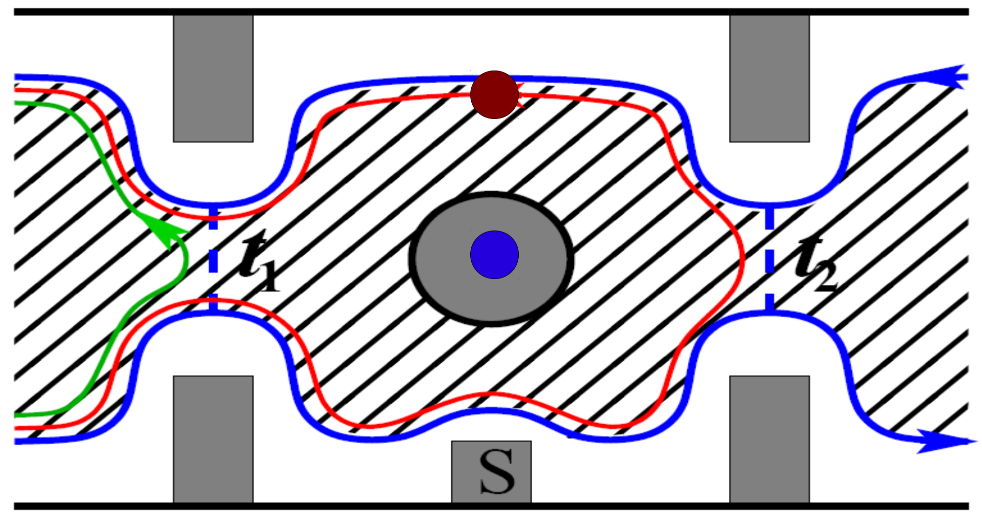
A Simple Quantum Computation



$\langle\Psi|$

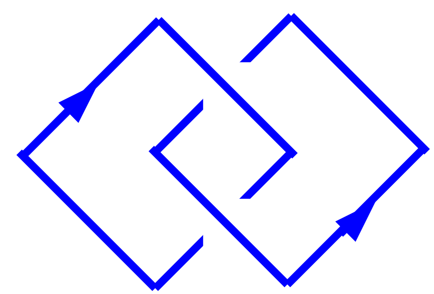
M_n

$|\Psi\rangle$

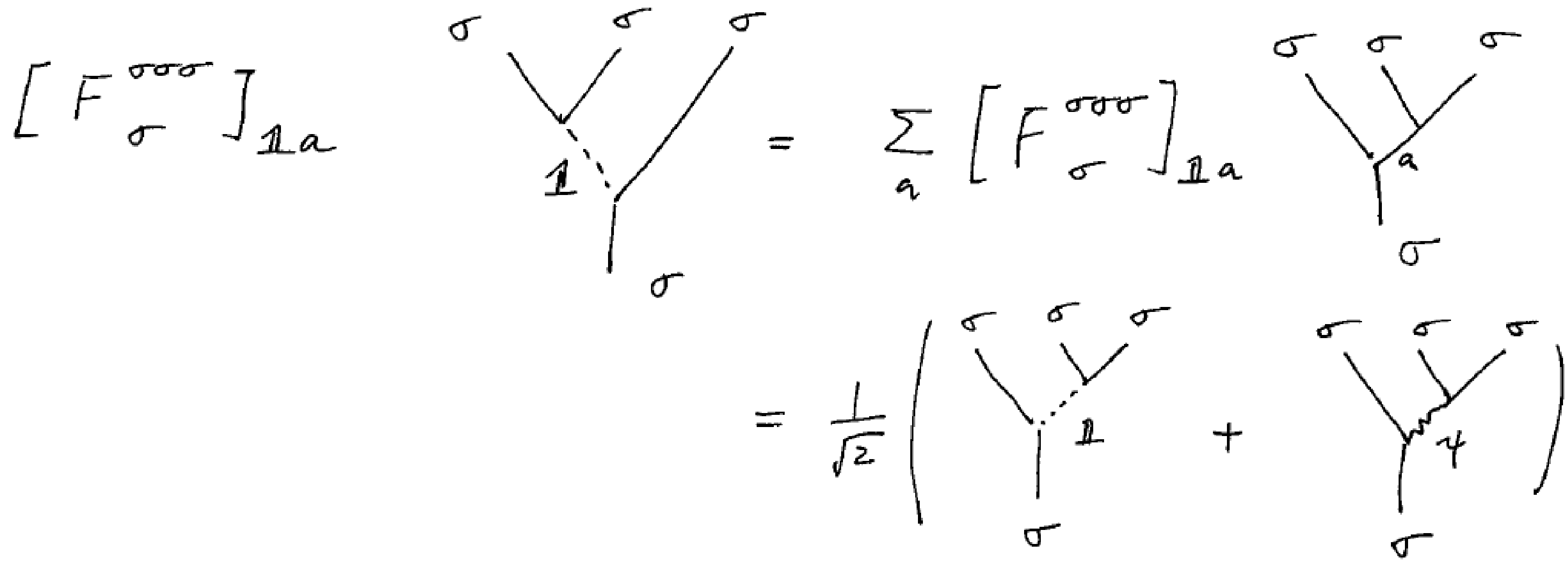


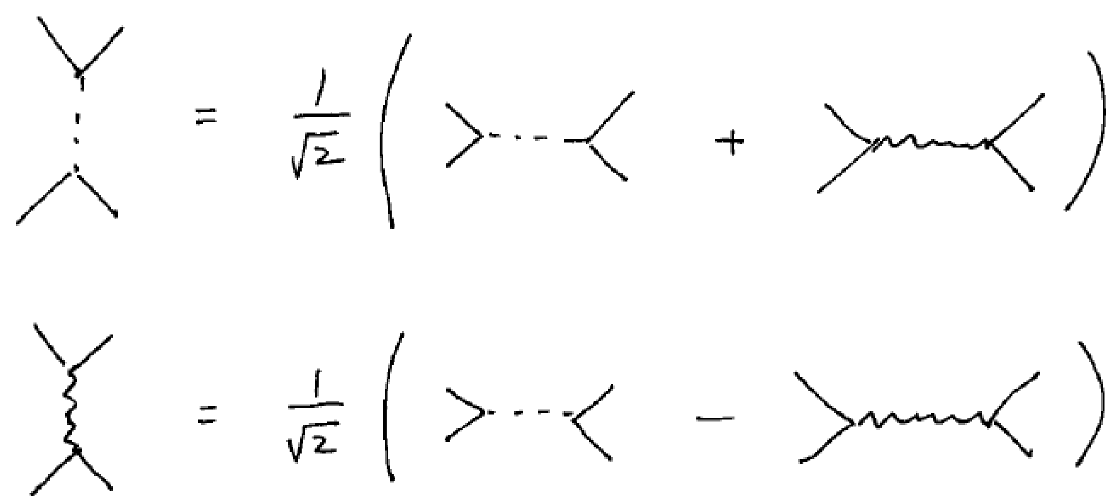
$$G \propto \left| t_1 U_1 + t_2 U_2 |\Psi\rangle \right|^2 = |t_1|^2 + |t_2|^2 + 2\,\Re\left\{ t_1^* t_2 e^{i\phi} \langle \Psi | M_n | \Psi \rangle \right\}$$

$\langle \Psi | M_n | \Psi \rangle$

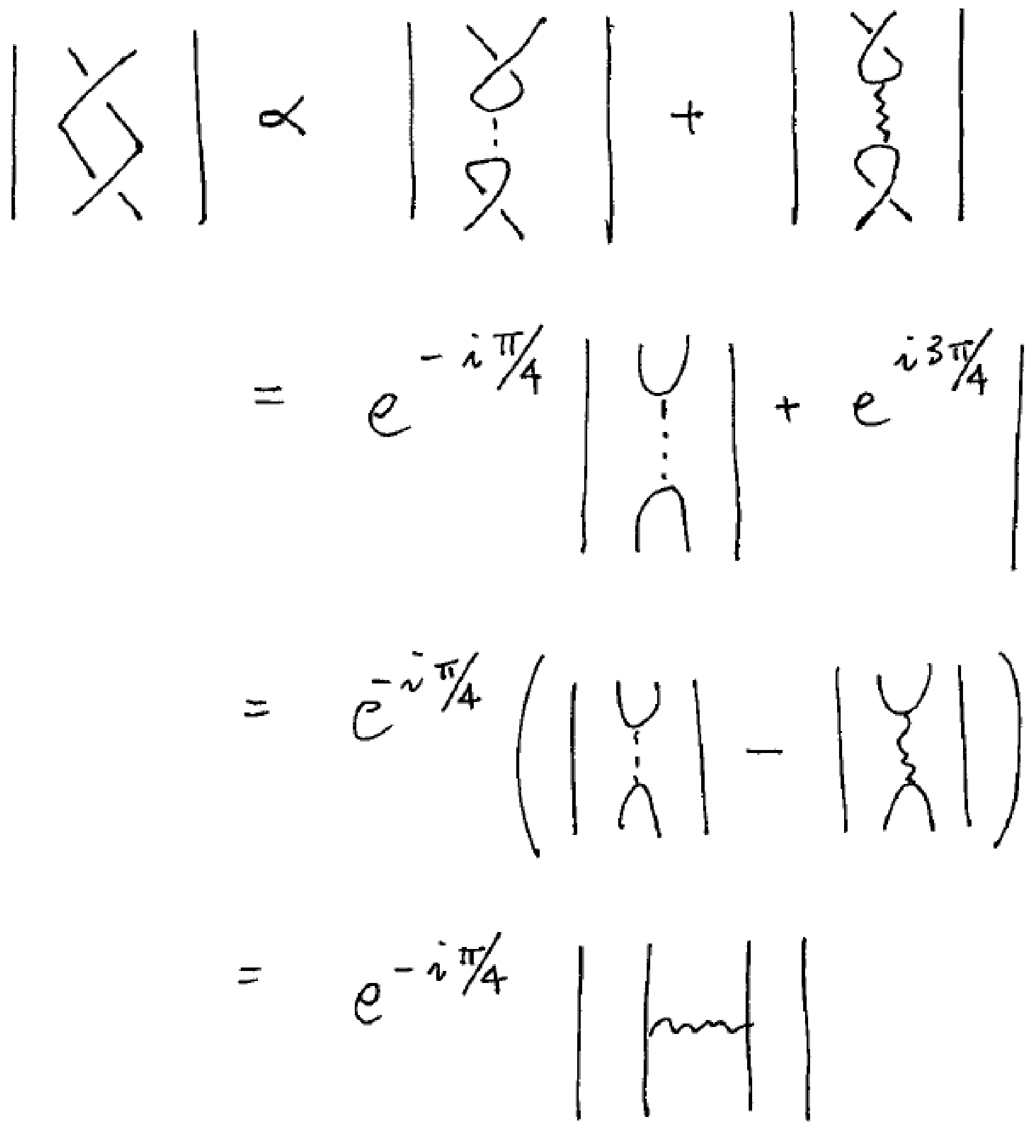


Calculating with F-Matrix



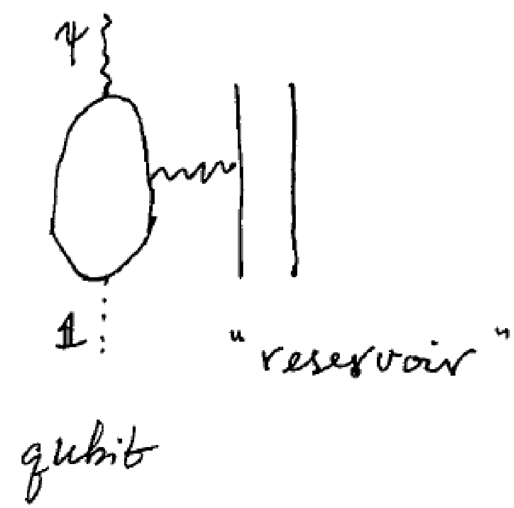


NOT Gate

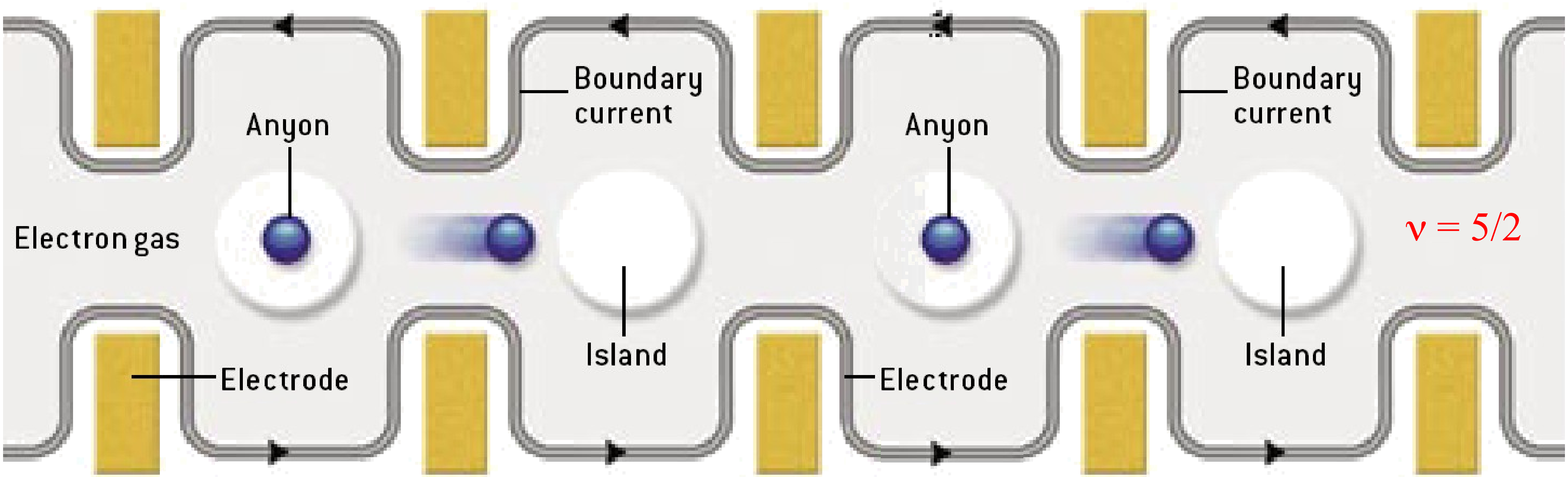


$$R_{\mathbb{1}}^{\sigma\sigma} = e^{-i\pi/8}$$

$$R_{\psi}^{\sigma\sigma} = e^{i3\pi/8}$$



Initialize Anyons

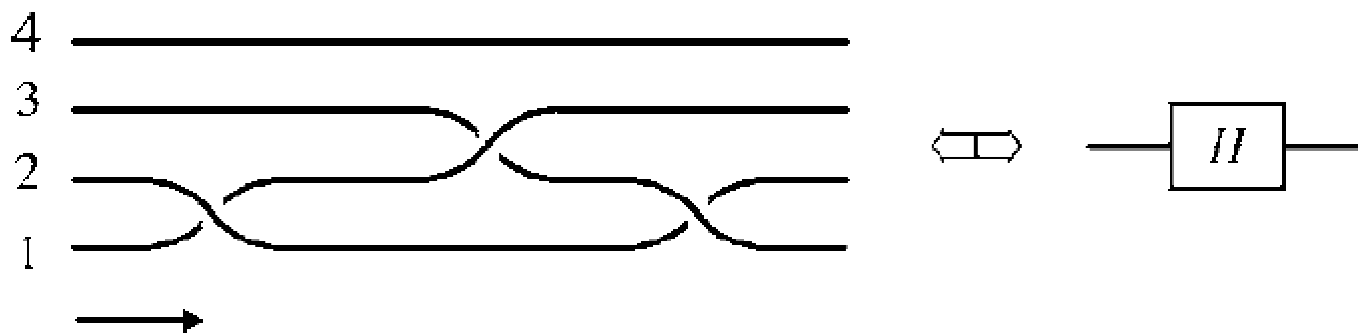


$\nu = 5/2$

Das Sarma, Freedman & Nayak (2005)

