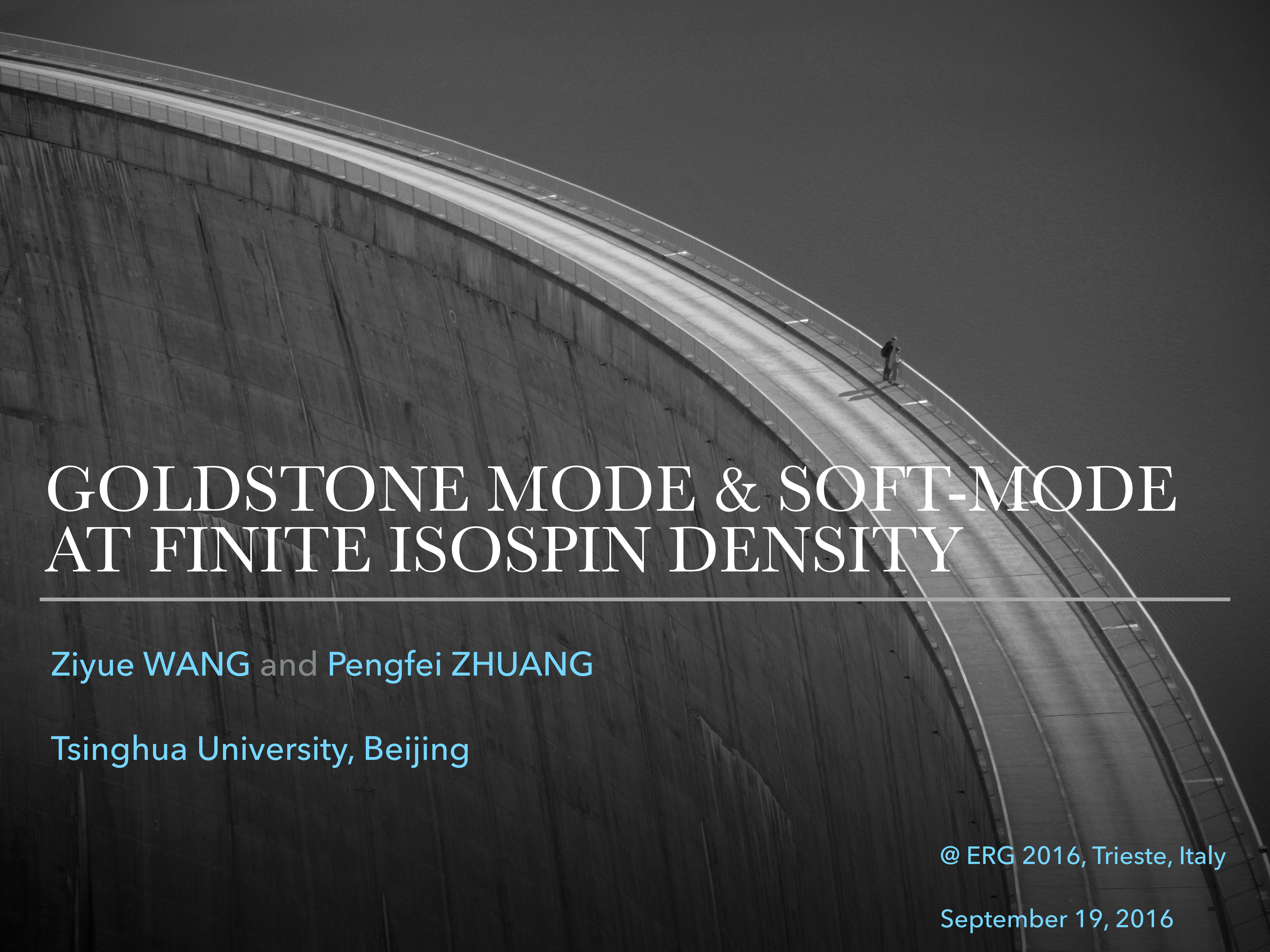


GOLDSTONE MODE & SOFT-MODE AT FINITE ISOSPIN DENSITY

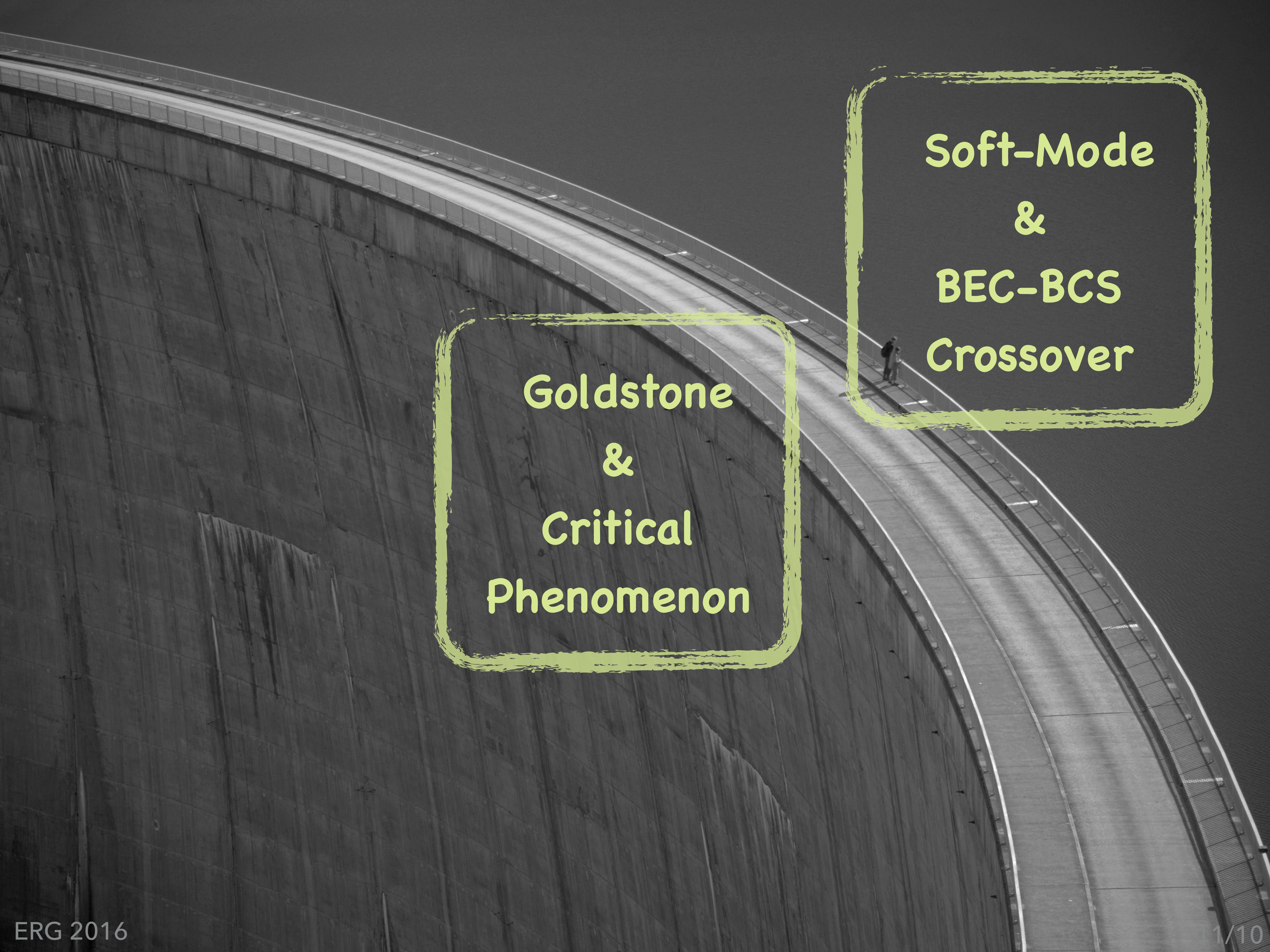
Ziyue WANG and Pengfei ZHUANG

Tsinghua University, Beijing

@ ERG 2016, Trieste, Italy

September 19, 2016

Goldstone
&
Critical Phenomenon



Soft-Mode
&
BEC-BCS
Crossover

Goldstone
&
Critical
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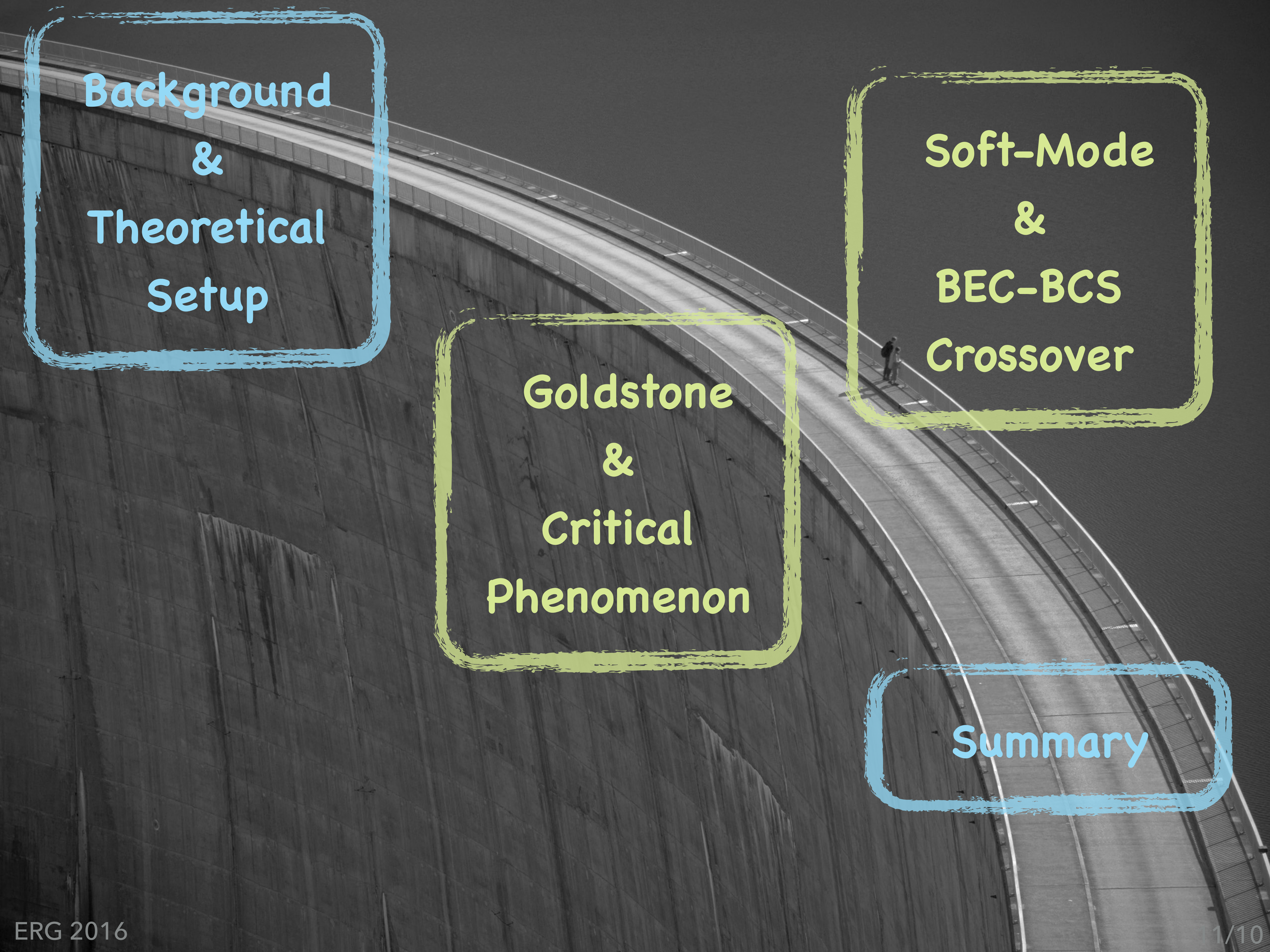

Background
&
Theoretical
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Goldstone
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Critical
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Soft-Mode
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Summary

Background
&
Theoretical
