NPRG approach to quantum criticality

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Outline

- Introduction : (continuous) quantum phase transitions
- Thermodynamics near a QCP
- Dynamics near a QCP
  - scalar (Higgs) susceptibility
  - conductivity

T=0 quantum phase transitions

- Transverse field Ising model $\hat{H} = -J \sum_{\langle i,j \rangle} \hat{\sigma}_i^z \sigma_j^z - h \sum_i \hat{\sigma}_i^x$

J/h

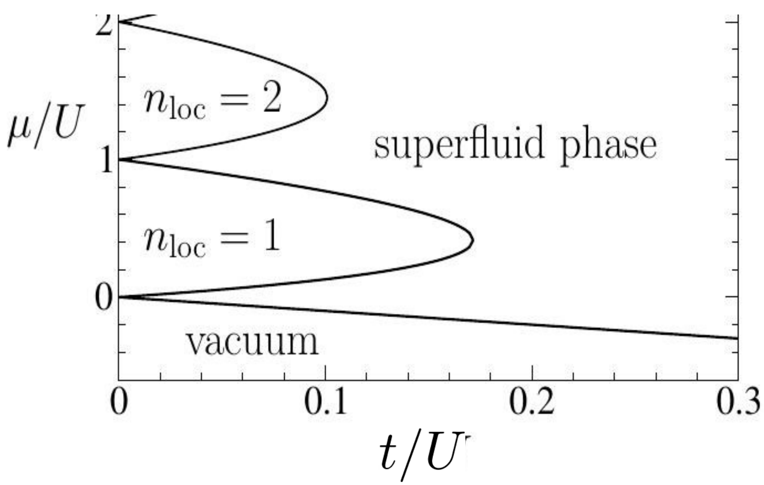
paramagnetic ground state
all spins aligned with h

$(J/h)_c$

ferromagnetic ground state
all spins parallel to z axis

- Bose-Hubbard model

$$\hat{H} = -t \sum_{\langle i,j \rangle} (\hat{\psi}_i^\dagger \hat{\psi}_j + \text{h.c.}) - \mu \sum_i \hat{\psi}_i^\dagger \hat{\psi}_i + \frac{U}{2} \sum_i \hat{\psi}_i^\dagger \hat{\psi}_i^\dagger \hat{\psi}_j \hat{\psi}_i$$



- Low-energy QFT and universality
  - transverse-field Ising model: quantum O(1) model
  - BH model: dilute Bose-gas or quantum O(2) model

Quantum O(N) model

(cold atoms in optical lattices, quantum AFs, etc.)

$$S = \int_0^{\beta\hbar} d\tau \int d^d r \, \frac{1}{2}(\nabla\varphi)^2 + \frac{1}{2c^2}(\partial_\tau\varphi)^2 + \frac{r_0}{2}\varphi^2 + \frac{u_0}{4!}(\varphi^2)^2$$

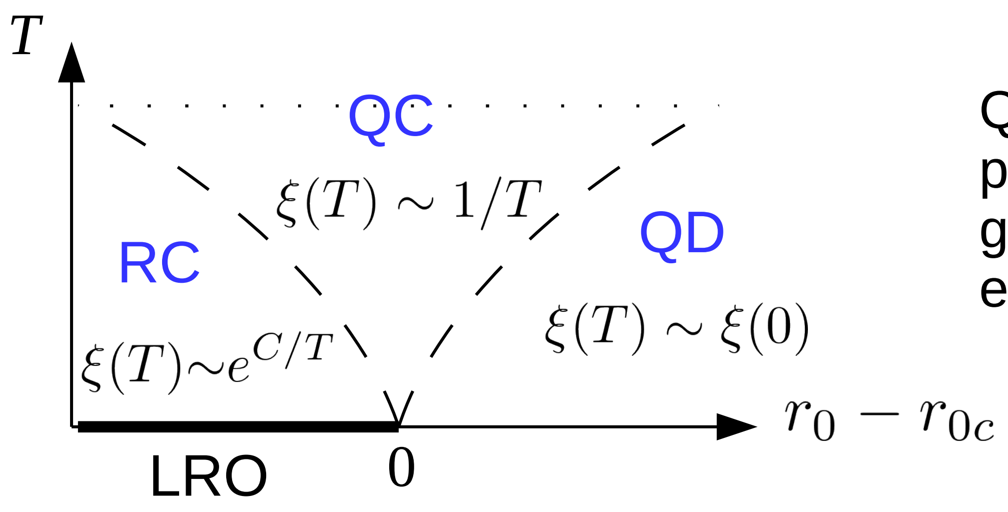
- T=0 : classical (d+1)-dimensional O(N) model $\quad \xi \sim |r_0 - r_{0c}|^{-\nu}$

  QPT for $r_0=r_{0c}$ ((d+1)D Wilson-Fisher fixed point) $\quad \xi_\tau \sim \xi^z \sim |r_0 - r_{0c}|^{-z\nu}$

  $z = 1$

- T>0 : finite size $L_\tau = \beta\hbar$ in imaginary time

(2D, N>2)



Quantum Critical regime : physics determined by critical ground state and its thermal excitations

What do we want do understand/calculate ?