Braiding Example: Hadamard Gate

- Braiding diagram for the Hadamard gate

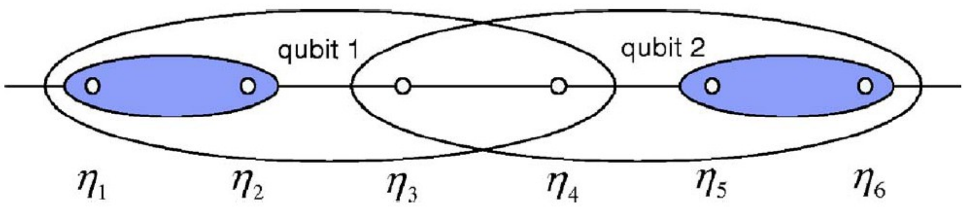


$$H = R_{12}^{-1} R_{23} R_{12}^{-1} \simeq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The Ising model is not universal; it cannot generate all single-qubit gates!

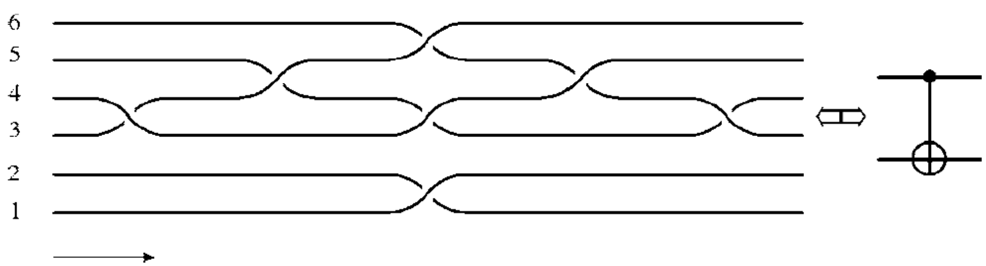
Braiding Example: CNOT Gate

$$R_{12}^{(6)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \quad R_{23}^{(6)} = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & -i \\ -i & 0 & 1 & 0 \\ 0 & -i & 0 & 1 \end{pmatrix}$$



$$R_{34}^{(6)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_{45}^{(6)} = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}} \begin{pmatrix} 1 & -i & 0 & 0 \\ -i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & -i & 1 \end{pmatrix} \quad R_{56}^{(6)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

Generates representation of the braid group B_6



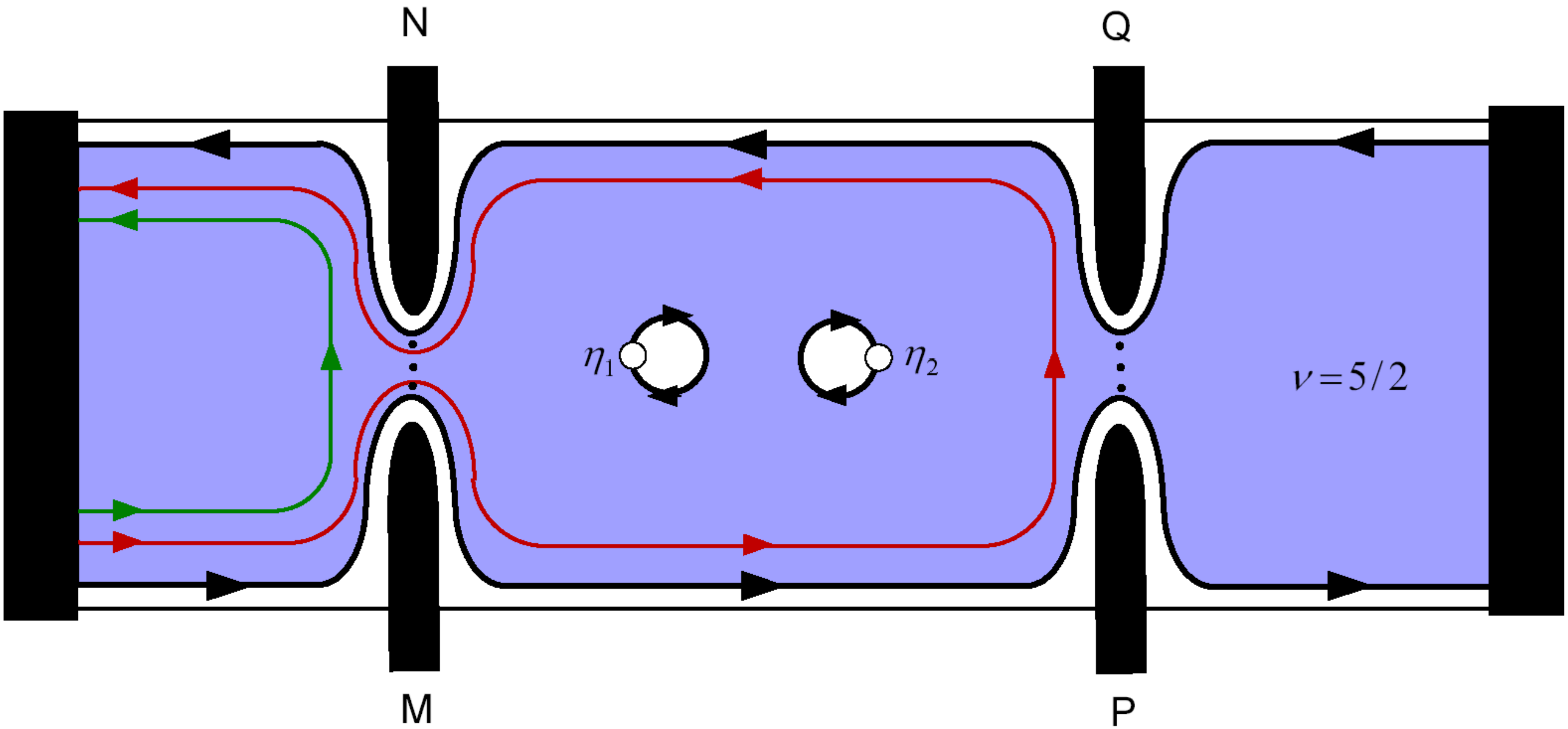
$$\text{CNOT} = R_{34}^{-1} R_{45} R_{34} R_{12} R_{56} R_{45} R_{34}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Universal Quantum Gate Set

- A set of universal quantum gates is any set of gates to which any operation possible on a quantum computer can be reduced, that is, any other unitary operation can be expressed as a finite sequence of gates from the set. We only require that any quantum operation can be approximated by a sequence of gates from this finite set. Moreover, for the specific case of single qubit gates, the Solovay-Kitaev theorem guarantees that this can be done efficiently.
- From a more mathematical point of view, the Solovay-Kitaev theorem is a remarkable general statement about how quickly the group SU(d) is "filled in" by a universal set of gates.
- One simple set of universal quantum gates is the Hadamard gate H, the π/8-gate R(π/4), and the controlled-NOT gate.

$$R_\theta = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$

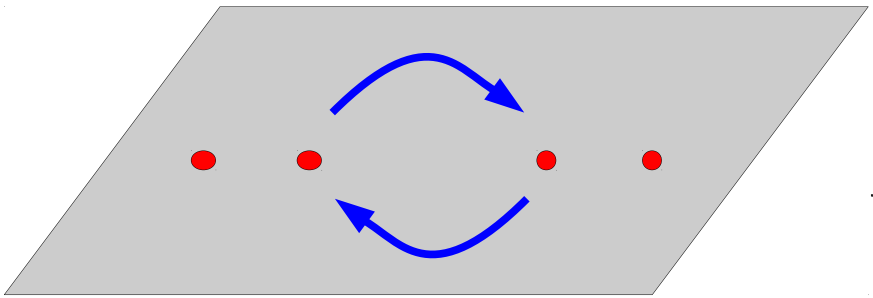
Measuring Anyons



$$\sigma_{xx}^{|0\rangle} \propto |t_{MN} + i t_{PQ}|^2. \qquad \sigma_{xx}^{|1\rangle} \propto |t_{MN} - i t_{PQ}|^2.$$

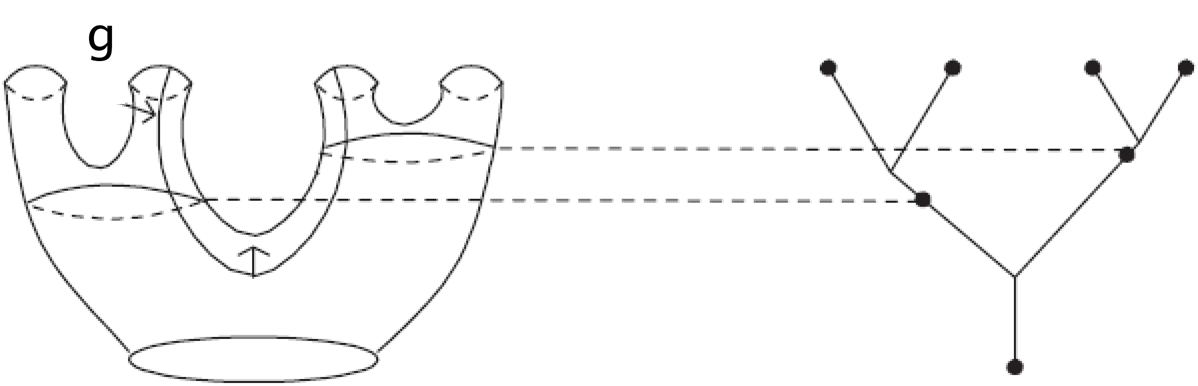
Das Sarma, Freedman & Nayak, PRL 94, 166802 (2005)

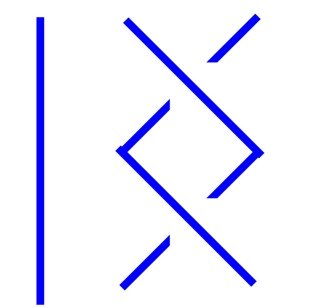
#4: Pictorial Messages



$$\Psi_a \to M_{ab} \Psi_b$$

Planer graph with punctures <==> Condensate with quasiparticles





initialization/
measurement (inverse process)

braiding = computing

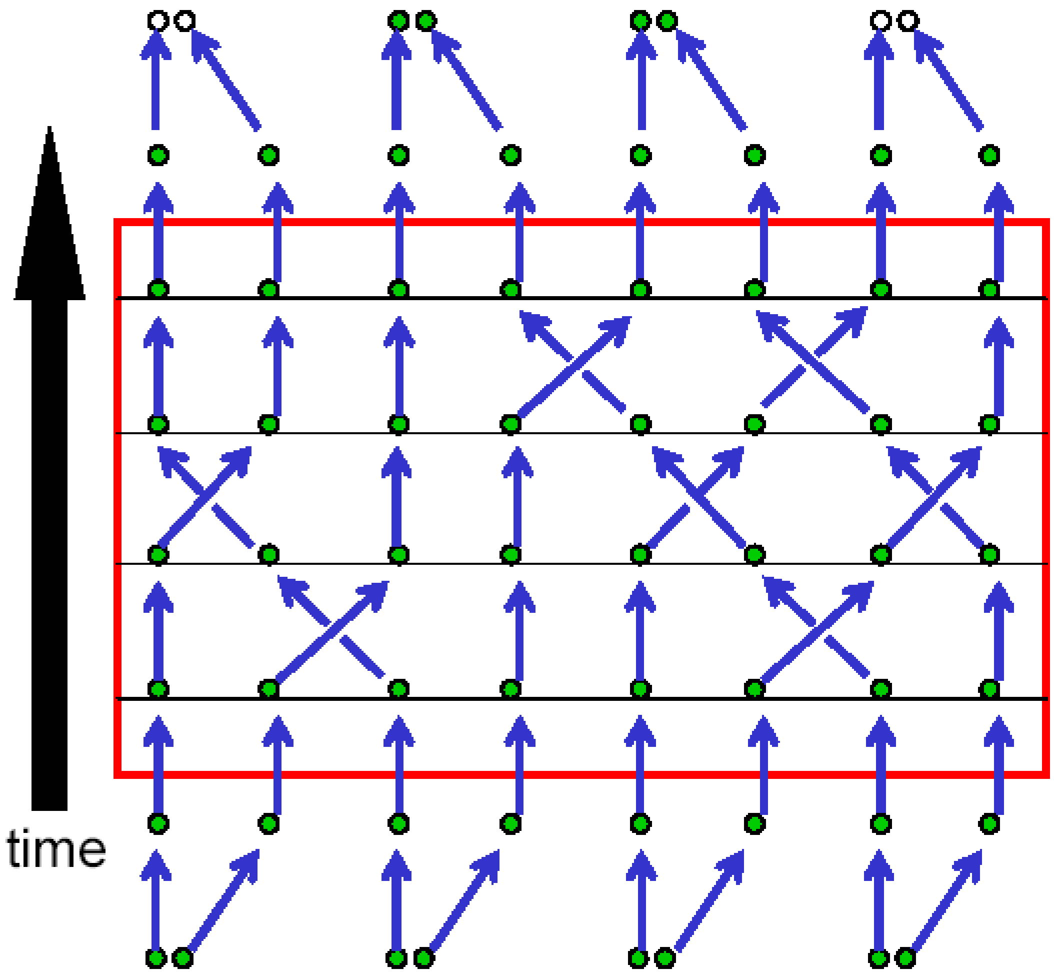
Advantages: GS degeneracy and braiding operation robust against local perturbation