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Pion Superfluidity @ Finite Isospin Density

- Motivation:
  - isospin imbalance in the interior of neutron stars

    [S. Barshay et al, Phys. Lett. B 47, 107 (1973)] [V. A. Khodel et al, Phys. Rev. Lett. 93, 151101 (2004)]
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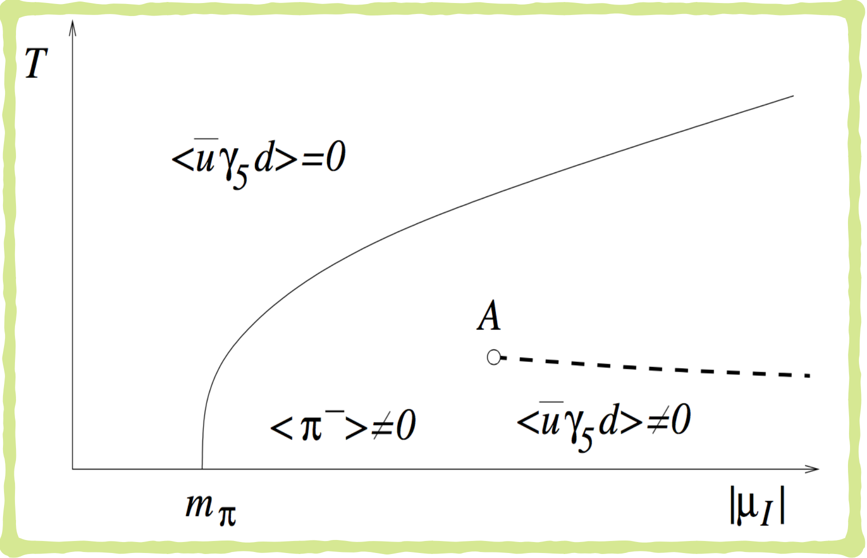

[D.T. Son and M.A. Stephanov, PRL.86,592]

Perturbative QCD

Background

&

Theoretical

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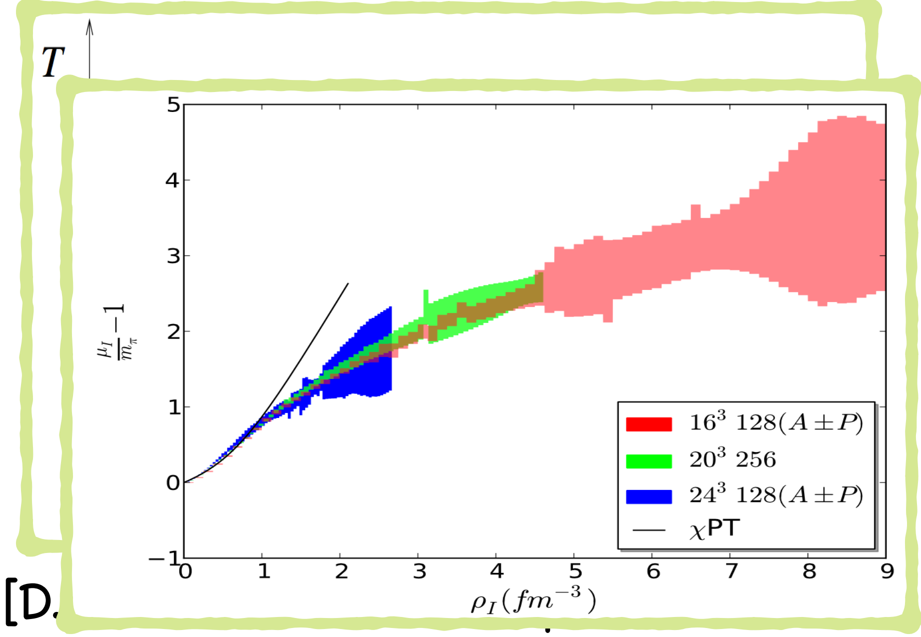

Perturbative QCD

Lattice Simulation

[D. 592]

[W. Detmold, et al., Phys. Rev. D 86, 054507]

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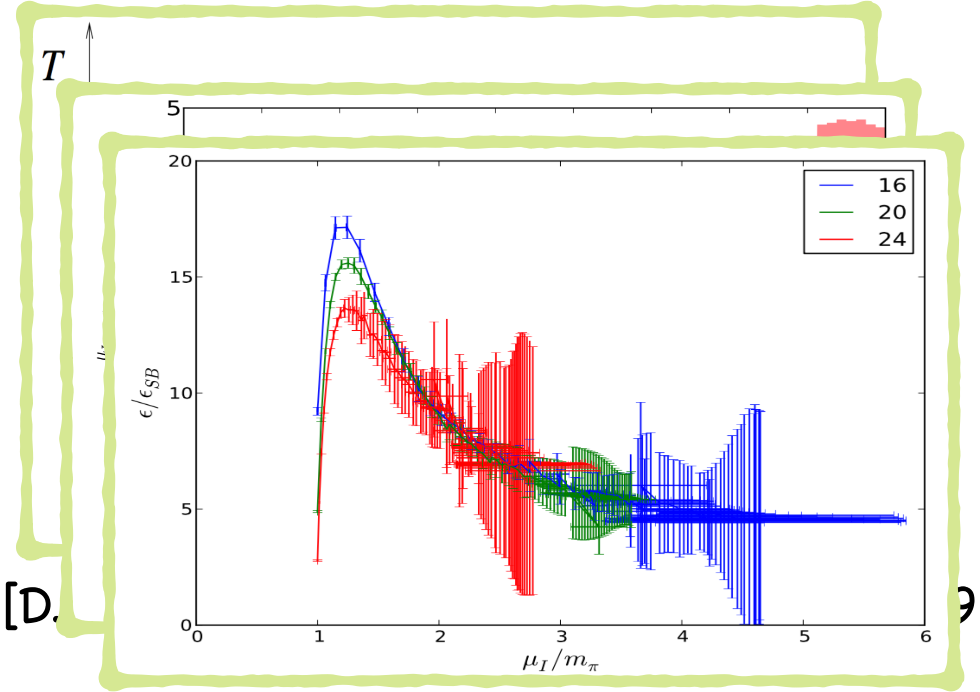