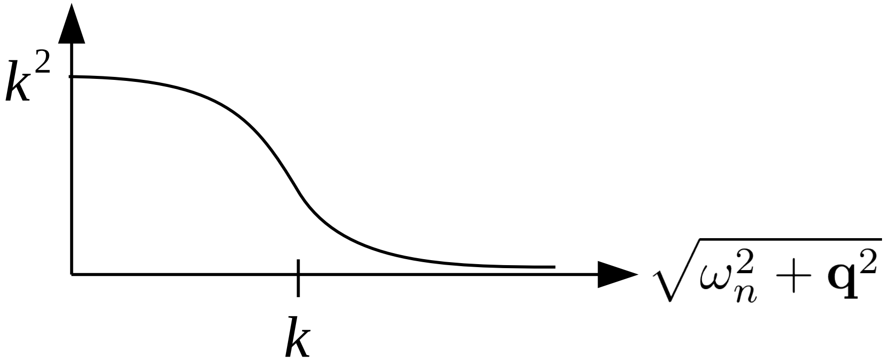
T=0 and T>0 properties near QCP

- thermodynamics : $$P(T) = P(0) + \frac{T^3}{c^2}\mathcal{F}_N\left(\frac{\Delta}{T}\right)$$

- time-dependent correlation functions (real time) $$\chi^R(\omega) \sim \Delta^{-x}\Phi_N\left(\frac{\omega}{T}, \frac{\Delta}{T}\right)$$

Non-perturbative functional RG for quantum O(N) model

- $S \to S + \frac{1}{2} T \sum_{\omega_n} \int_{\mathbf{q}} \varphi(-\mathbf{q}, -i\omega_n) R_k(\mathbf{q}, i\omega_n) \varphi(\mathbf{q}, i\omega_n)$



$$\mathcal{Z}_k[J] = \int \mathcal{D}[\varphi] \, e^{-S - \Delta S_k + \int_{\mathbf{r},\tau} J \cdot \varphi}$$

- Scale-dependent effective action $\quad \phi = \langle \varphi \rangle$

$$\Gamma_k[\phi] = -\ln \mathcal{Z}_k[J] + \int J \cdot \phi - \Delta S_k[\phi]$$

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \mathrm{Tr} \left\{ \partial_k R_k (\Gamma^{(2)}[\phi] + R_k)^{-1} \right\}$$ (Wetterich'93)

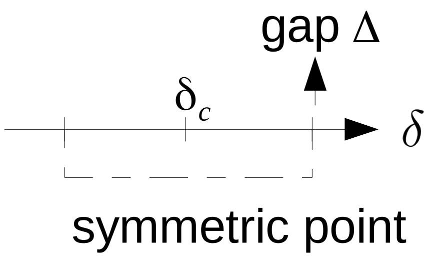
with $\Gamma_\Lambda[\phi] = S[\phi]$

- Derivative expansion

$$\rho = \frac{\phi^2}{2}$$

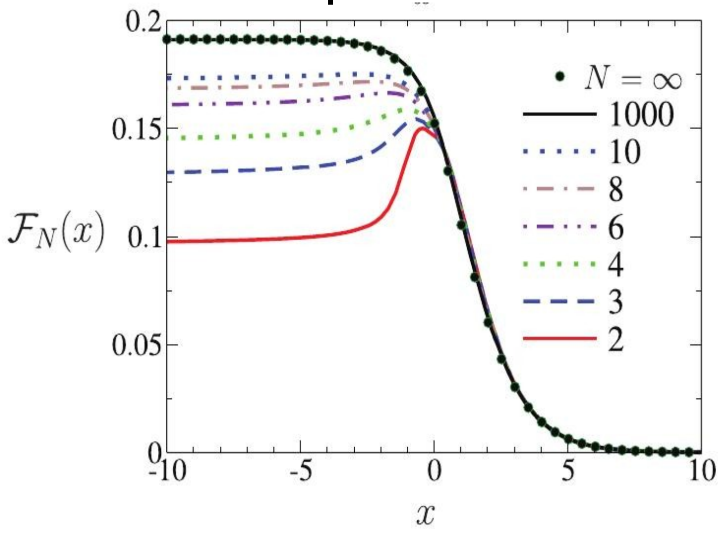
$$\Gamma_k[\phi] = \int_0^\beta d\tau \int d^2r \Big\{ \frac{Z_k^x(\rho)}{2}(\nabla\phi)^2 + \frac{Z_k^\tau(\rho)}{2}(\partial_\tau\phi)^2 + \frac{Y_k^x(\rho)}{4}(\nabla\rho)^2 + \frac{Y_k^\tau(\rho)}{4}(\partial_\tau\rho)^2 + U_k(\rho) \Big\}$$