Topological Quantum Computation

(1) *n* qubits
(2) initial state
(3) quantum gates
(4) classical control
(5) readout

Kitaev

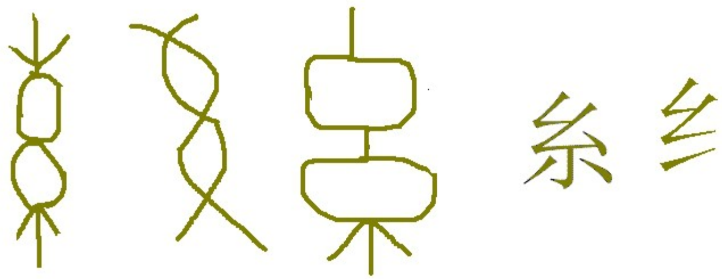
Freedman



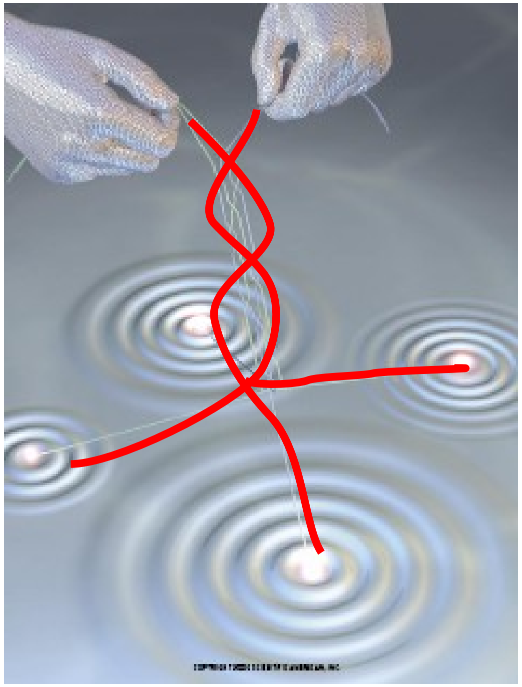
Next: compute with Fibonacci anyons

I Ching of Knots

- *I Ching* (~1100 BC): Ancient people tied knots on cords to keep record, while people during later periods replaced with writing
- 《易 · 系辞》载：“上古结绳而治，后世圣人易之以书契”



纠缠 ↔ entanglement

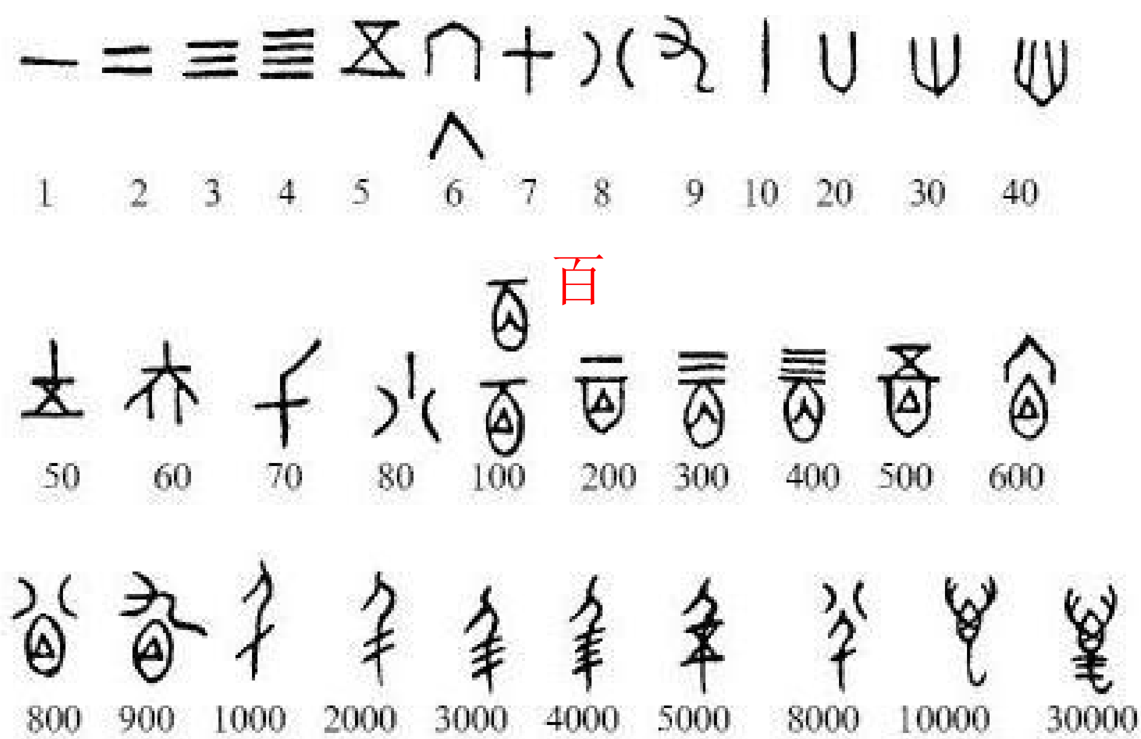


Counting in Oracle Bone Inscriptions

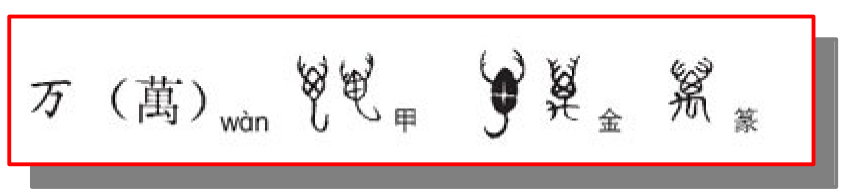
- Decimal system in China (over 3000 years ago)

一 二 三 四 五 六 七 八 九 十

百



千

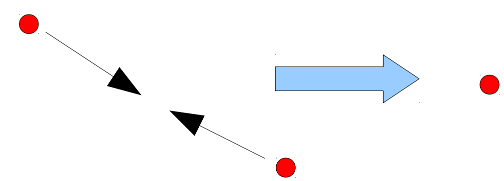


Fibonacci Anyons

- Suppose we have <u>only</u> two types of anyons
  - A trivial anyon I (or 0): representing the ground state of the system (vacuum)
  - A non-trivial anyon τ (or 1) – must be the antiparticle of itself

- Anyons can be fused to a new one

Two possibilities: non-Abelian! $\longrightarrow$ $\tau \times \tau = I + \tau$



or nothing

Ising: $\sigma \times \sigma = I + \psi$

$$F = \begin{pmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{pmatrix} \qquad R = \begin{pmatrix} e^{-i4\pi/5} & 0 \\ 0 & e^{i3\pi/5} \end{pmatrix}$$

$$\phi = \frac{\sqrt{5}+1}{2} \qquad (\phi^2 = 1 + \phi)$$

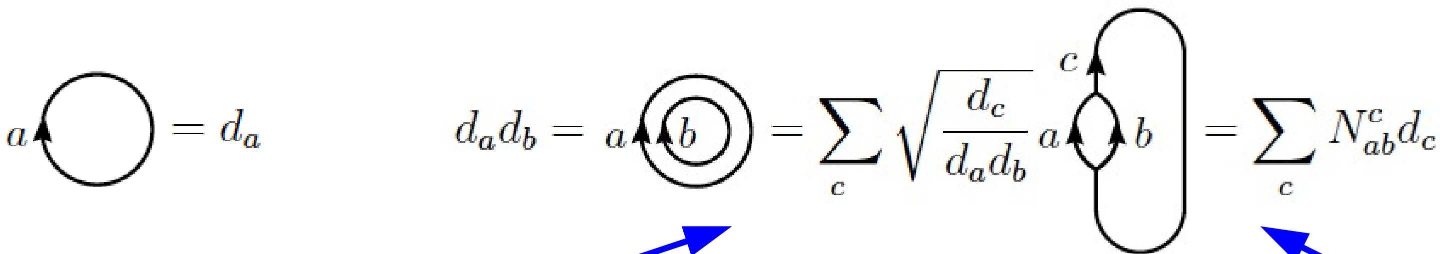
k = 3 Read-Rezayi state; non-Abelian spin-singlet state (Ardonne & Schoutens)

Quantum Dimension

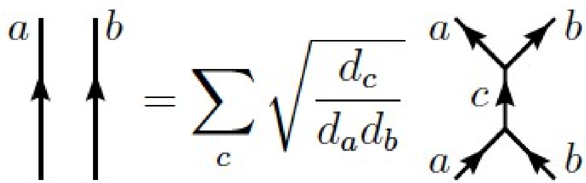
$$\tau\times\tau\times\tau\times\tau=(I+\tau)\times\tau\times\tau=(\tau\times\tau)+(\tau\times\tau\times\tau)=(I+\tau)+(I+\tau+\tau)$$

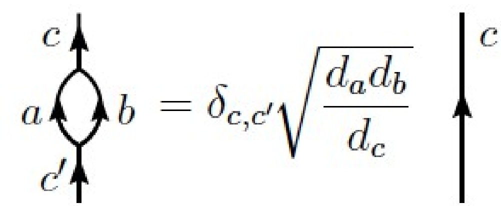
$$V_{n+1}=V_{n-1}+V_n \qquad Dim(V_n)\sim\phi^n, \quad \phi=(\sqrt{5}+1)/2$$

Dimension of V_n: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...



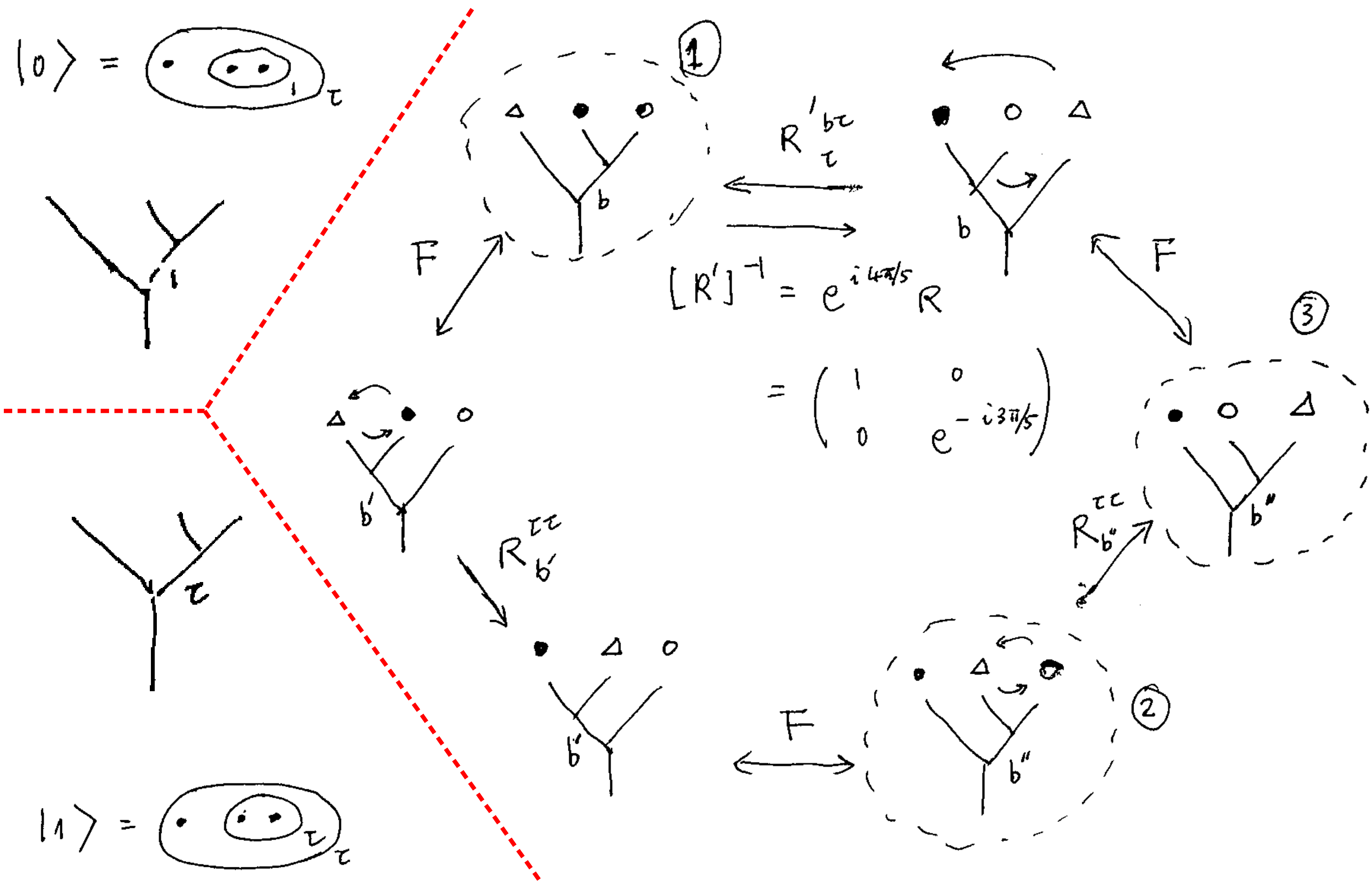
$\phi^2 = 1+\phi$





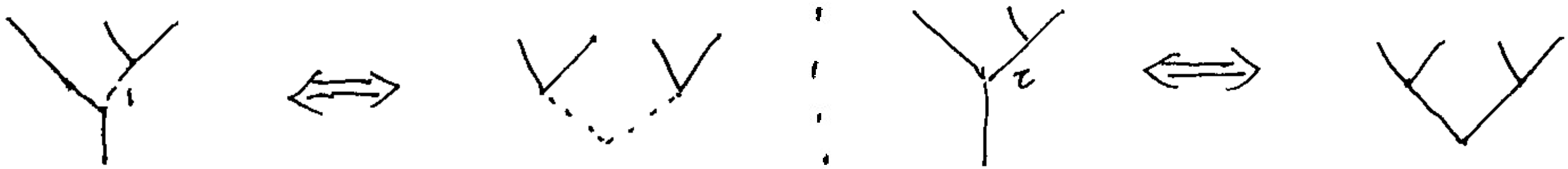
$$\mathbb{I}_{ab}=\sum_c |a,b;c\rangle\langle a,b;c| \qquad \langle a,b;c|a,b;c'\rangle=\delta_{c,c'}$$

Hexagon Equation

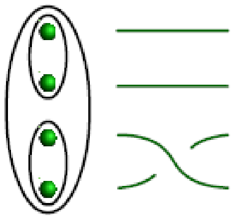


Single Qubit and Elementary Braids

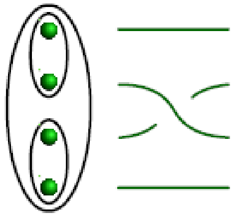
- Either three or four anyons can encode one qubit of information.