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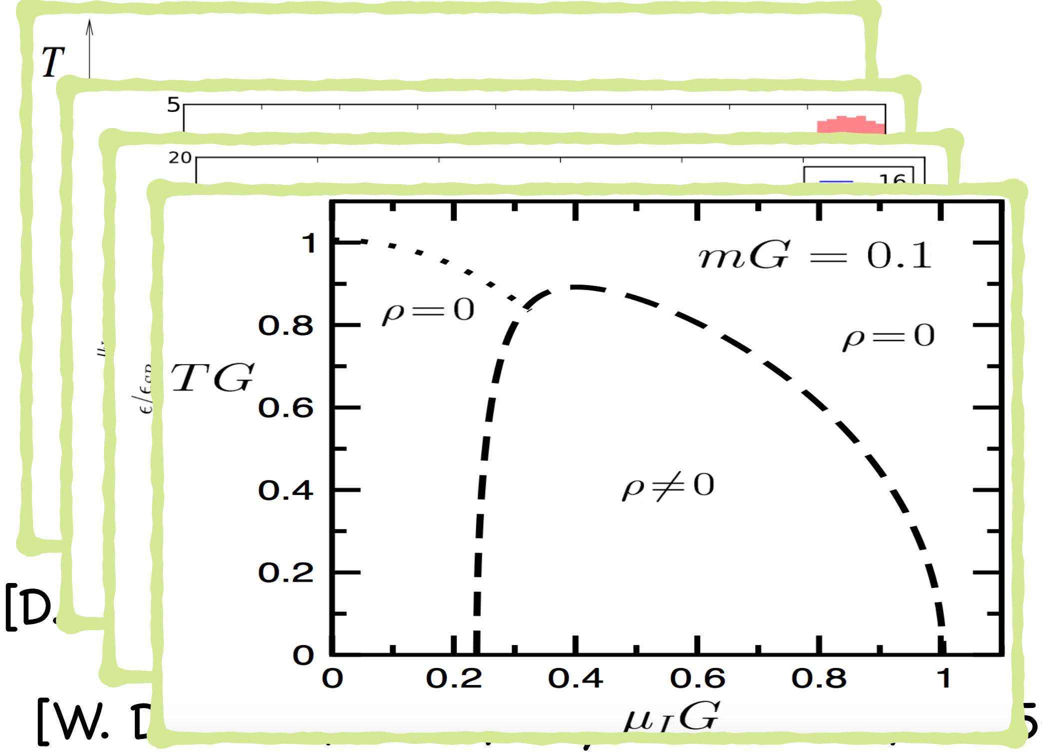

[D.

[W. D 507]

[B. Klein et al., Phys. Rev. D 72, 015007]

Perturbative QCD

Lattice Simulation

Random Matrix

Background

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Theoretical

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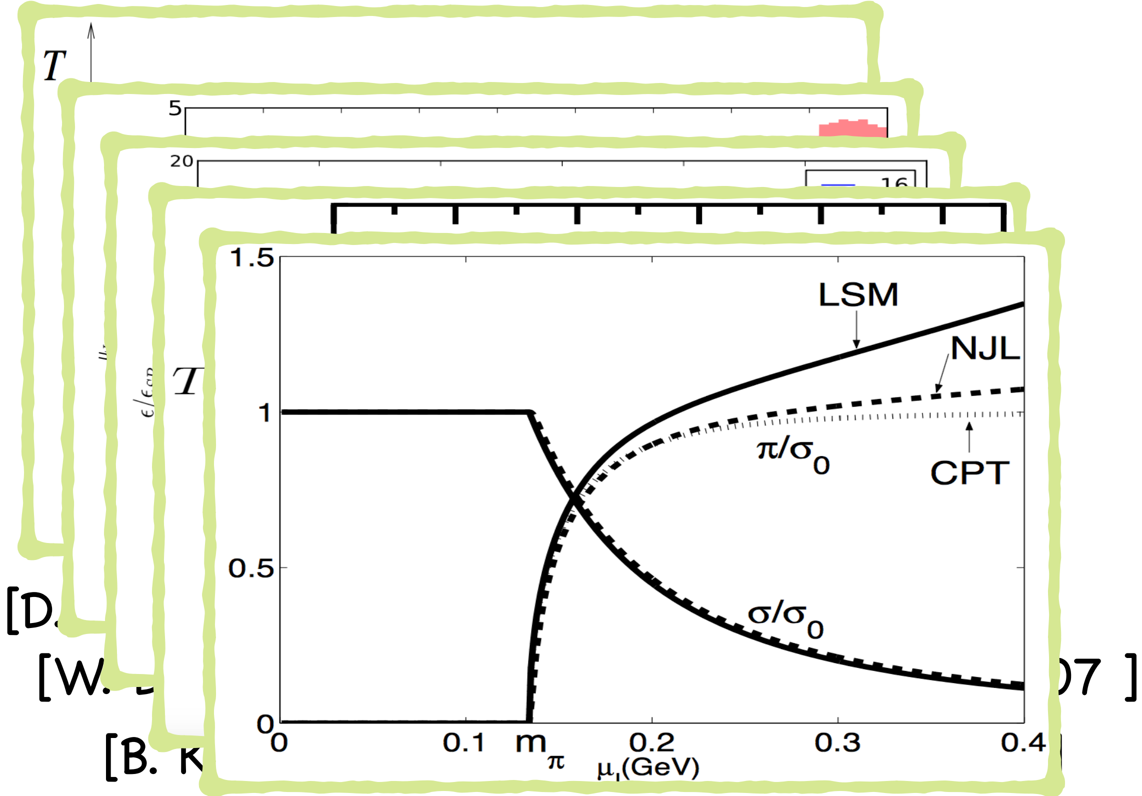

[L. He, et al., Phys. Rev. D 71, 116001]

Perturbative QCD

Lattice Simulation

Random Matrix

Nambu- Jona-Lasinio model

Background

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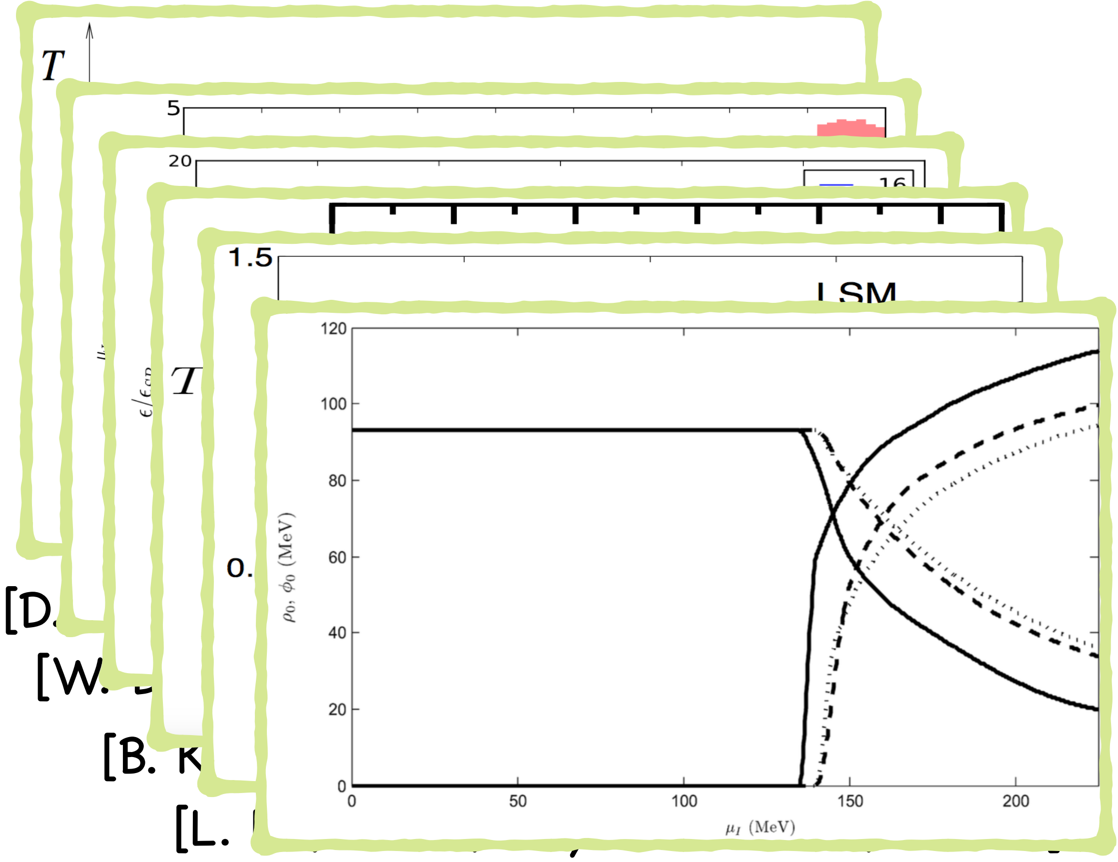

[D.

[W.