Pressure : $$P(T) = -\lim_{k\to 0} U_k(\rho_{0,k}) = P(0) + N\frac{T^3}{c^2}\mathcal{F}_N\left(\frac{\Delta}{T}\right)$$

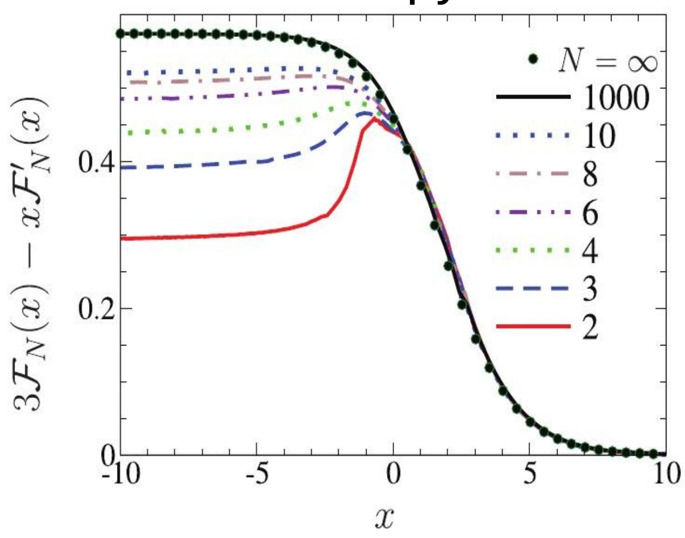


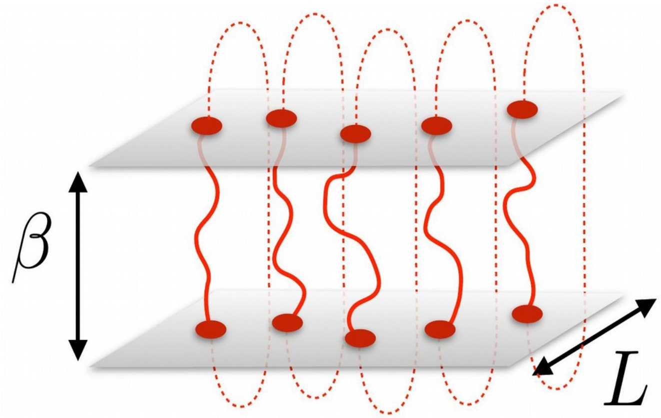
- Rançon, Kodio, ND, Lecheminant, PRB'13