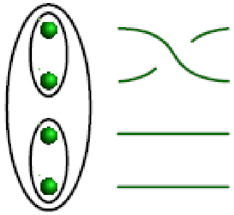
- A braid represents the worldline of anyons in the (2+1)-dim spacetime.



$$\sigma_1 = \begin{bmatrix} e^{-i4\pi/5} & 0 \\ 0 & -e^{-i2\pi/5} \end{bmatrix}$$



$$\sigma_2 = \begin{bmatrix} -\tau e^{-i\pi/5} & -\sqrt{\tau}e^{i2\pi/5} \\ -\sqrt{\tau}e^{i2\pi/5} & -\tau \end{bmatrix} \qquad \tau = \frac{\sqrt{5}-1}{2}$$

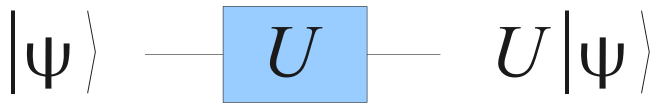


$$\sigma_3 = \begin{bmatrix} e^{-i4\pi/5} & 0 \\ 0 & -e^{-i2\pi/5} \end{bmatrix}$$

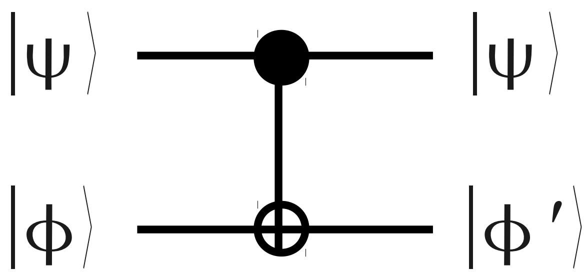
Identical to σ_1

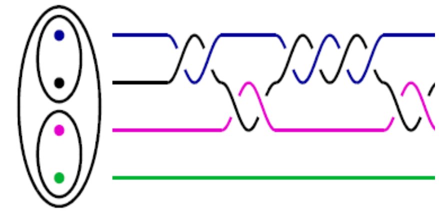
Universal Quantum Gates

- Single-qubit gates (rotation)



- At least a two-qubit gate, such as CNOT



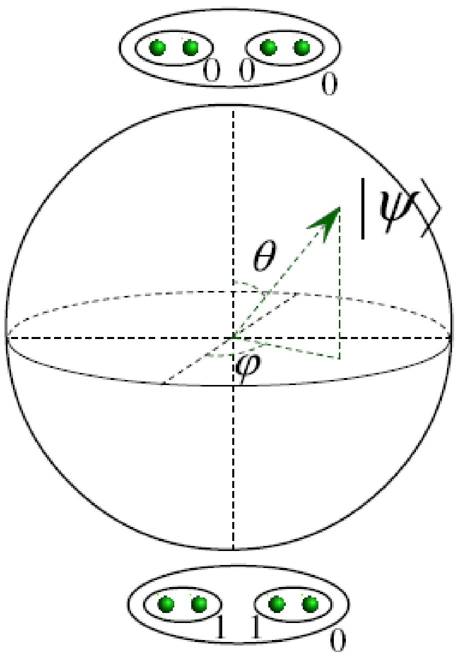


$$U^{-1} = \sigma_1^{-2}\sigma_2^{2}\sigma_1^{4}\cdots$$

Goal: Efficiently find a sequence that approximates the target gate within a given error ε.

- Any N-qubit gates can be realized by the set of universal gates
- Freedman et al. proved TQC is as powerful as conventional QC; implemented by Bonesteel and co-workers using Fibonacci anyons.
- Textbook discussion on conventional quantum gate construction (e.g., Nielsen & Chung)

Single-Qubit Gates: Brute-Force Search



$$|\psi\rangle = \cos\theta\,|\,0\,\rangle + \sin\theta\, e^{i\varphi}|\,1\,\rangle$$

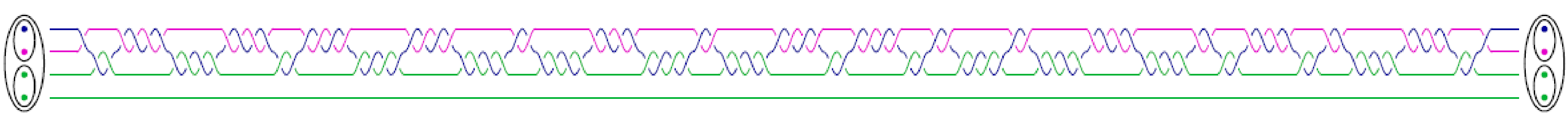
$$|\psi\rangle = e^{i\alpha}(\cos\theta\ |((11)_0(11)_0)_0\rangle + \sin\theta\ e^{i\varphi}|((11)_1(11)_1)_0\rangle)$$

- We have σ_1 (or σ_3), σ_2, and inverses σ_1^{-1}, σ_2^{-1}
- Each exchange has 3 possibilities (no return)
- Finding the best braid in $\sim 3^N$ possibilities
- Exhaustive search: non-polynomial time
- Error to the desired gate (semi-empirical)

$$\sigma_i^{10} = 1$$

$$\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1 = 1$$

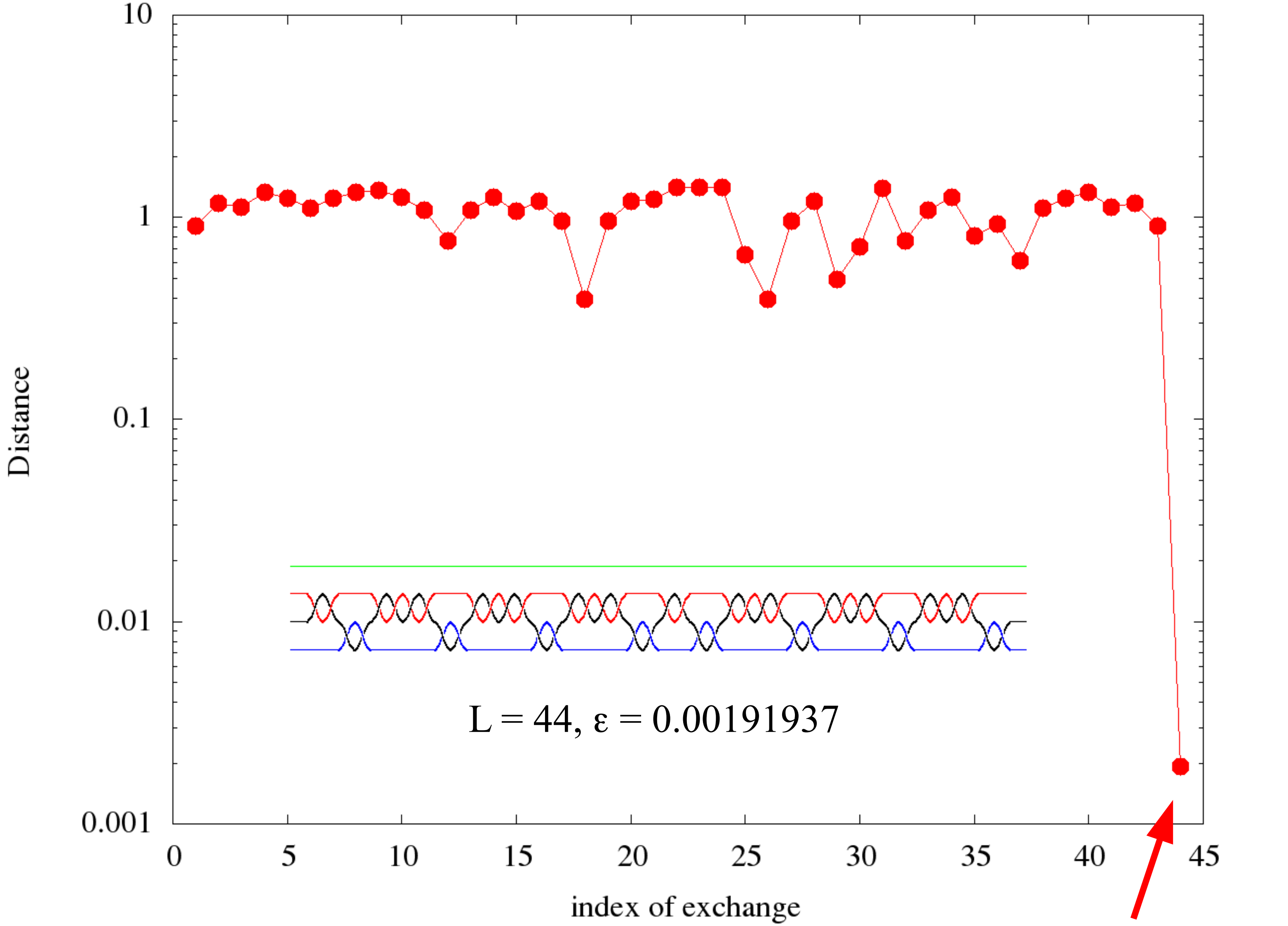
$$\epsilon \sim e^{-L/\xi},\quad \xi \approx 7.3 \quad \textit{for identity}$$



$$\approx \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$$\sigma_3\sigma_2^2\sigma_3^4\sigma_2^4\sigma_3^4\sigma_2^{-2}\sigma_3^{-4}\sigma_2^{-4}\sigma_3^{-4}\sigma_2^4\sigma_3^{-2}\sigma_2^4\sigma_3^4\sigma_2^{-4}\sigma_3^{-2}\sigma_2^4\sigma_3^{-4}\sigma_2^{-2}\sigma_3^{-4}\sigma_2^{-2}\sigma_3^{-2}\sigma_2^{-4}\sigma_3^{-2}\sigma_2^4\sigma_3^4\sigma_2^4\sigma_3^2\sigma_2^{-2}\sigma_3^4\sigma_2^{-2}\sigma_3^2\sigma_2^4\sigma_3^4\sigma_2^{-2}\sigma_3^{-1}$$

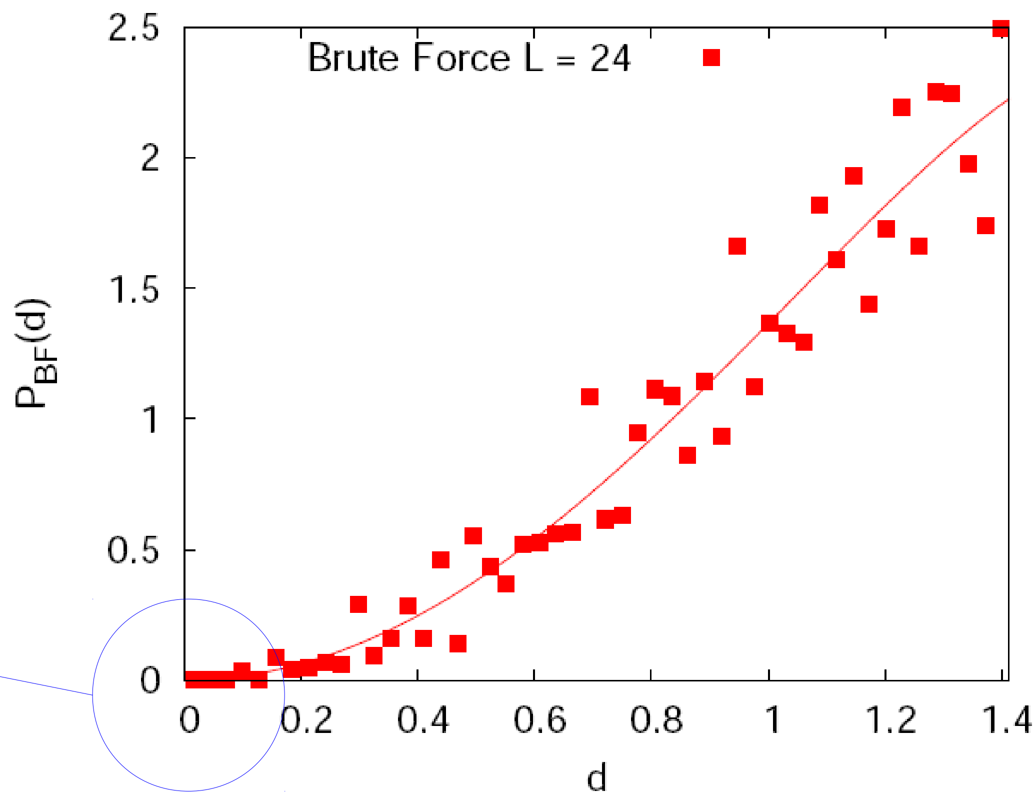
Exchange-by-Exchange Distance



Distance between matrices U and V is defined as the square root of the highest eigenvalues of (U-V)*(U-V)

Distance Distribution for a Fixed Length

Distribution of distance to the identity for all weaves (a subset of braids in which only one anyon moves) with a length 24:



How to enhanced the sampling at small *d*?

$$g = e^{i\hat{m}\cdot\vec{\sigma}(\phi/2)} = \begin{bmatrix} \cos(\phi/2) + im_z \sin(\phi/2) & m_y \sin(\phi/2) + im_x \sin(\phi/2) \\ -m_y \sin(\phi/2) + im_x \sin(\phi/2) & \cos(\phi/2) - im_z \sin(\phi/2) \end{bmatrix}$$

$$d = 2\sin(\phi/4) \qquad P_{BF}(d) = \frac{4}{\pi} d^2 \sqrt{1 - d^2/4}$$

assuming that the braids distributed uniformly in the space of unitary matrices

Randomly Uniform Approximation

- Assumption: The matrix representations of long enough braids distribute randomly in the space of unitary matrices (3-sphere). There is no local correlation.

$$g = e^{i \hat{m}\cdot\vec{\sigma}(\phi/2)}$$

  - Total number of weaves for a fixed braid length L:

$$N(L) \sim \alpha^{L/2}, \quad \alpha \approx 2.732 < 3$$

$$\sigma_i^{10} = 1 \qquad \sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1 = 1$$

  - Average volume per weave on the 3-sphere:

$$[\epsilon(L)]^3 \sim 1/N(L) \sim \alpha^{-L/2}$$

  - Average error: $\epsilon(L) \sim \alpha^{-L/6}$

$$\sigma_1^{n_1}\sigma_2^{n_2}\sigma_1^{n_3}\sigma_2^{n_4}\cdots\sigma_1^{n_{m-1}}\sigma_2^{n_m}$$

$$n_i = \pm 2, \pm 4$$

$$L = \sum_i |n_i|$$

$$N(L) \sim \left(1 + \sqrt{3}\right)^{L/2}$$

or $\quad L \sim \ln(1/\epsilon)$

$$T \sim (1/\epsilon)^3$$

inefficient!