[B. K

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]

[E. E. Svanes et al., Nucl. Phys. A 857, 16]

Perturbative QCD

Lattice Simulation

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Linear-Sigma model

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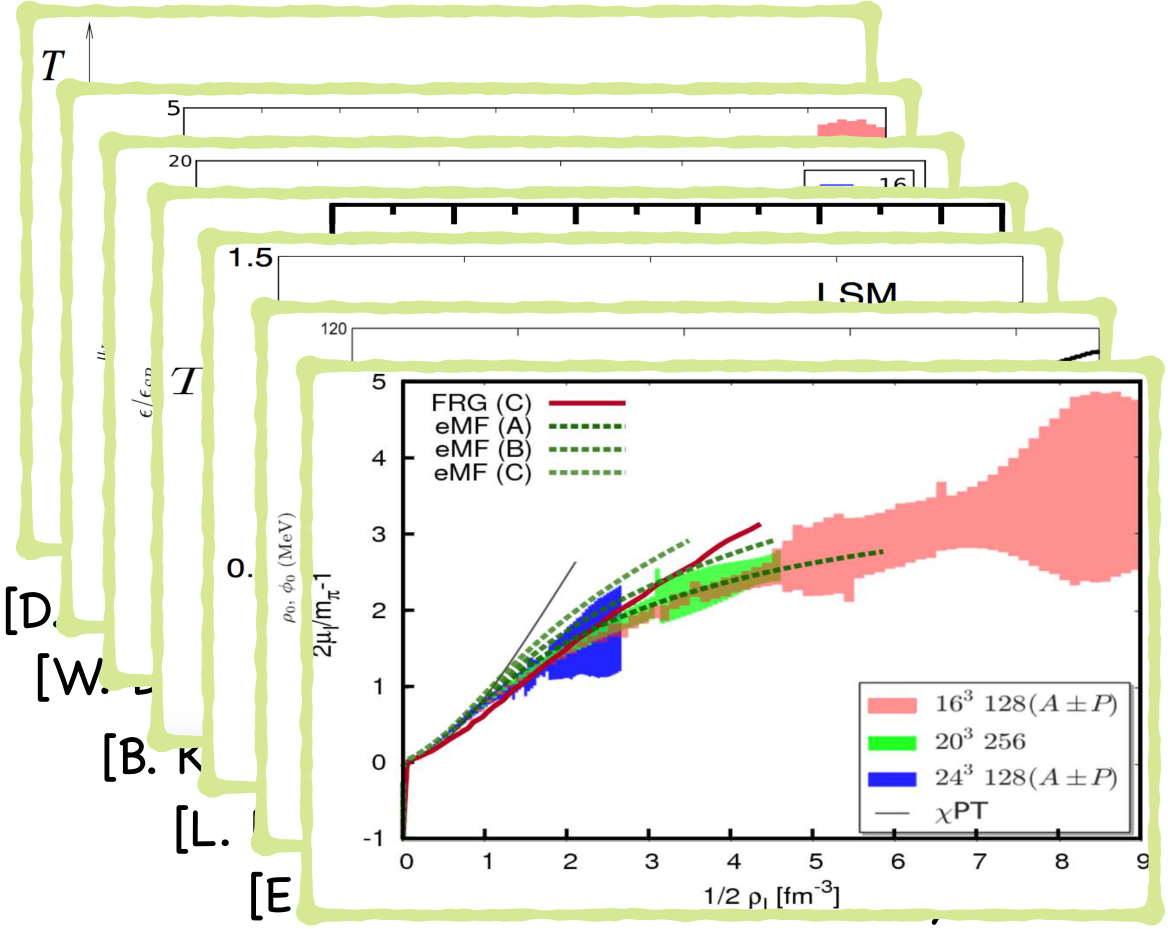

[D.

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[E 16]

[K. Kamikado, et al.,Phys. Lett. B 718, 1044]

- Perturbative QCD
- Lattice Simulation
- Random Matrix
- Nambu– Jona-Lasinio model
- Linear-Sigma model
- Quark-Meson model

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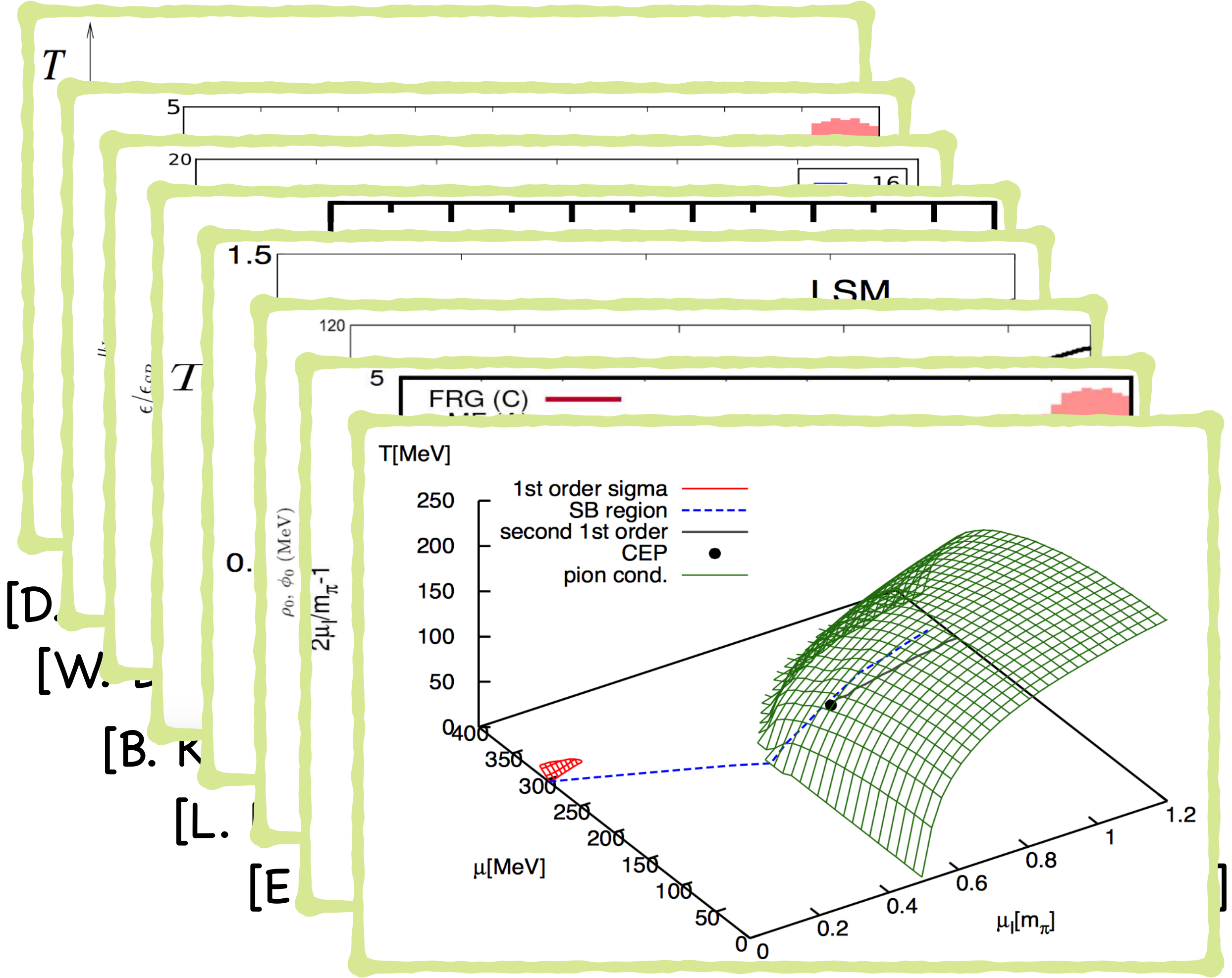

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Perturbative QCD

Lattice Simulation

Random Matrix

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Linear-Sigma model

Quark-Meson model

Second order phase transition on phase boundary

Background
&
Theoretical
Setup

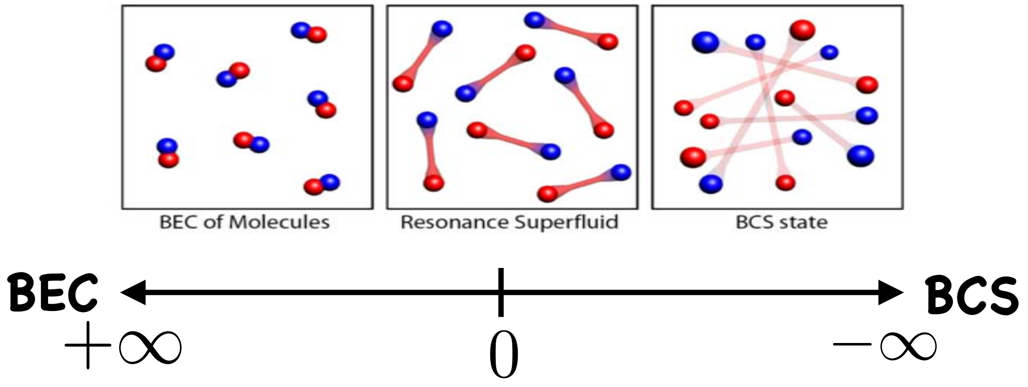
Pion Superfluidity @ Finite Isospin Density