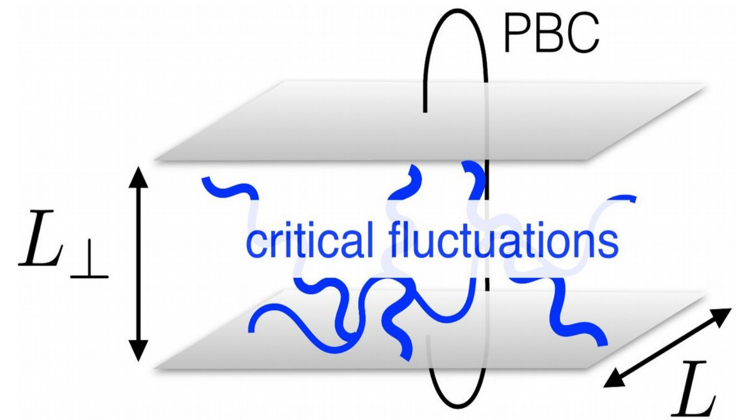
pressure



entropy





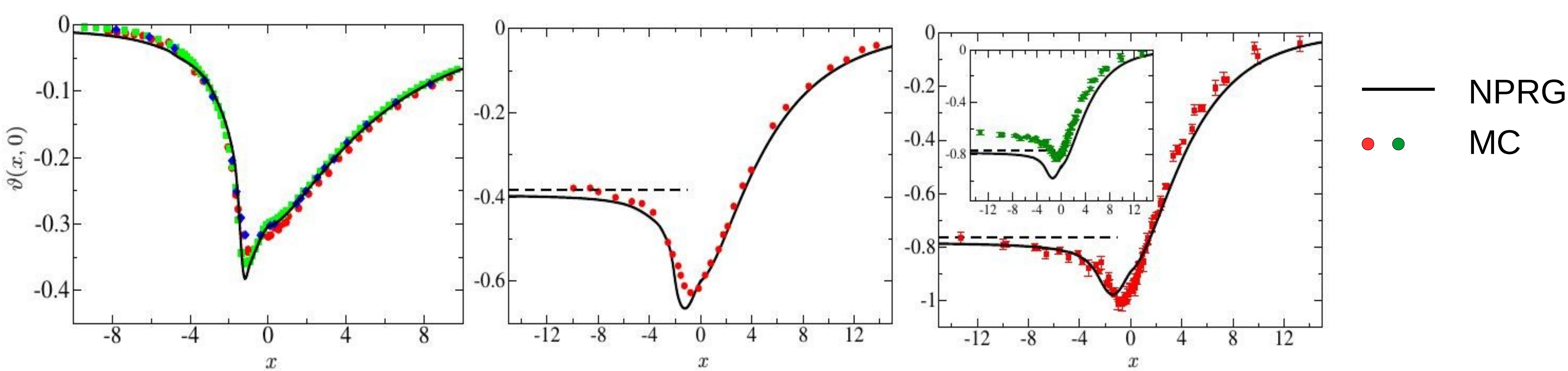


quantum-classical mapping

2D quantum system at finite temperature (equation of state)

$$L_\perp \sim \hbar\beta = \frac{\hbar}{k_B T}$$

3D classical system in finite geometry with periodic boundary conditions (Casimir effect)



A. Rançon, L.-P. Henry, F. Rose, D. Lopes Cardozo, ND, P. Holdsworth, and T. Roscilde, arXiv:1606.03205

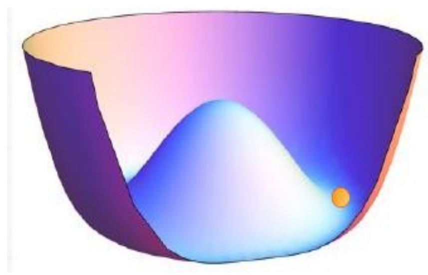
See Adam Rançon's talk (Tuesday afternoon)

Correlation functions

- Examples
  - o.p. susceptibility: $G_i(\mathbf{r},\tau) = \langle \varphi_i(\mathbf{r}\tau)\varphi_i(00)\rangle$
  - Scalar (Higgs) susceptibility: $\chi_s(\mathbf{r},\tau) = \langle \varphi^2(\mathbf{r}\tau)\varphi^2(00)\rangle$
  - Conductivity : $\chi^{ab}_{\mu\nu} = \langle j^a_\mu(\mathbf{r}\tau) j^b_\nu(00)\rangle$
- Difficulties
  - Strongly interacting theory (QCP)
  - Frequency/momentum dependence
  - Analytical continuation $\chi^R(\omega) = \chi(i\omega_n \to i\omega + i0^+)$

Longitudinal vs scalar susceptibililty
(ordered phase)

- Mean field



$$G_{\|}(\mathbf{p}, i\omega_n) = \frac{1}{\omega_n^2 + \mathbf{p}^2 + 2\Delta^2}$$

$$G_{\perp}(\mathbf{p}, i\omega_n) = \frac{1}{\omega_n^2 + \mathbf{p}^2} \qquad \text{(N-1 GBs)}$$

- Beyond mean field : transverse and longitudinal fluctuations are coupled

$$\mathrm{G}_{\|}(\mathbf{p}, i\omega_n) \sim \frac{1}{(\omega_n^2 + \mathbf{p}^2)^{(3-d)/2}}$$

- Scalar susceptibility [Podolsky, Auerbach, Arovas, PRB'11]

$$\chi_s(\mathbf{r}, \tau) = \langle \varphi(\mathbf{r}\tau)^2 \varphi(00)^2 \rangle$$

Higgs resonance ?

Spectral function : $\chi_s''(\omega) = \Im[\chi_s^R(\mathbf{q}=0, \omega)] \sim \Delta^{3-2/\nu} \Phi\left(\frac{\omega}{\Delta}\right)$

How to compute the scalar susceptibility (T=0) ?

- BMW approximation [Blaizot et al.'06, Benitez et al.'09]

$$\partial_k U_k(\rho) \to \Gamma_k^{(2)}$$
$$\partial_k \Gamma_k^{(2)} \to \Gamma_k^{(3)},\ \Gamma_k^{(4)}$$

BMW : $\Gamma^{(3)}_{ijl}(\mathbf{p},0,-\mathbf{p}) = \frac{\partial}{\partial \phi_j}\Gamma^{(2)}_{il}(\mathbf{p},-\mathbf{p})$

- Scalar susceptibility [Rose, Léonard, ND, PRB'15]

$$S \to S + \Delta S_k + \int_{\mathbf{r},\tau} (J\cdot\varphi + h\varphi^2) \to \mathcal{Z}_k[J,h] \to \Gamma_k[\phi,h]$$

$$\chi_s(\mathbf{p}) = -\Gamma^{(0,2)}(\mathbf{p}) + \Gamma_i^{(1,1)}(\mathbf{p})\Gamma^{(0,2)-1}(\mathbf{p})\Gamma_j^{(1,1)}(\mathbf{p})$$