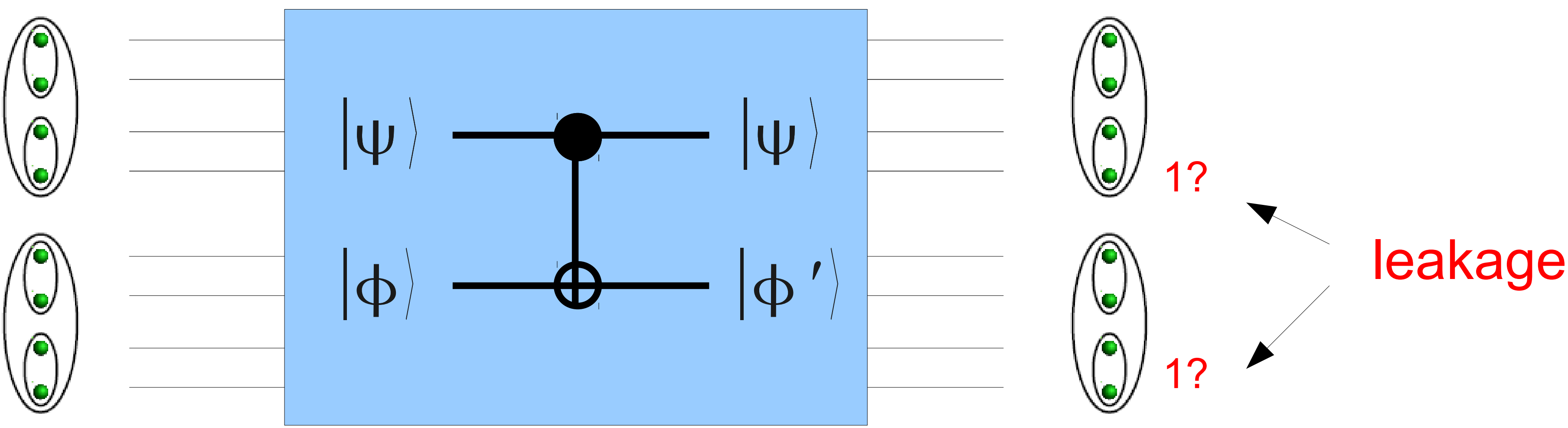
Burrello et al., 2011

Two-Qubit Gates

- Single-qubit gate: 3 free parameters [SU(2)]

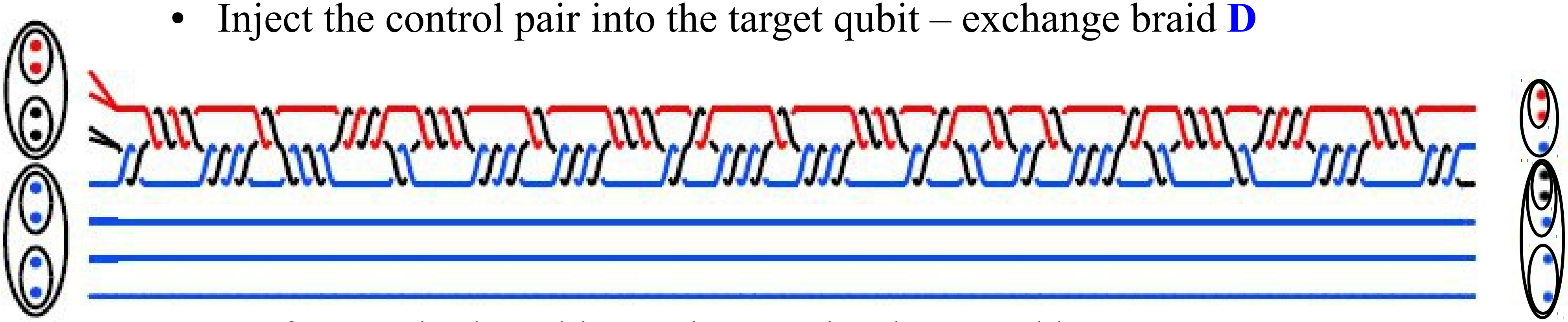
$$e^{i\alpha}\begin{bmatrix} \sqrt{1-b^2}e^{-i\beta} & be^{i\gamma} \\ -be^{-i\gamma} & \sqrt{1-b^2}e^{i\beta} \end{bmatrix}$$

- Two-qubit gate: too many parameters

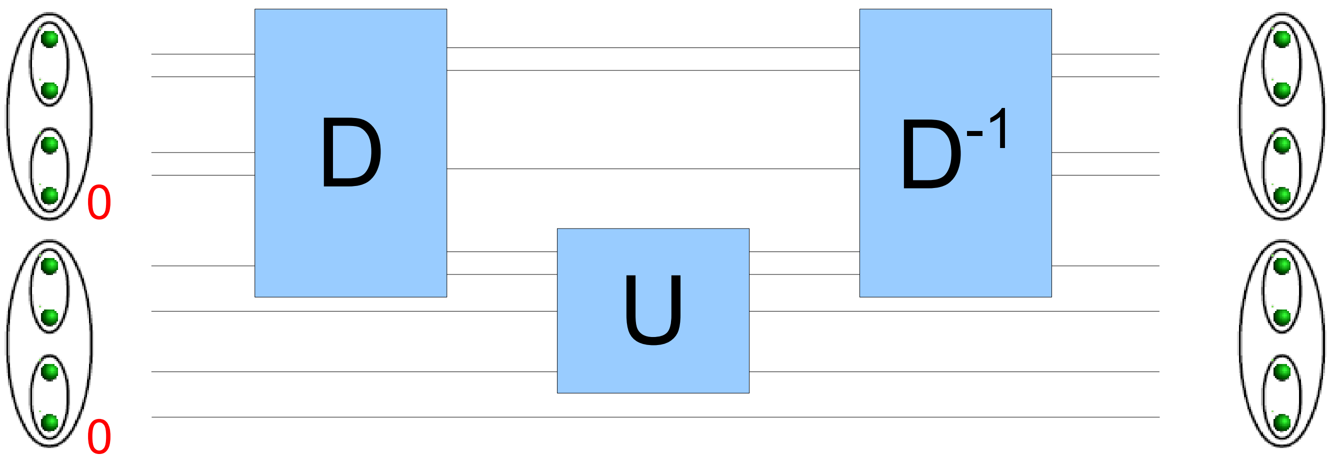


Decomposing Two-Qubit Gates

- Idea proposed by Bonesteel et al. (2005) – leakage error $\sim 10^{-3}$
- [Xu & Wan, 08] Reduces leakage error significantly, $\sim 10^{-9}$
    - Inject the control pair into the target qubit – exchange braid **D**



    - Perform a single-qubit rotation **U** – implemented by a weave
    - Extract the control pair use the inverse of the inverse of the exchange braid $\mathbf{D}^{-1}$

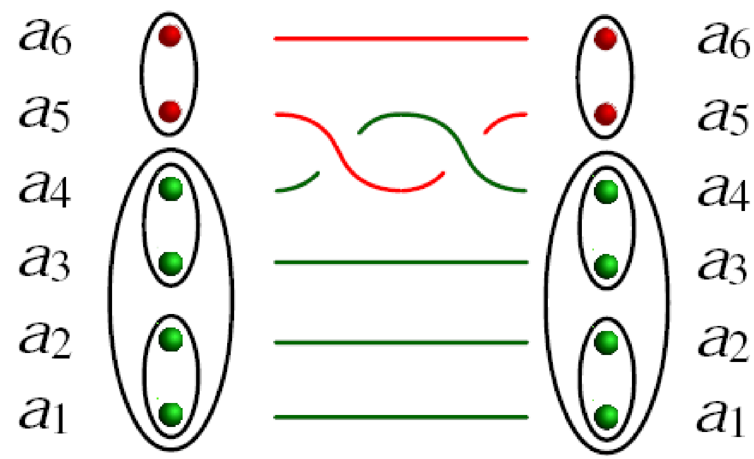


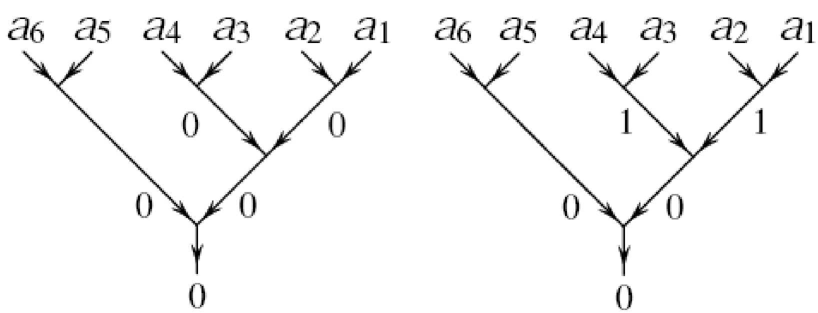
Generic controlled-gates with leakage error $\sim 10^{-9}$.

Leakage Error

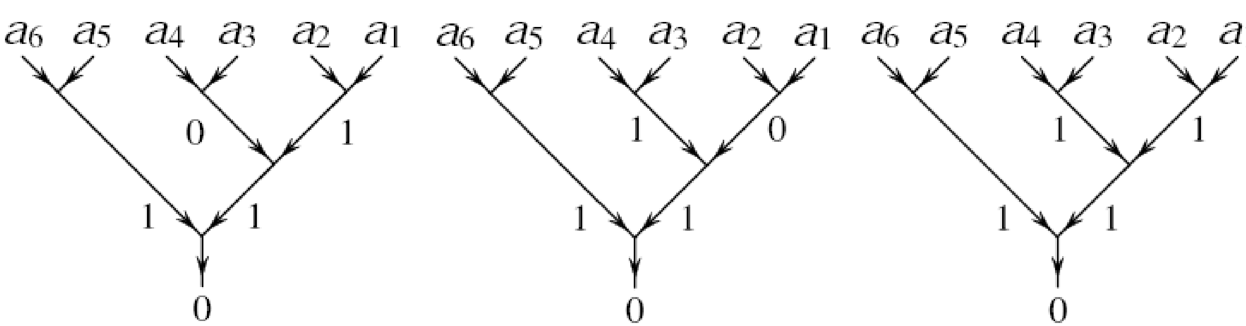
Create a pair of anyons out of vacuum (so fuse to 0).

Note they could also be stray anyons thermally excited.



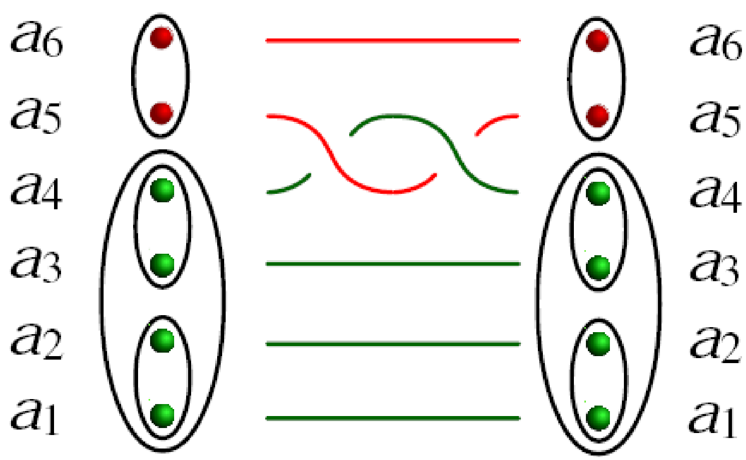


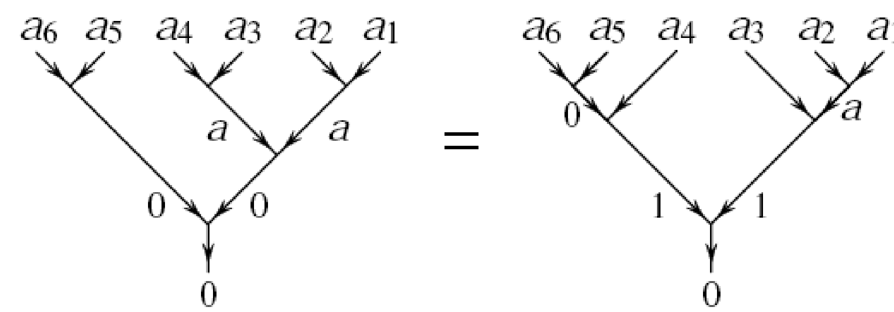
Computing basis



Non-computing basis

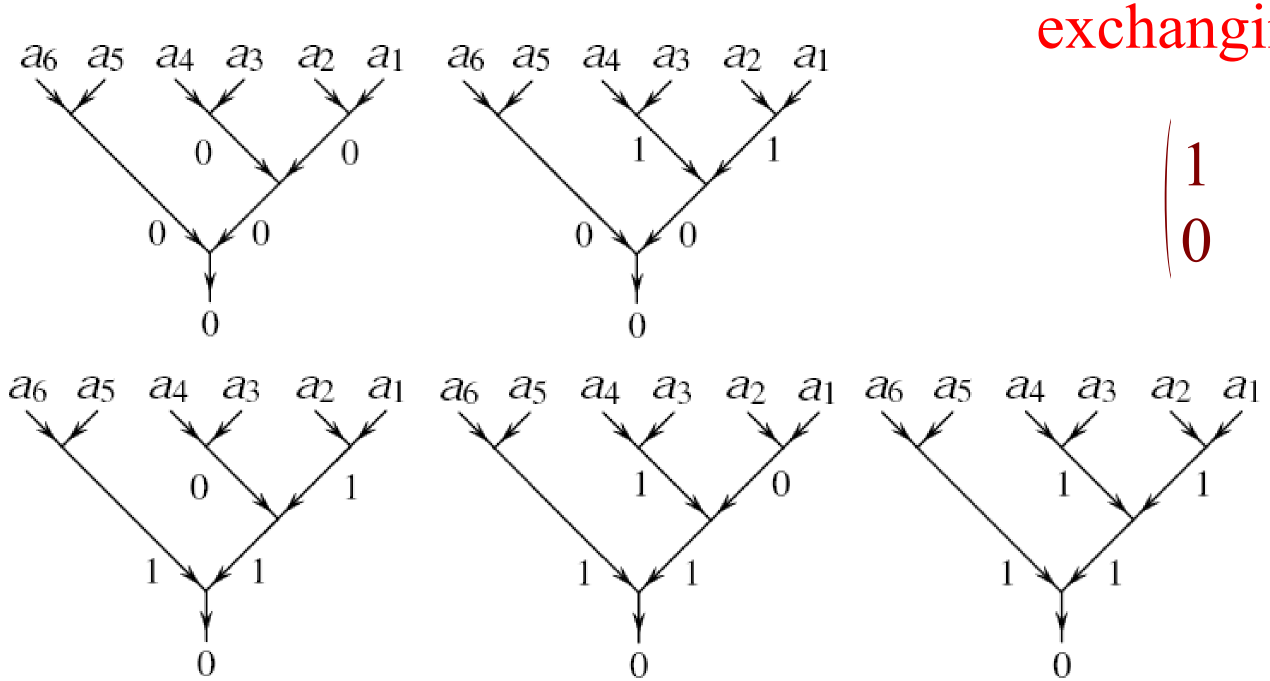
Leakage-Error Analysis





What kind of braids (of a_4, a_5, a_6) leave the left qubit in state 0, after exchanging a_4 and a_5?