- BEC-BCS crossover in pion superfluidity

- ultra cold atom & condensed matter :

attractive strength

$$1/k_F a$$



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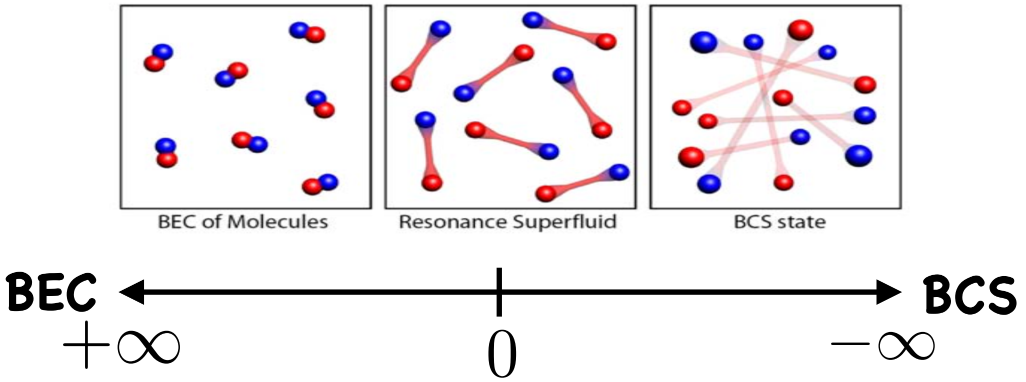
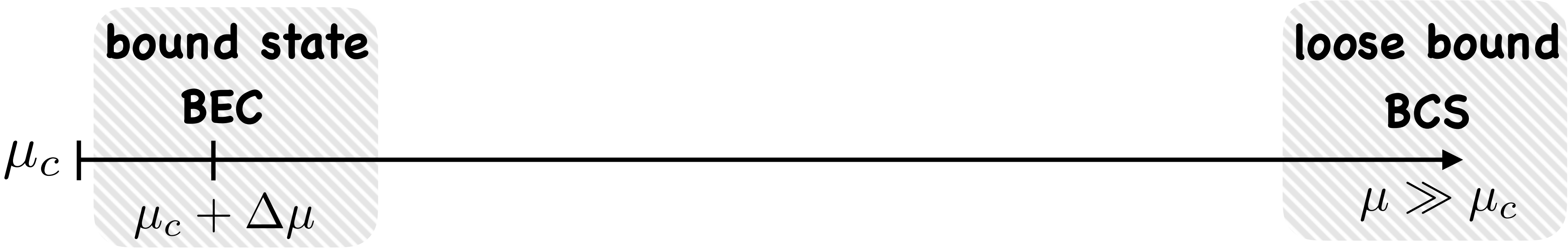


- QCD system (relativistic) : color-superconductivity & pion superfluidity...

density induced: **Tc < T***

T < Tc :



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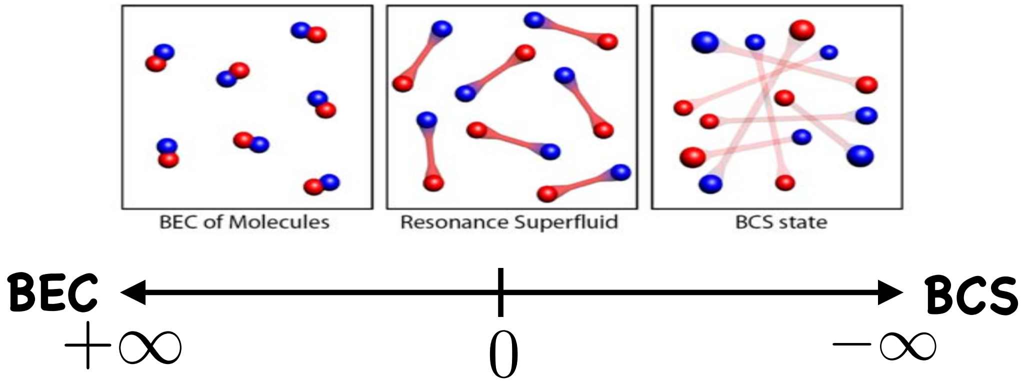

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T < Tc : **same symmetry, same order parameter, no phase transition**

bound state
BEC

μ_c

$\mu_c + \Delta\mu$

...crossover...

loose bound
BCS

$\mu \gg \mu_c$

Background
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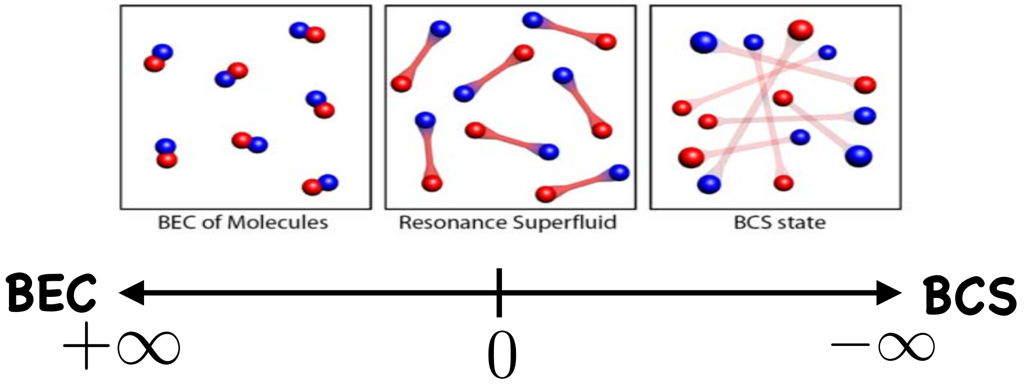

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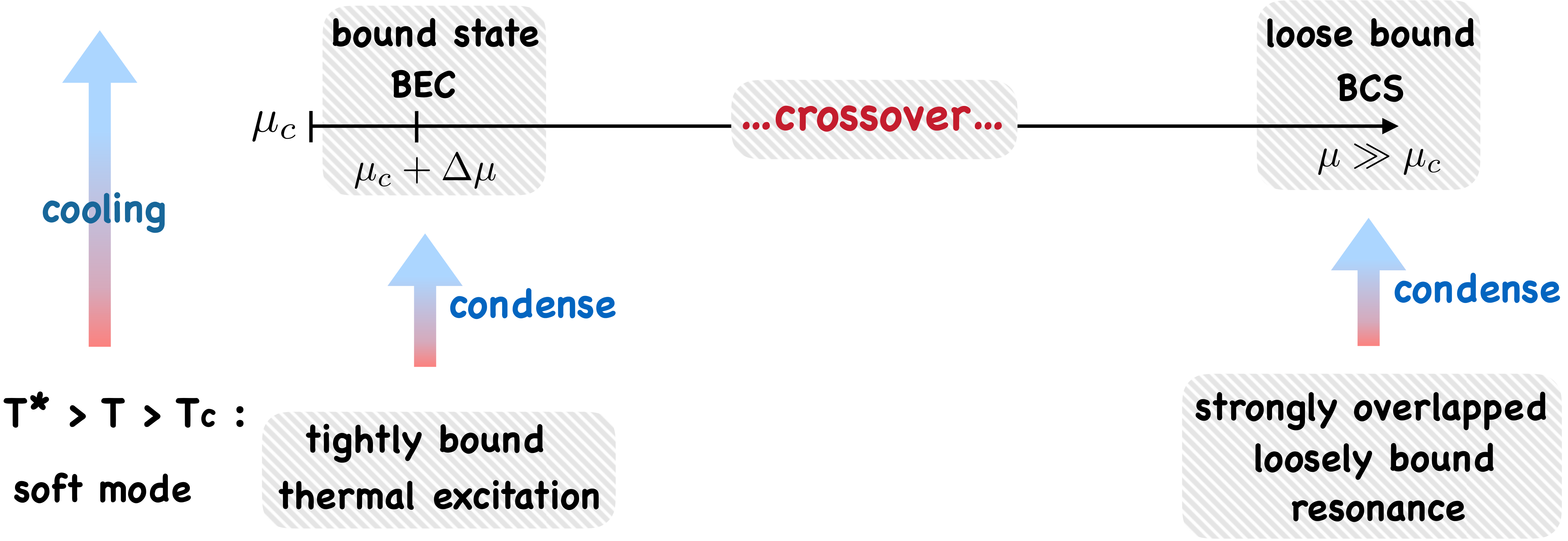

T* > T > Tc :

soft mode

tightly bound
thermal excitation

strongly overlapped
loosely bound
resonance

Pion Superfluidity @ Finite Isospin Density

- BEC-BCS crossover in pion superfluidity

- ultra cold atom & condensed matter :

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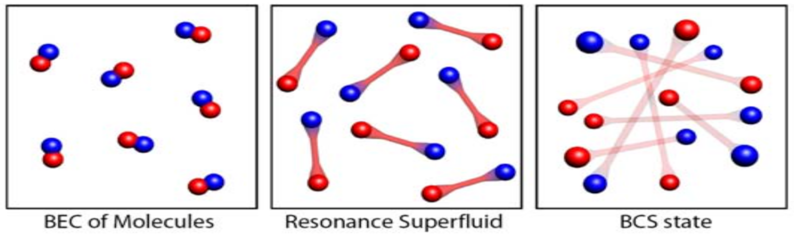