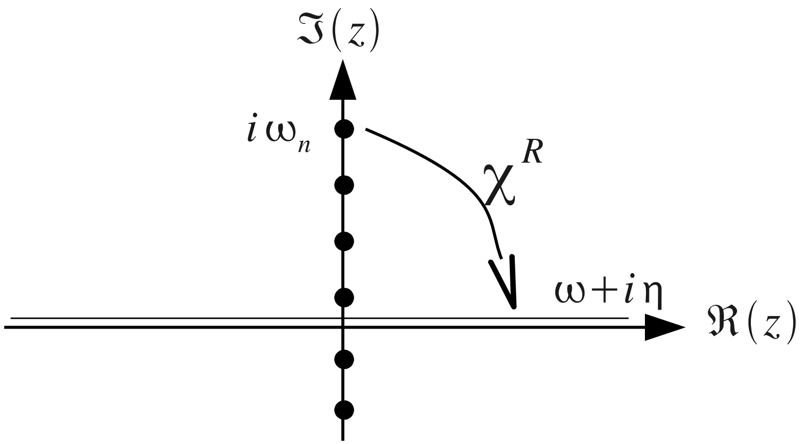
Generalized vertices : $\Gamma^{(n,m)}_{k,i_1\cdots i_n}[\phi] = \left.\frac{\delta^{n+m}\Gamma_k[\phi,h]}{\delta\phi_{i_1}\cdots\delta\phi_{i_n}\delta h\cdots\delta h}\right|_{h=0}$

BMW : closed equations for $\Gamma_k^{(2,0)},\ \Gamma_k^{(0,2)}\ ,\Gamma_k^{(1,1)}$

Analytical continuation

- we compute $\chi(i\omega_n) = \int \frac{d\omega}{\pi} \frac{\Im[\chi^R(\omega)]}{\omega - i\omega_n}$

  we want $\chi^R(\omega) = \chi(i\omega_n \to \omega + i0^+)$



- Padé approximant [Vidberg-Serene'77]

  $\chi(z)$ known for *M* Matsubara frequencies: $z_1 = i\omega_{n_1}, \cdots, z_M = i\omega_{n_M}$

$$\chi(z) = \cfrac{a_1}{1 + \cfrac{a_2(z - z_1)}{1 + \cfrac{\cdots}{a_M(z - z_{M-1})}}} \equiv \frac{P_{M-1}(z)}{Q_M(z)}$$

$$\chi^R(\omega) = \frac{P_M(\omega + i\epsilon)}{Q_{M-1}(\omega + i\epsilon)}$$

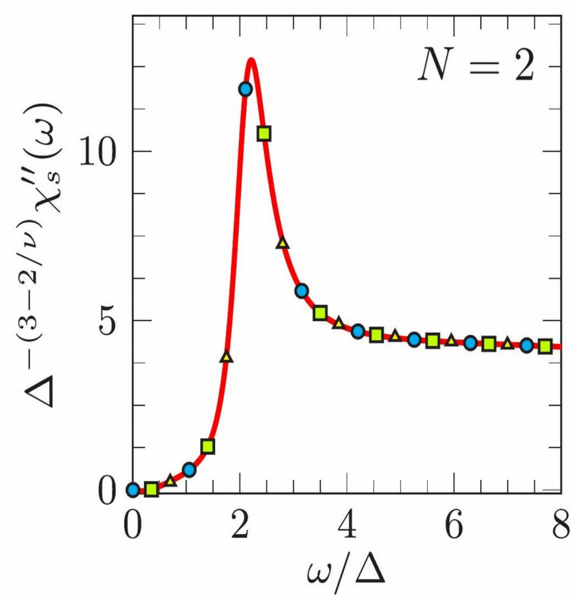
works well at *T*=0 and with data without numerical noise

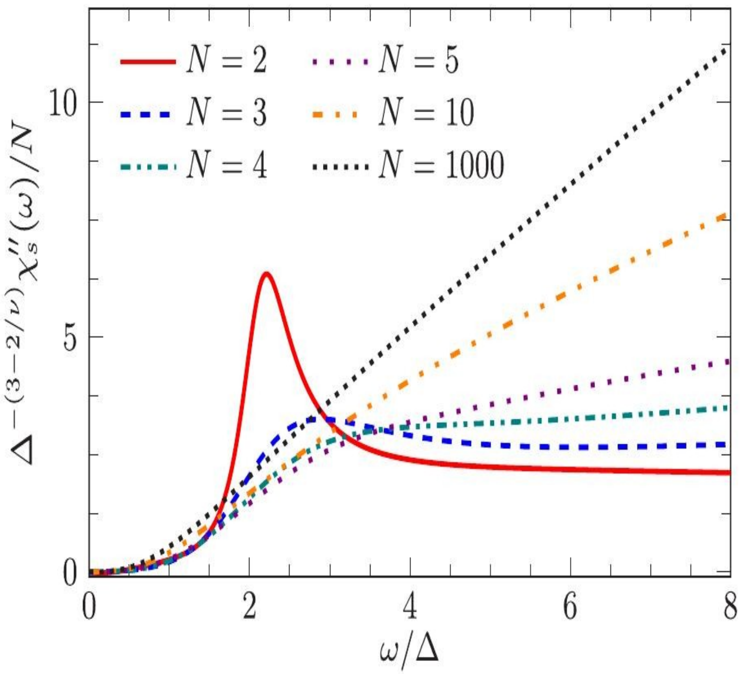
- PHYSICAL REVIEW D **93,** 125018 (2016)