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$$



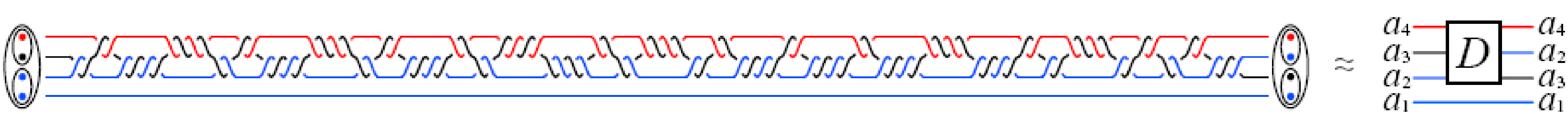
$$\begin{pmatrix}e^{i\alpha} & 0\\ 0 & e^{i\beta}\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = e^{i\alpha}\begin{pmatrix}1\\ 0\end{pmatrix}$$

Phase Gates

- Let us look for diagonal matrices, rather than the identity matrix; this means we introduce a phase error.

$$e^{i\alpha}\begin{bmatrix} \sqrt{1-b^2}e^{-i\beta} & be^{i\gamma} \\ -be^{-i\gamma} & \sqrt{1-b^2}e^{i\beta} \end{bmatrix}$$

- For small b, γ is irrelavent. Targeting less parameter – higher accuracy!



$$\sigma_2^{-2}\sigma_3^{-2}\sigma_2^{-4}\sigma_3^{4}\sigma_2^{2}\sigma_3^{-2}\sigma_2^{-4}\sigma_3^{4}\sigma_2^{2}\sigma_3^{-2}\sigma_2^{-4}\sigma_3^{2}\sigma_2^{2}\sigma_3^{-4}\sigma_2^{4}\sigma_3^{2}\sigma_2^{2}\sigma_3^{4}\sigma_2^{-2}\sigma_3^{2}\sigma_2^{-4}\sigma_3^{-2}\sigma_2^{-4}\sigma_3^{2}\sigma_2^{-4}\sigma_3^{4}\sigma_2^{-4}\sigma_3^{-2}\sigma_2^{-2}\sigma_3^{4}\sigma_2^{-2}\sigma_3^{-2}\sigma_2^{2}\sigma_3^{-2}\sigma_2^{-3} = e^{i\theta}\begin{bmatrix} r & \delta_1 \\ \delta_2 & re^{i\varphi} \end{bmatrix}$$

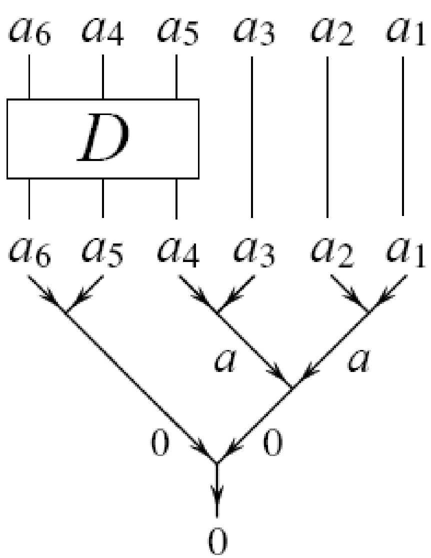
$$|\delta_1| = |\delta_2| = \sqrt{1-r^2} = 1.56\times 10^{-10}$$

- But how do we use it? What about the phase?

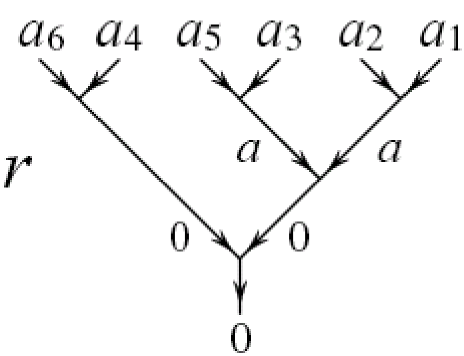
Exchange Braid

- Apply the diagonal gate (with the irrelevant phase) to the leakage model

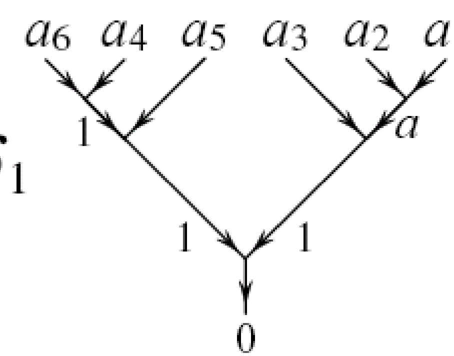
Low leakage! $|\delta_1| = 1.56\times 10^{-10}$

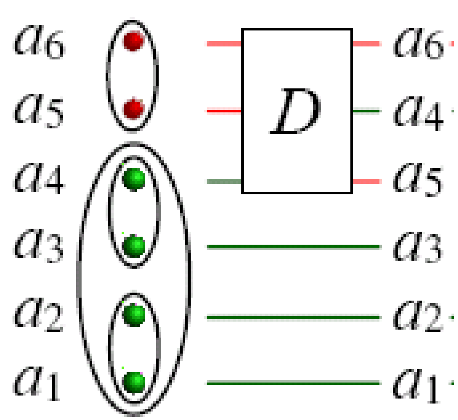


$= e^{i\theta} r$



$+ \; e^{i\theta}\delta_1$



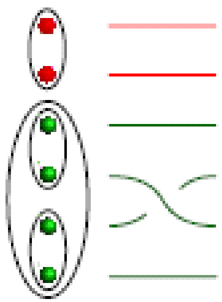


$$
e^{i\theta}\left[\begin{array}{cc|ccc}
r & 0 & 0 & \epsilon_1 & 0 \\
0 & r & \epsilon_2 & 0 & \epsilon_3 \\
\hline
0 & \epsilon_5 & & & \\
\epsilon_4 & 0 & & A & \\
0 & \epsilon_6 & & &
\end{array}\right]
$$

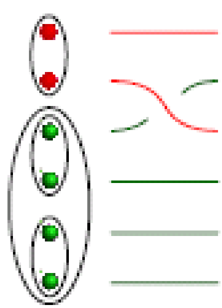
$$
|\epsilon_1| = \sqrt{|\epsilon_2|^2 + |\epsilon_3|^2} = |\epsilon_4| = \sqrt{|\epsilon_5|^2 + |\epsilon_6|^2} = |\delta_1| = \sqrt{1-r^2} \approx 1.56\times 10^{-10}
$$

5-Dimensional Representation

- One calculate the braiding matrix in an enlarged space, including non-computing bases.

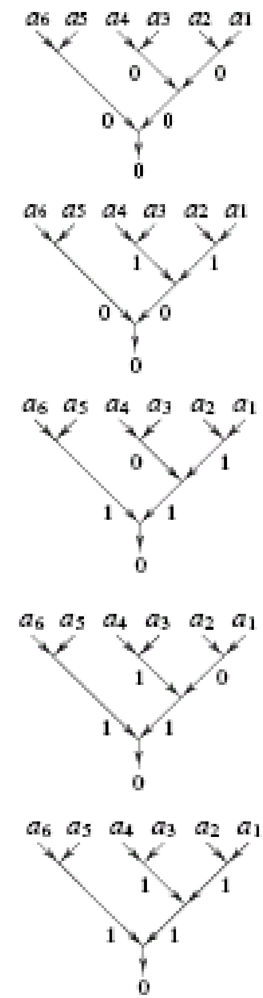


$$\sigma_2 = \begin{bmatrix} -\tau e^{-i\pi/5} & -\sqrt{\tau}e^{i2\pi/5} & 0 & 0 & 0 \\ -\sqrt{\tau}e^{i2\pi/5} & -\tau & 0 & 0 & 0 \\ 0 & 0 & -\tau e^{-i\pi/5} & -\tau e^{i2\pi/5} & \tau^{3/2}e^{i2\pi/5} \\ 0 & 0 & -\tau e^{i2\pi/5} & -\tau e^{-i\pi/5} & \tau^{3/2}e^{i2\pi/5} \\ 0 & 0 & \tau^{3/2}e^{i2\pi/5} & \tau^{3/2}e^{i2\pi/5} & \nu \end{bmatrix}$$

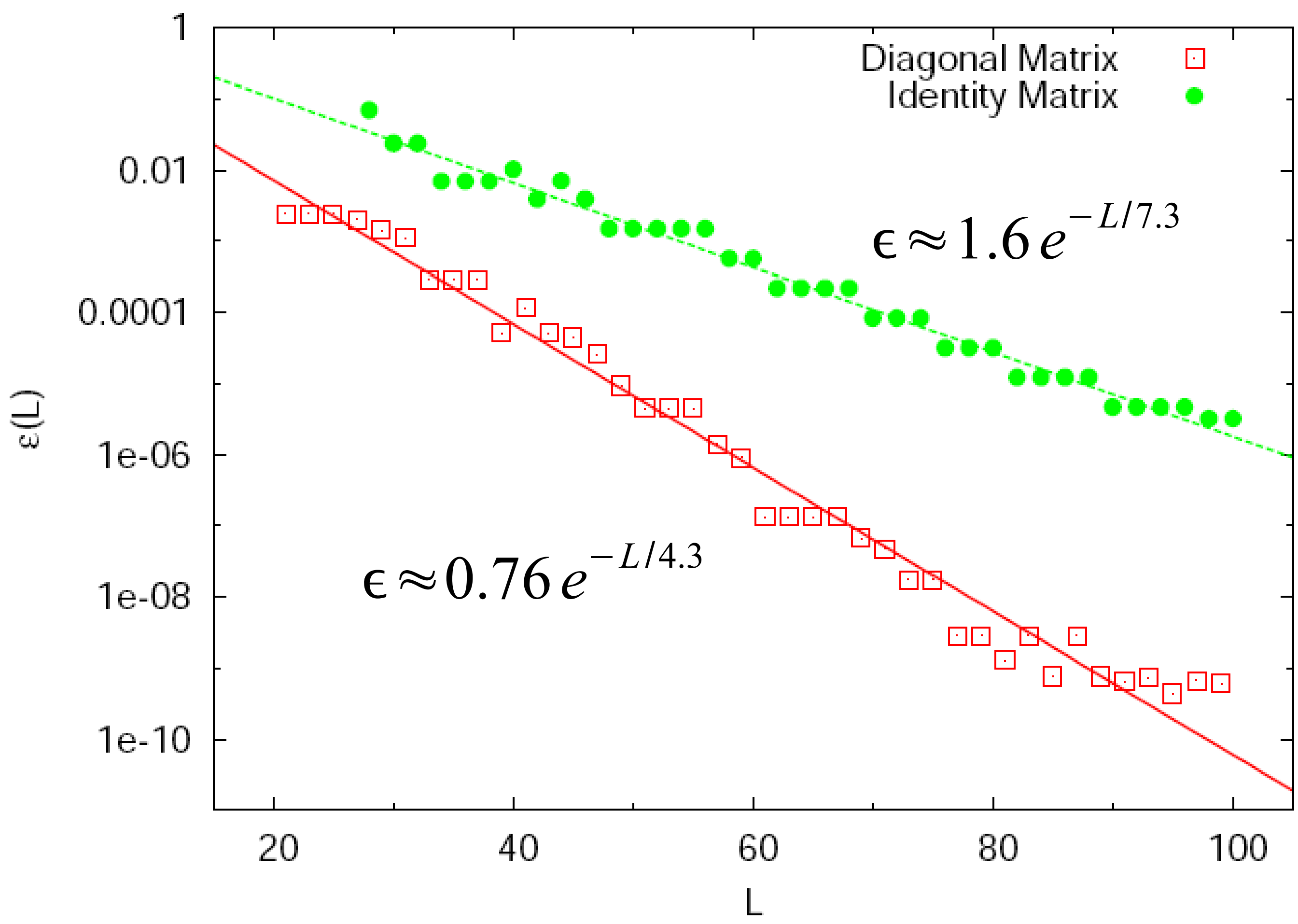


$$\sigma_4 = \begin{bmatrix} -\tau e^{-i\pi/5} & 0 & 0 & -\sqrt{\tau}e^{i2\pi/5} & 0 \\ 0 & -\tau e^{-i\pi/5} & -\tau e^{i2\pi/5} & 0 & \tau^{3/2}e^{i2\pi/5} \\ 0 & -\tau e^{i2\pi/5} & -\tau e^{-i\pi/5} & 0 & \tau^{3/2}e^{i2\pi/5} \\ -\sqrt{\tau}e^{i2\pi/5} & 0 & 0 & -\tau & 0 \\ 0 & \tau^{3/2}e^{i2\pi/5} & \tau^{3/2}e^{i2\pi/5} & 0 & \nu \end{bmatrix}$$