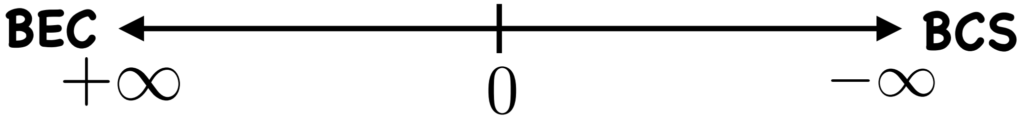

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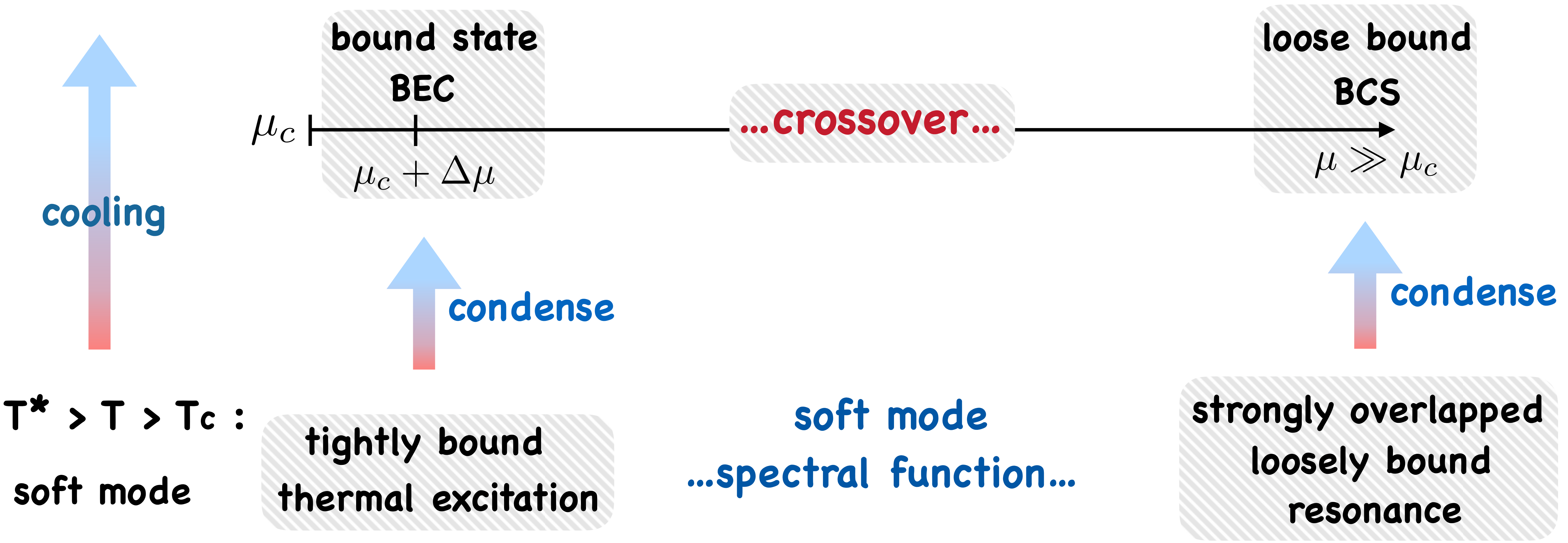

T* > T > Tc :

soft mode

tightly bound thermal excitation

soft mode …spectral function…

strongly overlapped loosely bound resonance

&

Theoretical Setup

Functional Renormalization Group (FRG)

• strongly coupled many-body system	phase transition
• scale transformation (fixed point)	critical phenomenon
• based on effective action	$\Gamma \rightarrow \Gamma^{(2)} \ldots \Gamma^{(n)}$
• Analytical continuation	real-time observables

& Theoretical Setup

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- Analytical continuation	real-time observables

Linear Sigma Model & Quark Meson Model

$$\Gamma_k[\Phi] \;=\; \int_x \Big\{ \bar{\Psi} S(i\partial)\Psi + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \pi_0)^2 + U_k(\vec{\varphi}^2) - c\sigma + \Big[(\partial_\mu - 2\delta_{\mu 4}\mu_I)\pi_-\Big]\Big[(\partial_\mu + 2\delta_{\mu 4}\mu_I)\pi_+\Big] \Big\}$$

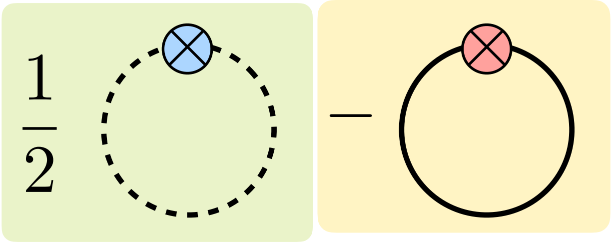
$$S(i\partial) \;=\; \begin{pmatrix} i(\not{\partial} + \gamma_4 \mu_I) + ih\,(\sigma + i\gamma_5 \pi_0) & -\sqrt{2}h\gamma_5\pi_- \\ -\sqrt{2}h\gamma_5\pi_+ & i(\not{\partial} - \gamma_4\mu_I) + ih\,(\sigma - i\gamma_5\pi_0) \end{pmatrix}.$$

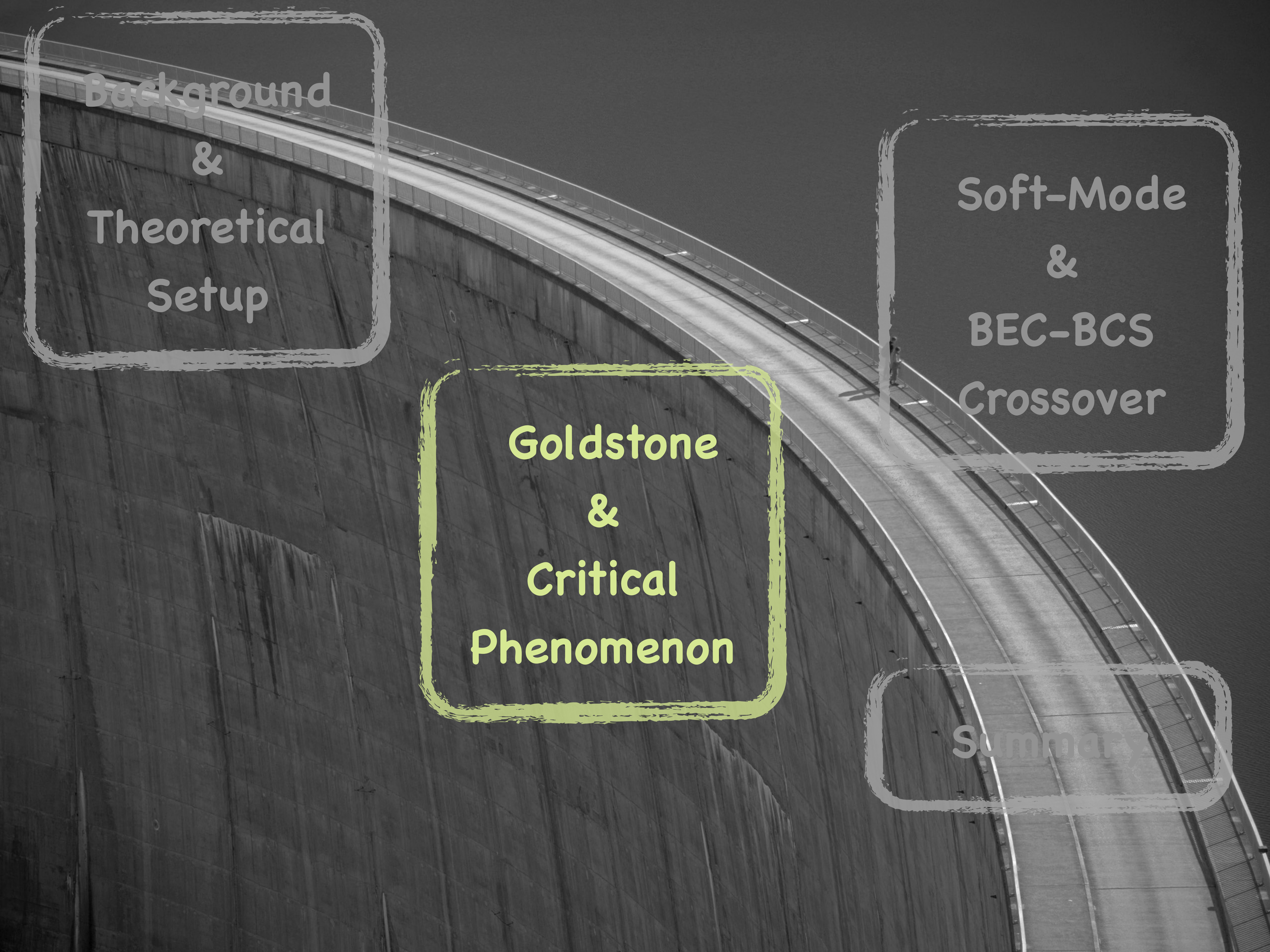
Two flavor chiral effective model $\quad \Psi = \begin{pmatrix} u \\ d \end{pmatrix}$

Symmetry breaking pattern: O(4) — O(3) — O(2) — Z(2)

RG Flow Equation

$$\partial_k \Gamma_k =$$





Background
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Goldstone
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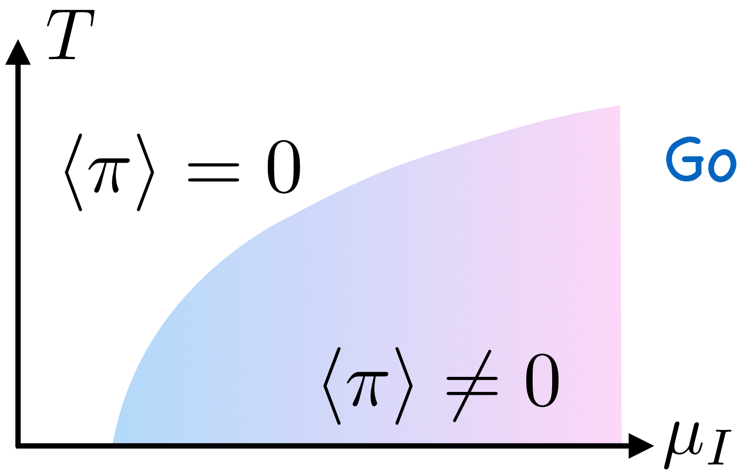
Summary