  Bound states of the ϕ^4 model via the nonperturbative renormalization group

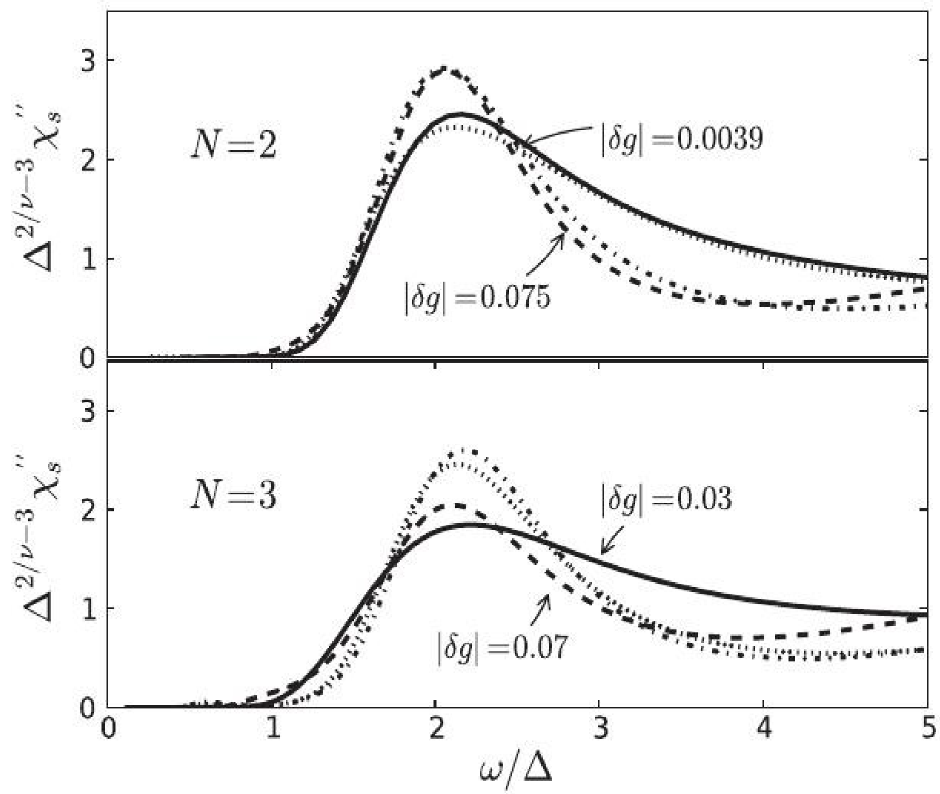
  F. Rose,[1] F. Benitez,[2,3] F. Léonard,[1] and B. Delamotte[1]

NPRG vs Monte Carlo (T=0)





[Rose, Léonard, ND, PRB'15]



[Gazit, Podolsky, Auerbach, PRL'13]

experimental observation (cold atoms)
Endres et al., Endres et al. Nature 2012

Higgs mass m_H/Δ

			$N=3$	$N=2$
Mean field			$\sqrt{2}$	$\sqrt{2}$
MC[1]	2013	O(N) model	2.2(3)	2.1(3)
QMC[2]	2013	Bose-Hubbard		3.3(8)
NPRG[3]	2014	O(N) model		2.4
ϵ expansion[4]	2015	O(N) model	1.64	1.67
NPRG-BMW[5]	2015	O(N) model	2.7	2.2
QMC[6]	2015	bilayer Heisenberg	2.6(4)	
Exact diag.[7]	2016	XY model		2.1(2)
Exact diag.[8]	2016	bilayer Heisenberg	2.7	

Conductivity

- Methods
  - QMC but analytical continuation not possible for $\omega \leq T$
  - Holographic models (AdS/CFT correspondence) but relationship to condensed-matter systems not clear
  - NPRG [F. Rose & ND, arXiv:1609xxxxx]
- NPRG (T=0) See Félix Rose's talk this afternoon

$$S = \int_{\mathbf{r}} \frac{1}{2}[(\partial_\mu - A_\mu)\varphi]^2 + \frac{r_0}{2}\varphi^2 + \frac{u_0}{4!}(\varphi^2)^2$$

$$\Delta S_k = \frac{1}{2}\int_{\mathbf{r},\mathbf{r}'} \varphi(\mathbf{r}) \cdot R_k\big(-(\partial - A)^2\big)\varphi(\mathbf{r}')$$