$$\nu = -\tau(1+\tau^3)e^{-i2\pi/5} + \tau^3 e^{-i4\pi/5}.$$



Improvement in the Brute-Force Performance



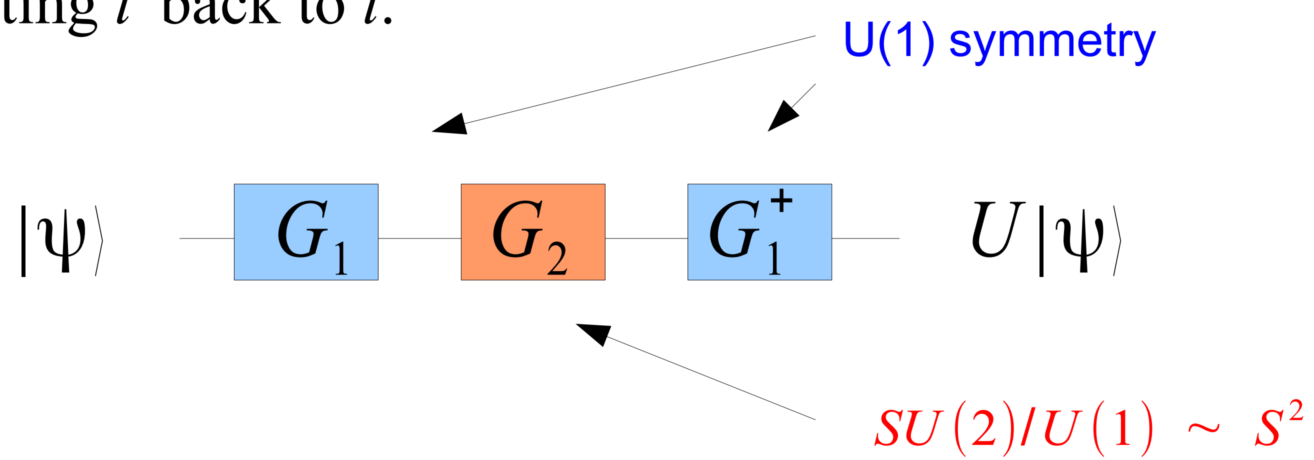
Search for one point on the bloch sphere

Search for any point on a circle on the Bloch sphere

Leakage error reduction by several orders of magnitude

Single-Qubit Construction Again

- Single-qubit construction hides an SU(2) symmetry. A rotation around an arbitrary axis l by an angle θ on a Bloch sphere can be carried out by first rotating l to another direction l', then rotating around l' by an angle θ, and finally rotating l' back to l.



- Implementation: Instead of search for a gate G, we search a pair of gates G_1 and G_2, such that $G \approx G_1 G_2 G_1^+$

$$G_{1,2} = e^{i\alpha_{1,2}} \begin{bmatrix} \sqrt{1-b_{1,2}^2}\, e^{-i\beta_{1,2}} & b_{1,2} e^{i\gamma_{1,2}} \\ -b_{1,2} e^{-i\gamma_{1,2}} & \sqrt{1-b_{1,2}^2}\, e^{i\beta_{1,2}} \end{bmatrix}$$

Geometric Redundancy for Single-qubit Gates

- We first rotate the axis of rotation, then rotate around the axis, and finally rotate the axis back – physically, this means that we have a geometric redundancy in search, due to the SU(2) rotation symmetry.
  - We can search G_1 and G_2 separately
  - Both searches are achievable in lower (than 3) dimensions
  - i.e., we can fix

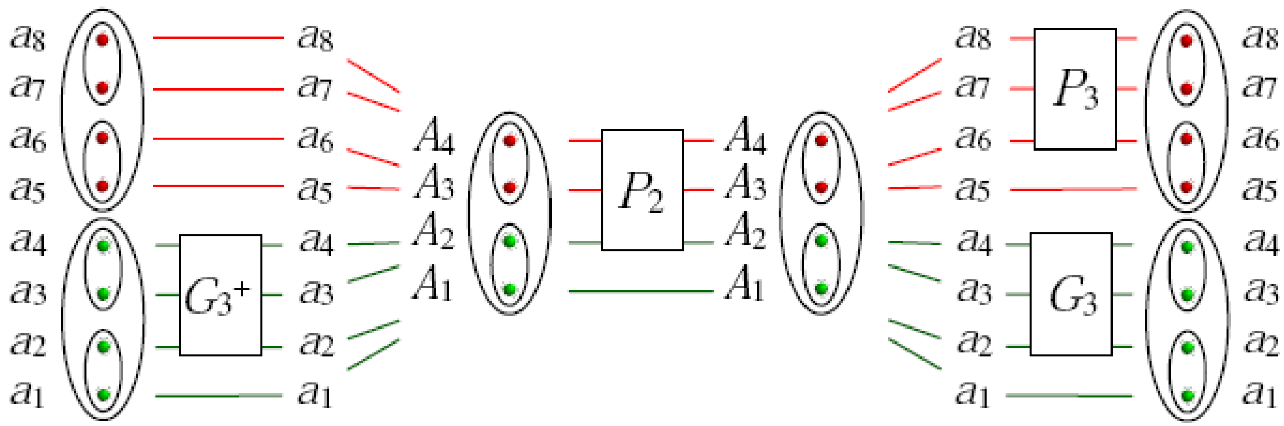
    G_1 up to a U(1) rotation, and
    G_2 up to SU(2) / U(1) ~ S^2

$$P = \begin{bmatrix} e^{-i\beta} & 0 \\ 0 & e^{i\beta} \end{bmatrix}$$

$$(1-b_2^2)^{1/2}\cos\beta_2 = \cos\beta,$$
$$b_1 = \frac{b_2}{\sqrt{2\sin^2\beta + 2(1-b_2^2)^{1/2}\sin\beta_2\sin\beta}},$$
$$\beta_1 + \gamma_1 = \gamma_2 + (k+1/2)\pi,$$

- Outcome: Generic single-qubit gates with error (distance) ~10^{-10} with braids of ~300 exchanges (length) – Xu & XW (2009).
  - Hormozi et al. (07): 4 $\times 10^{-5}$ for a braid of length 220 with Solovay-Kitaev algorithm

Arbitrary Controlled-rotation Gate



$$e^{i(\alpha_3-\beta_3)}\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{R}} \\ 0 & 0 & & \end{array}\right]$$

$$P_2 = e^{i\alpha_2}\begin{bmatrix} e^{-i\beta_2} & 0 \\ 0 & e^{i\beta_2} \end{bmatrix}$$

$$P_3 = e^{i\alpha_3}\begin{bmatrix} e^{-i\beta_3} & 0 \\ 0 & e^{i\beta_3} \end{bmatrix}$$

$$e^{i\left(\frac{\alpha_2-\beta_2}{2}+2\beta_3\right)} G_3 \begin{bmatrix} e^{-i(\alpha_2-\beta_2)/2} & 0 \\ 0 & e^{i(\alpha_2-\beta_2)/2} \end{bmatrix} G_3^+$$

- Example: CNOT with precision 5×10^{-10} – 280 interchanges of double braids and 208 of single braids

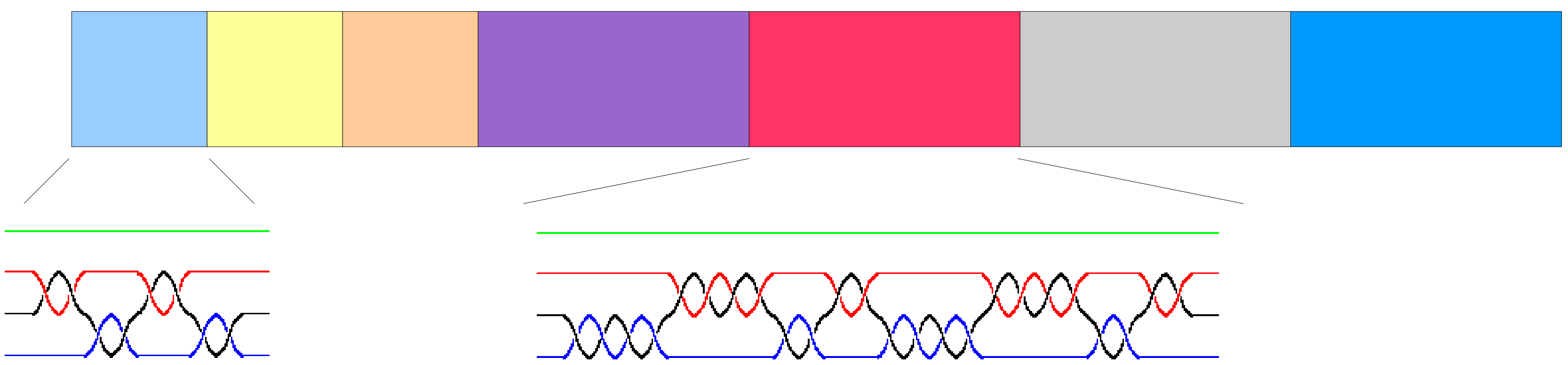
#5: Messages on Topological Quantum Gates

- Three or four Fibonacci anyons can encode one qubit of information.
- Quantum gates can be achieved by braiding anyons; in particular, moving one or one pair of anyons is enough to generate all quantum gates.
- Braids for quantum gates can be compiled into sequences of two elementary exchanges and their inverses.
- The construction of two-qubit gates can be mapped to that of single-qubit gates. But at least one high-precision phase gate is needed to eliminate leakage errors.
- In the brute-force search for braids geometrical redundancy can be explored to boost the efficiency.

Next: reducing the computational complexity of search

Renormalization Group Scheme

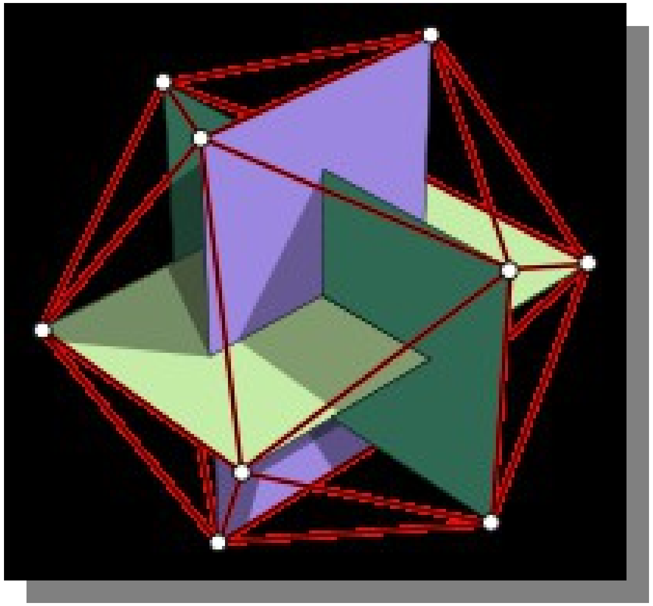
1. Start from a collection of braids of certain length
2. Find the cluster of braids that approximates the target best
3. Moving on to a collection of longer braids (finer in distance) matching the residual error
4. Repeat 2-3, and stop when the desired error scale is reached



Icosahedral Group

- The following Cartesian coordinates define the vertices of an icosahedron with edge-length 2, centered at the origin:

$$(0,\pm 1,\pm \phi),(\pm 1,\pm \phi,0),(\pm \phi,0,\pm 1)$$



$$\phi=\frac{1+\sqrt{5}}{2}$$

- The icosahedral group is the largest finite subgroup of SU(2). It is composed by the 60 rotations around the axes of symmetry of the icosahedron.

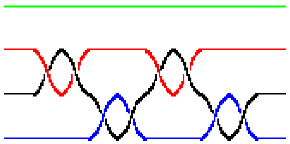
- There are 6 axes of the 5th order, 10 of the 3rd, and 15 of the 2nd.

$$I_{60}=\{g_{0,} g_{1,} g_{2,} \cdots, g_{59}\} \qquad g_0=e$$

- We approximate all group elements by braids of various length.

Braid Representations for the Identity e

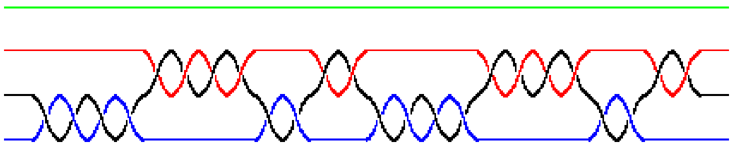
- L = 8, e = 0.236068



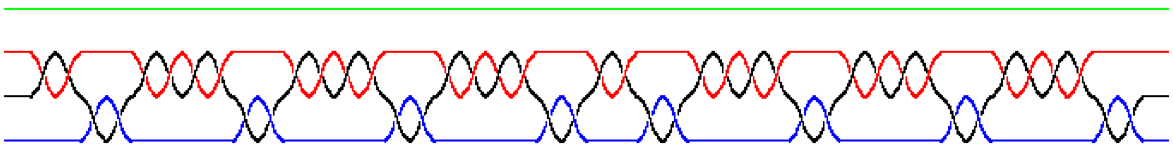
$$\tilde{g}_0(8)=\sigma_2^{-2}\sigma_1^{2}\sigma_2^{-2}\sigma_1^{2}=g_0\, e^{i\Delta_0^{(8)}}$$

$\Delta_0^{(8)}$: a Hermitian matrix characterizing error

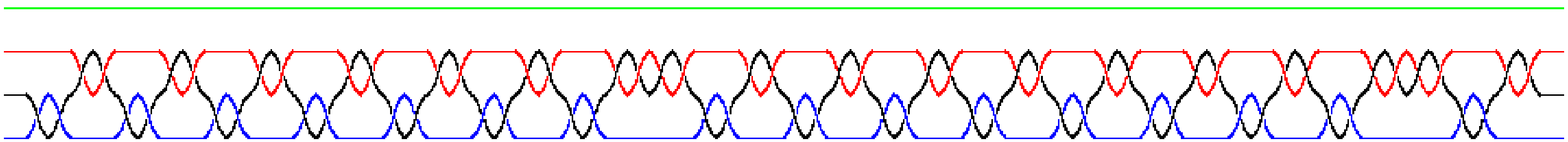
- L = 24, e = 0.0344419



- L = 44, e = 0.00191937



- L = 68, e = 0.000304193



The braid representations can be computed and stored once for all.
Hence no additional cost to the search later.