Goldstone Mode & Critical Phenomenon

- Symmetry breaking pattern: O(4) – O(3) – O(2) – Z(2)
- Second order phase transition
- Critical exponents & Universality class

Goldstone mode



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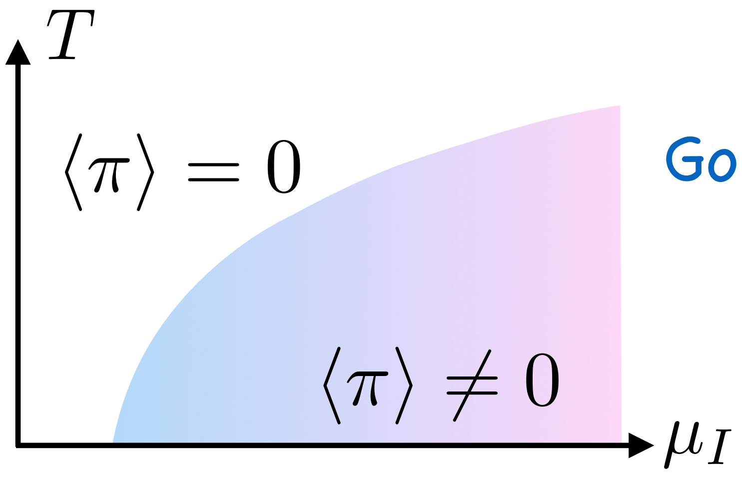

Goldstone mode

- LINEAR SIGMA Model + ISOSPIN

  FRG & MEAN FIELD

ZyW, Pengfei ZHUANG, arXiv:1511.05279

- Fit critical exponents β, ν

  along the phase boundary @ $T_c - \Delta T$

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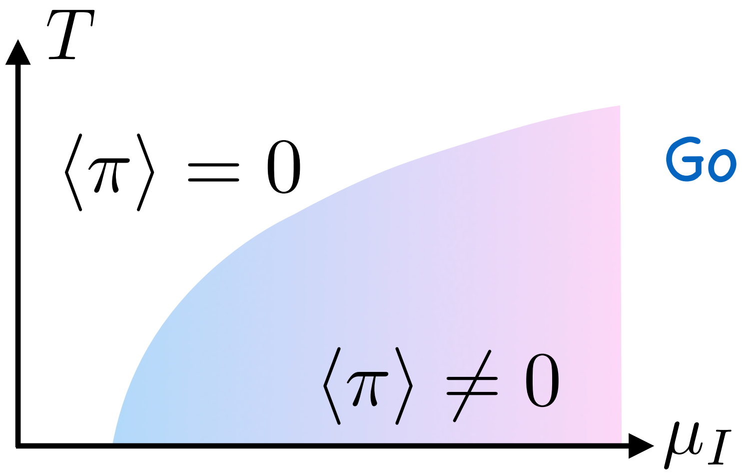

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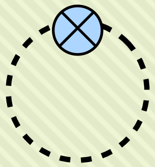
- Compare with a general O(N) MODEL, N=2, FIXED POINT
- O(2) Universality Class with a dimension crossover

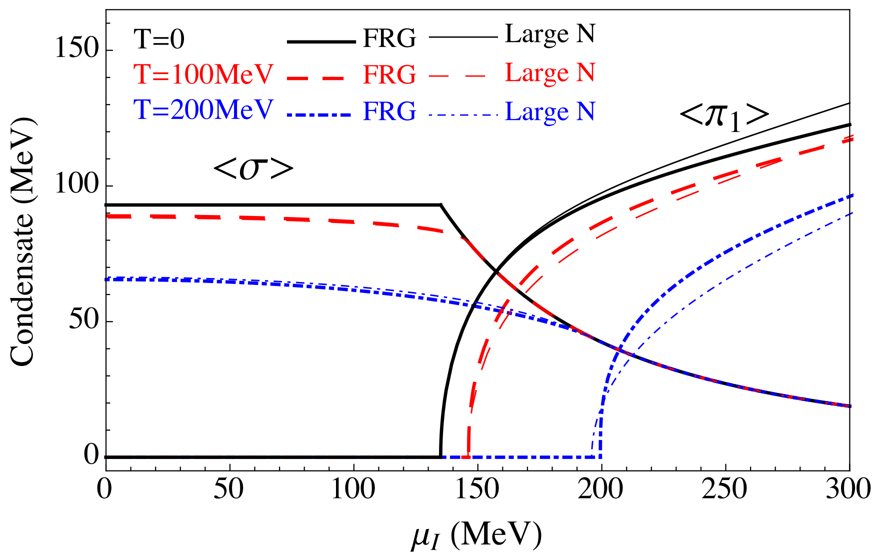
- **FRG :** $\partial_k \Gamma_k = \frac{1}{2}$ 