(d+1)-dimensional classical model

$$A_\mu = A_\mu^a T^a \quad (T^a: \text{ generators of SO(N)})$$

$$J_\mu^a = -\frac{\delta S}{\delta A_\mu^a} = \partial_\mu \varphi \cdot T^a \varphi - A_\mu \varphi \cdot T^a \varphi \equiv j_\mu^a - A_\mu \varphi \cdot T^a \varphi$$

conductivity
$$K_{\mu\nu}^{ab}(\mathbf{r},\mathbf{r}') = \langle j_\mu^a(\mathbf{r}) j_\nu^b(\mathbf{r}')\rangle - \delta_{\mu\nu}\delta(\mathbf{r}-\mathbf{r}')\langle T^a\varphi \cdot T^b \varphi\rangle$$

$$\sigma_{\mu\nu}^{ab}(\omega) = \frac{1}{i(\omega + i0^+)} K_{\mu\nu}^{ab}(p \to -i\omega + 0^+)$$

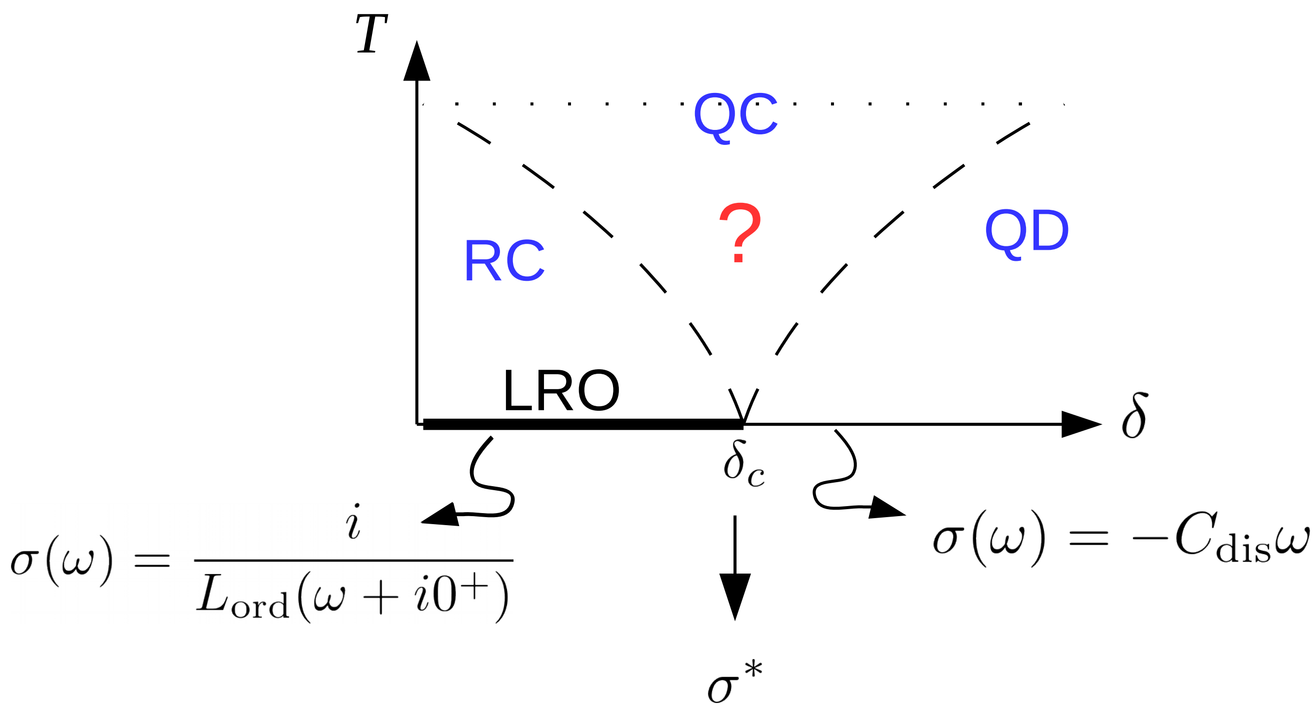
- Derivative expansion

$$\Gamma_k[\phi, A] = \int_{\mathbf{r}} \left\{ \frac{Z_k(\rho)}{2}[(\partial_\mu - A_\mu)\phi]^2 + \frac{Y_k(\rho)}{4}(\nabla\rho)^2 + U_k(\rho) + \frac{1}{4}X_{1,k}(\rho){F^a_{\mu\nu}}^2 + \frac{1}{4}X_{2,k}(\rho)(F^a_{\mu\nu}T^a\phi)^2 \right\}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^b_\mu - f_{abc}A^b_\mu A^c_\nu$

$$K^{ab}_{\mu\nu}(i\omega_n) = \delta_{\mu\nu}\delta_{ab}\{-\omega_n^2 X_1(\rho_0) + 2\rho_0\delta_{a\in\mathrm{A}}[-\omega_n^2 X_2(\rho_0) + Z(\rho_0)]\}$$

$$\sigma^{ab}_{\mu\nu}(\omega) = \frac{1}{i(\omega + i0^+)} K^{ab}_{\mu\nu}(i\omega_n \to \omega + i0^+)$$



$$\frac{\sigma^*}{\sigma_q} \text{ and } \frac{C_{\text{dis}}}{L_{\text{ord}}\sigma_q^2} \text{ are universal } [\sigma_q = q^2/h]$$