Connection to Random Matrix Theory

- Pseudogroup of braids (for small Δ_i)

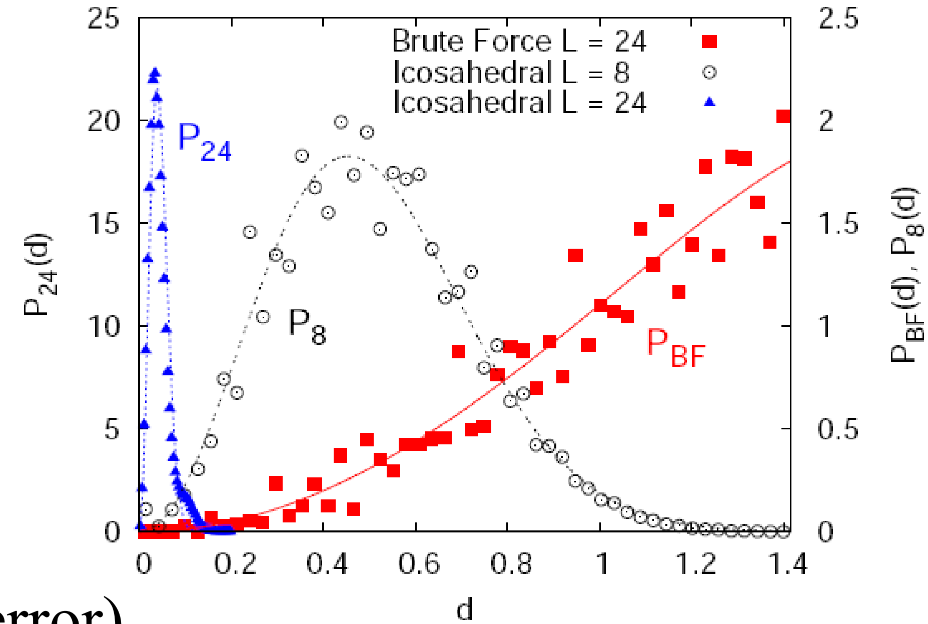
$$g_i g_j = g_k, \qquad \tilde{g}_i \tilde{g}_j = g_i e^{i\Delta_i} g_j e^{i\Delta_j} \approx g_k e^{i(g_j^{-1}\Delta_i g_j + \Delta_j)} \neq \tilde{g}_k = g_k e^{i\Delta_k}$$

- To approximate $g_i g_j \cdots g_{n+1} = e$

$$\tilde{g}_i \tilde{g}_j \cdots \tilde{g}_{n+1} = g_i e^{i\Delta_i} g_j e^{i\Delta_j} \cdots g_{n+1} e^{i\Delta_{n+1}} \equiv e^{iH_n}$$

$$\begin{aligned} H_n = {} & g_i \Delta_i g_i^{-1} + g_i g_j \Delta_j g_j^{-1} g_i^{-1} + \cdots \\ & + g_i g_j \cdots g_n \Delta_n g_n^{-1} \cdots g_j^{-1} g_i^{-1} \\ & + \Delta_{n+1} + O(\Delta^2) \end{aligned}$$

- We conjecture H_n is a random matrix in the Wigner-Dyson Gaussian Unitary Ensemble (s for eigenvalue/error)



$$P(s) = \frac{32}{\pi^2 s_0} \left(\frac{s}{s_0} \right)^2 e^{-(4/\pi)(s/s_0)^2}$$

n = 3 is large enough

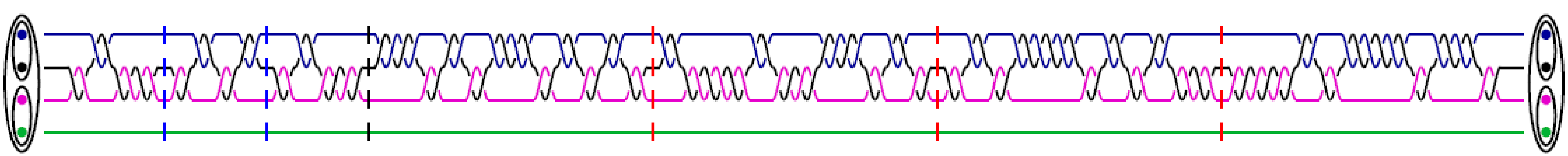
A single parameter s_0 controls the flow of the (distribution of) error.

Refined Realization

1. Load a collection of braids of certain length that approximate the elements of the icosahedral group.
2. Find the cluster of braids that approximates the target best
3. Replace with a collection of longer (finer in distance) braids that approximate the icosahedral group matching the residual error
4. Find the cluster of braids in the vicinity of the identity that when adding to the previous approximation reduces the residual error most
5. Repeat 3-4, and stop when the desired error scale is reached

First approximation

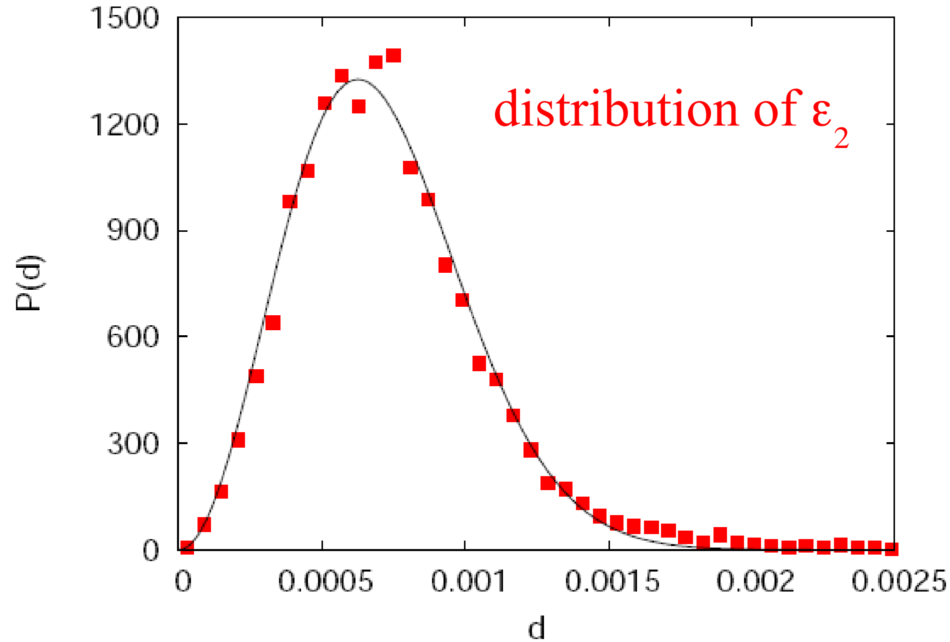
Correction: approximation to $g_i g_j \cdots g_{n+1} = e$



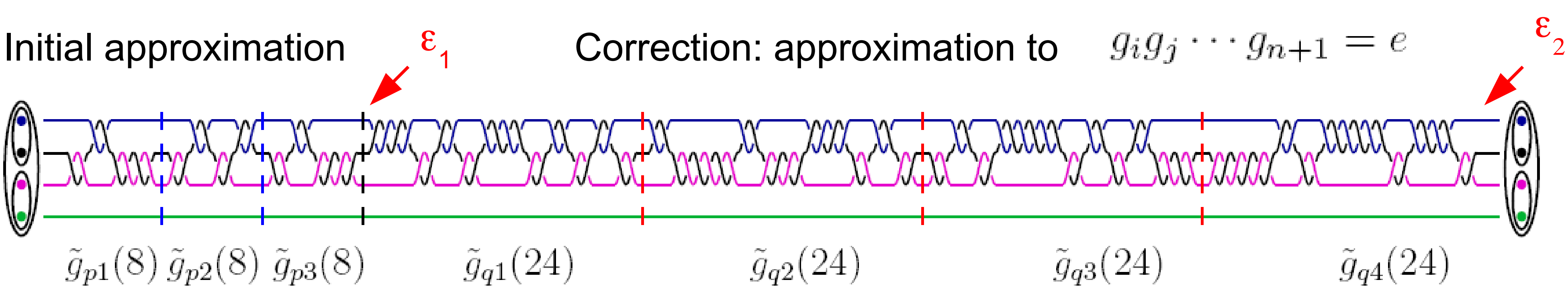
$\tilde{g}_{p1}(8)$ $\tilde{g}_{p2}(8)$ $\tilde{g}_{p3}(8)$ $\tilde{g}_{q1}(24)$ $\tilde{g}_{q2}(24)$ $\tilde{g}_{q3}(24)$ $\tilde{g}_{q4}(24)$

Understanding Error Renormalization

- First approximate by gluing 3 short (L = 8) segments (1 out of 60^3).
- Reduce the error (e_1) by gluing 4 (= n + 1) longer (L = 24) segments (1 out of 60^3).
- The resulting error (e_2) follows the Wigner-Dyson distribution.
- Average error reduction:



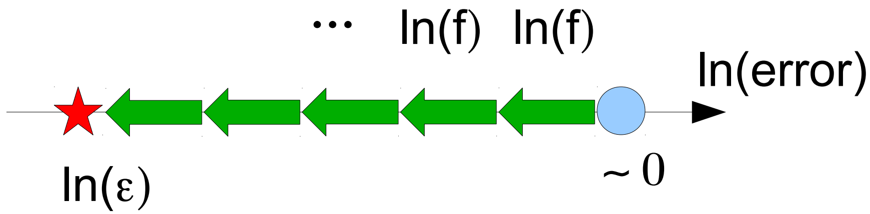
$$\langle e_1 \rangle / \langle e_2 \rangle \sim \quad f = \frac{60^{n/3}}{\sqrt{n+1}}$$



Scaling Analysis

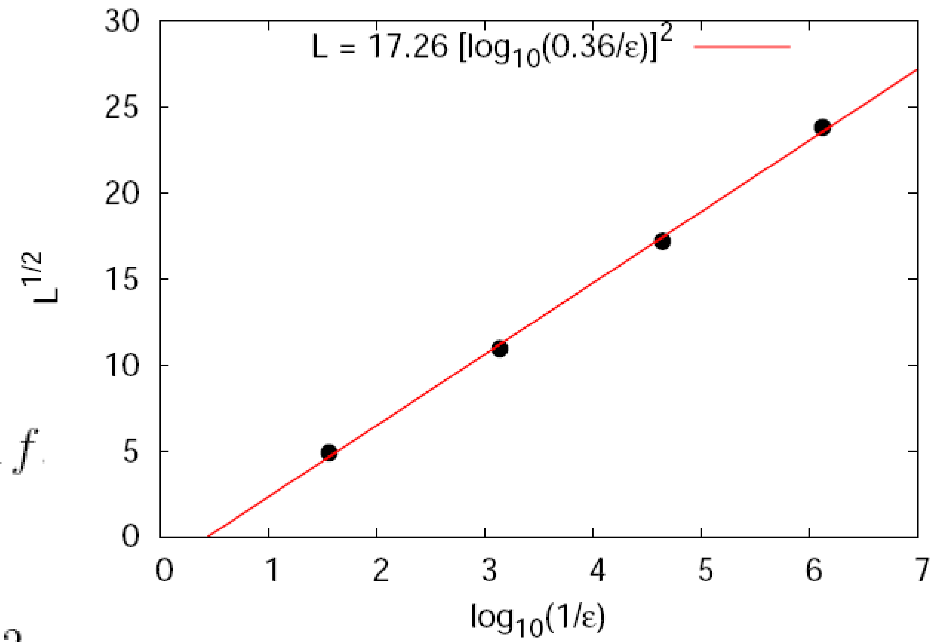
- The number of iteration for a given final error $\epsilon \ll 1$

$$\sim \frac{\ln(1/\varepsilon)}{\ln f} \qquad \ln(1/\varepsilon_i) \sim i \ln f$$



- Choose suitable braid segment length to match the residual error

$$l_i \sim L_0 \ln\left(1/\varepsilon_{i-1}\right)$$



- Each iteration increases the length by 4 (= n + 1) segments

$$L_i - L_{i-1} = 4L_0 \ln\left(1/\varepsilon_{i-1}\right) \sim 4L_0(i-1)\ln f$$

- Length of braid after q iterations

$$L_q \sim \sum_{i=1}^{q} 4L_0(i-1)\ln f \sim q^2 \sim \left(\ln\left(1/\varepsilon\right)\right)^2$$

- Time $\sim \ln\left(1/\varepsilon\right)$

Comparison with Other Algorithms

- Compiling with the RG-like algorithm

$$L_{\rm qh} \sim (\ln(1/\varepsilon))^2$$
$$T_{\rm qh} \sim N \sim \ln(1/\varepsilon)$$

- Brute-force search

$$L_{\rm bf} \approx L_0 \ln(1/\varepsilon),$$
$$T_{\rm bf} \sim (1/\varepsilon)^3.$$

- Solovay-Kitaev

$$L \sim (\ln(1/\varepsilon))^c \quad \text{with} \quad c = \frac{\ln 5}{\ln(3/2)} \approx 3.97$$
$$T \sim (\ln(1/\varepsilon))^d \quad \text{with} \quad d = \frac{\ln 3}{\ln(3/2)} \approx 2.71$$

$$U_{i+1} = A_i B_i A_i^{-1} B_i^{-1} U_i$$

Hormozi *et al.*

$$L \sim (\ln(1/\varepsilon))^2 \ln(\ln(1/\varepsilon))$$
$$T \sim (\ln(1/\varepsilon))^2 \ln(\ln(1/\varepsilon))$$

Dawson and Nielsen

It takes less than a second on a 3 GHz Intel E6850 processor to reach an average precision of 7 x 10^{-4} for an arbitrary gate.

Thanks to randomness in the building blocks, we save time in search exponentially.

#6: Importance of Algorithm

- In a classical computer, one can build up a circuit, e.g., to add two numbers using OR and NOT gates.
- In a quantum computer, the set of possible quantum gates forms a continuum, and it's not necessarily possible to use one gate set to simulate another exactly. Instead, some approximation may be necessary.
- We explore an algorithm that guarantees the efficient construction of any quantum gate, to a very good approximation.
  - From a practical point of view, this is important in compiling quantum algorithms (like Shor's) into a form that can be implemented fault-tolerantly.
  - From a more mathematical point of view, we give a general statement about how quickly the group SU(d) is "filled in" by a universal set of gates.
- This is also the importance of the textbook example – the Solovay-Kitaev algorithm.