[Fisher et al., PRL'89]

ω-dependence of conductivity (N=2 and T=0) : Gazit, Podolsky, Auerbach, PRB'13

- Universal ratio

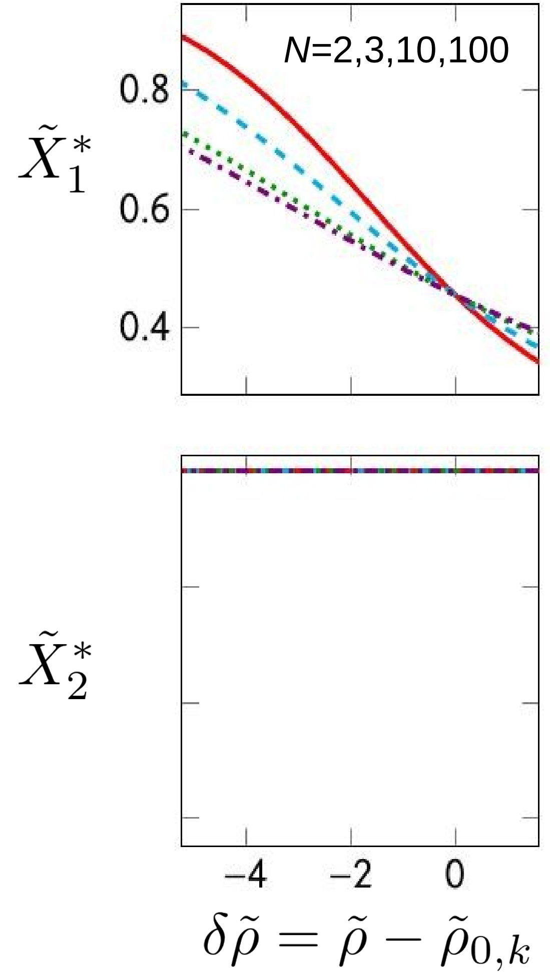
N	$\hbar\sigma_q/2\pi C_{\rm dis}\Delta$	$C_{\rm dis}/N L_{\rm ord}\sigma_q^2$
2	1.98	0.105
3	1.98	0.0742
4	1.98	0.0598
5	1.97	0.0520
6	1.97	0.0475
8	1.96	0.0731
10	1.96	0.0415
100	1.92	0.0413
1000	1.91	0.0416
∞	$\frac{6}{\pi} \simeq 1.910$	$\frac{1}{24} \simeq 0.4167$

MC
2.1(1)

Critical Capacitance and Charge-Vortex Duality Near the Superfluid-to-Insulator Transition

Snir Gazit, Daniel Podolsky, and Assa Auerbach

PRL'14



- ordered phase (N≥3) :

$$\sigma_{\mu\nu}^{ab}(\omega) = \delta_{\mu\nu}\delta_{ab}\begin{cases} \sigma_{\rm A}(\omega) & \text{if} \quad T^a\phi \neq 0 \\ \sigma_{\rm B}(\omega) & \text{if} \quad T^a\phi = 0 \end{cases}$$

$$\sigma_{\rm A}(\omega \to 0) = \frac{i}{L_{\rm ord}(\omega + i0^+)}$$

$$\sigma_{\rm B}(\omega \to 0) = \sigma_{\rm B}^* = \frac{\pi}{8}\sigma_q \quad \text{for all } N$$

Conclusion

- Non-perturbative functional RG is a powerful tool to study QCP's.
- Finite-temperature thermodynamic near a QCP is fully understood
  - Universal scaling function compares well with MC simulations of classical 3D systems in finite geometry (see A. Rançon's talk).
  - Pressure, entropy, specific heat are non-monotonous across the QCP; hence a clear thermodynamic signature of quantum criticality.
- Promising results for dynamic correlation functions (e.g. Higgs susceptibility and conductivity) but finite-temperature calculation still very challenging.
  - Derivation expansion needs to be improved to compute $\sigma(\omega)$ and prove the universality of $\sigma_B(\omega)$ [work in progress]
  - How to perform analytic continuation at finite temperature? Calculation with simplified propagators so that Matsubara sums can be performed explicitly (see, e.g., Tripolt, Strodthoff, von Smekal, Wambach)?