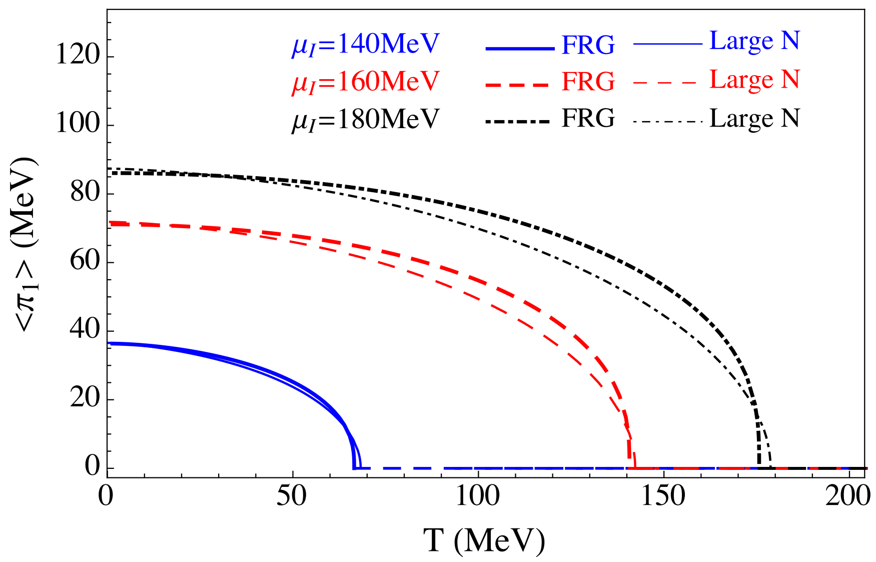
  Taylor expansion

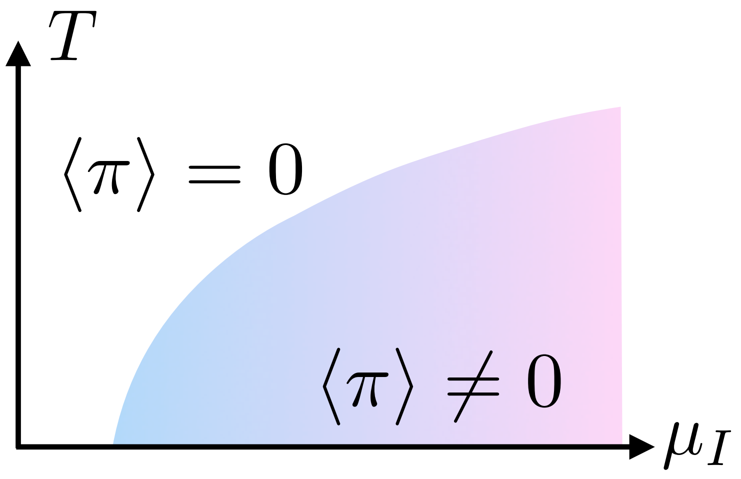
- **MEAN FIELD :**

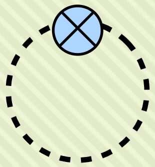
  Large-N expansion

Order Parameter







- **FRG:** $\partial_k \Gamma_k = \frac{1}{2}$ 

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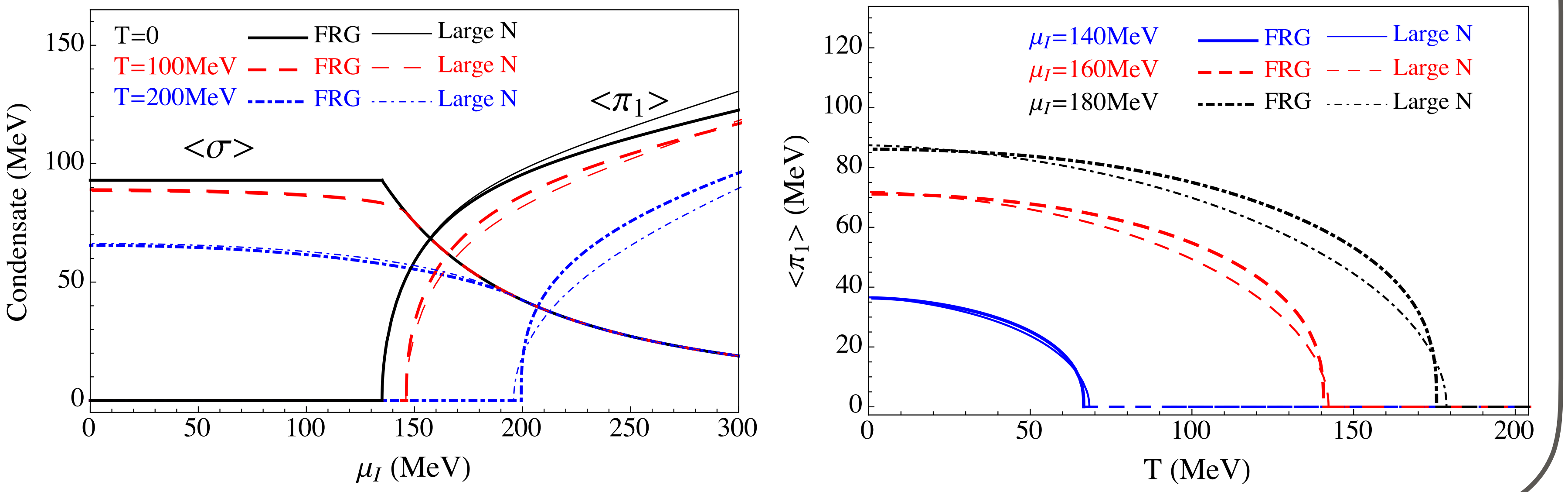
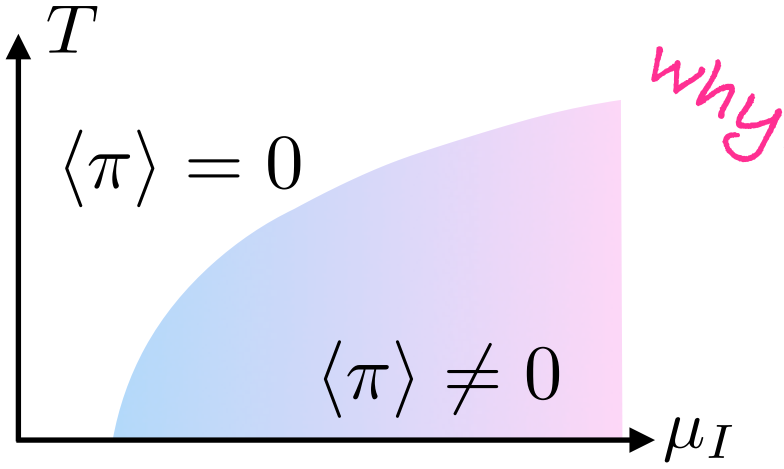


- **Fit critical exponent:**

$$\langle\pi\rangle \sim \left(\frac{\mu_I-\mu_I^c}{\mu_I^c}\right)^{\beta} \text{ or } \langle\pi\rangle \sim \left(\frac{T_c-T}{T_c}\right)^{\beta}$$

- **FRG:**

  β drops fast, then saturate at high T



why?!

Critical Exponent

- **FRG: 0.5 — 0.3**

T(MeV)	0	10	50	100	150	200	250
μ_I^c(MeV)	135.0	135.2	138.0	146.3	164.5	199.4	248.9
β	0.5	0.445	0.380	0.347	0.328	0.318	0.314

- **MEAN FIELD: 0.5**

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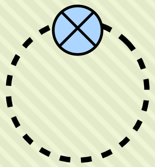
- **FRG:** $\partial_k \Gamma_k = \frac{1}{2}$ 

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- MEAN FIELD

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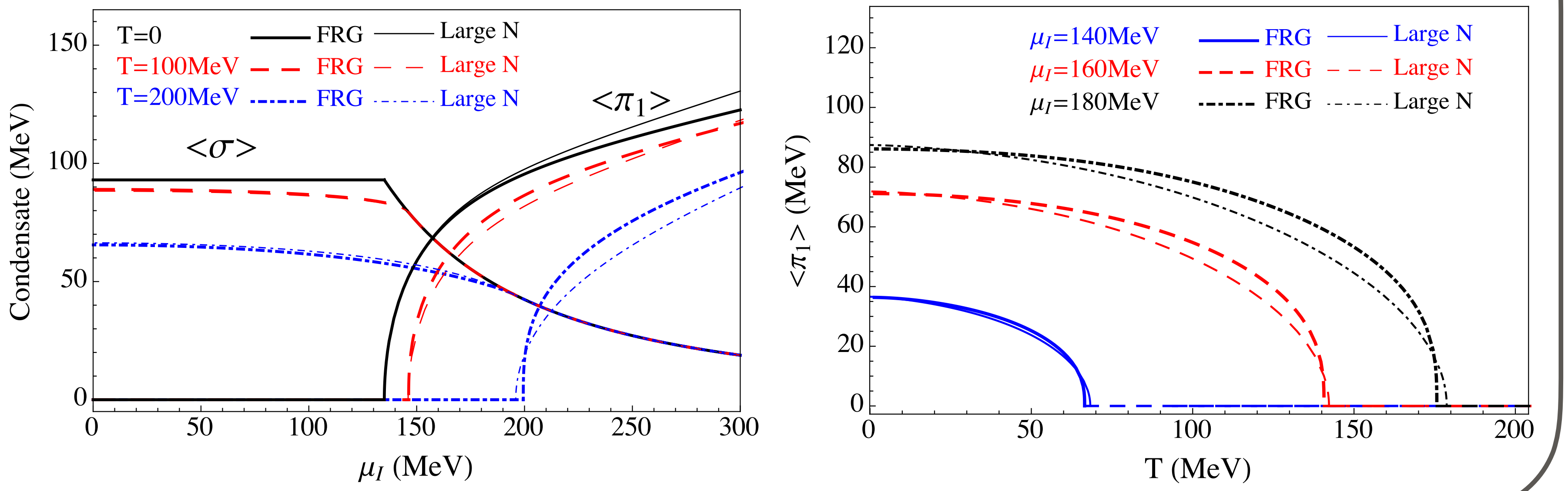

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- Dimension Reduction

$$\int_0^\infty d^d x \to \int_0^{T^{-1}} dt \int_0^\infty d^{d-1}\mathbf{x}$$

T=0 to high T limit: $S^1 \times R^{d-1} \to R^{d-1}$

$$d_{\text{eff}} = d \to d_{\text{eff}} = d - 1$$

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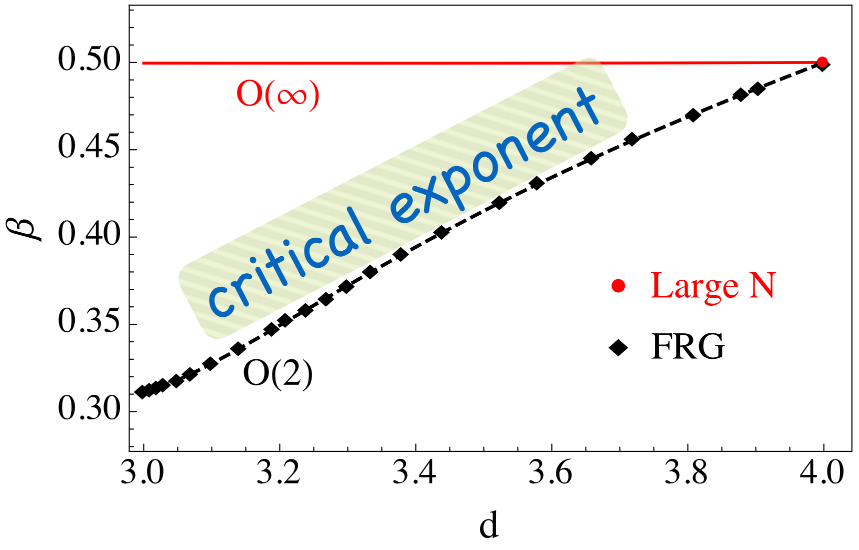
- Dimension Reduction

T=0 to high T limit: $d_{\mathrm{eff}} = d \to d_{\mathrm{eff}} = d - 1$

- PION SUPERFLUIDITY @ finite T
- General O(2) MODEL @ continuous d

Same symmetry

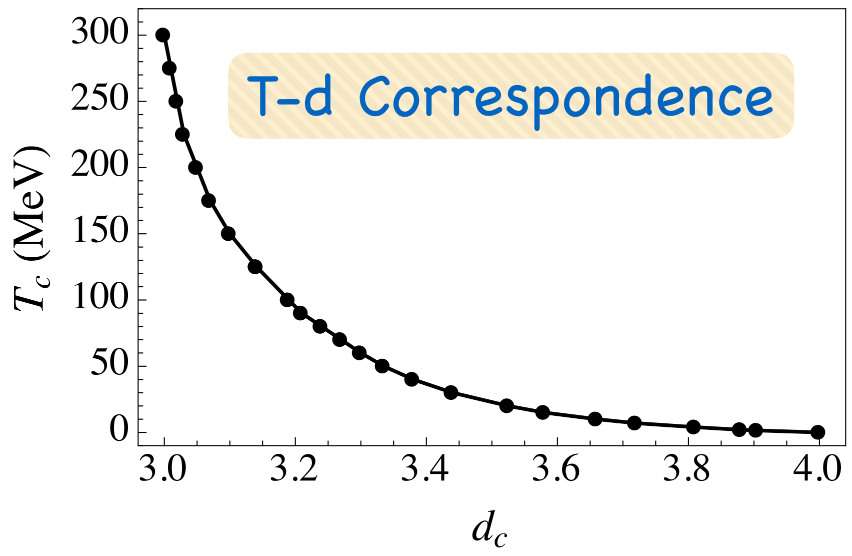
T-d Correspondence



O(2) universality class

with a

dimension crossover



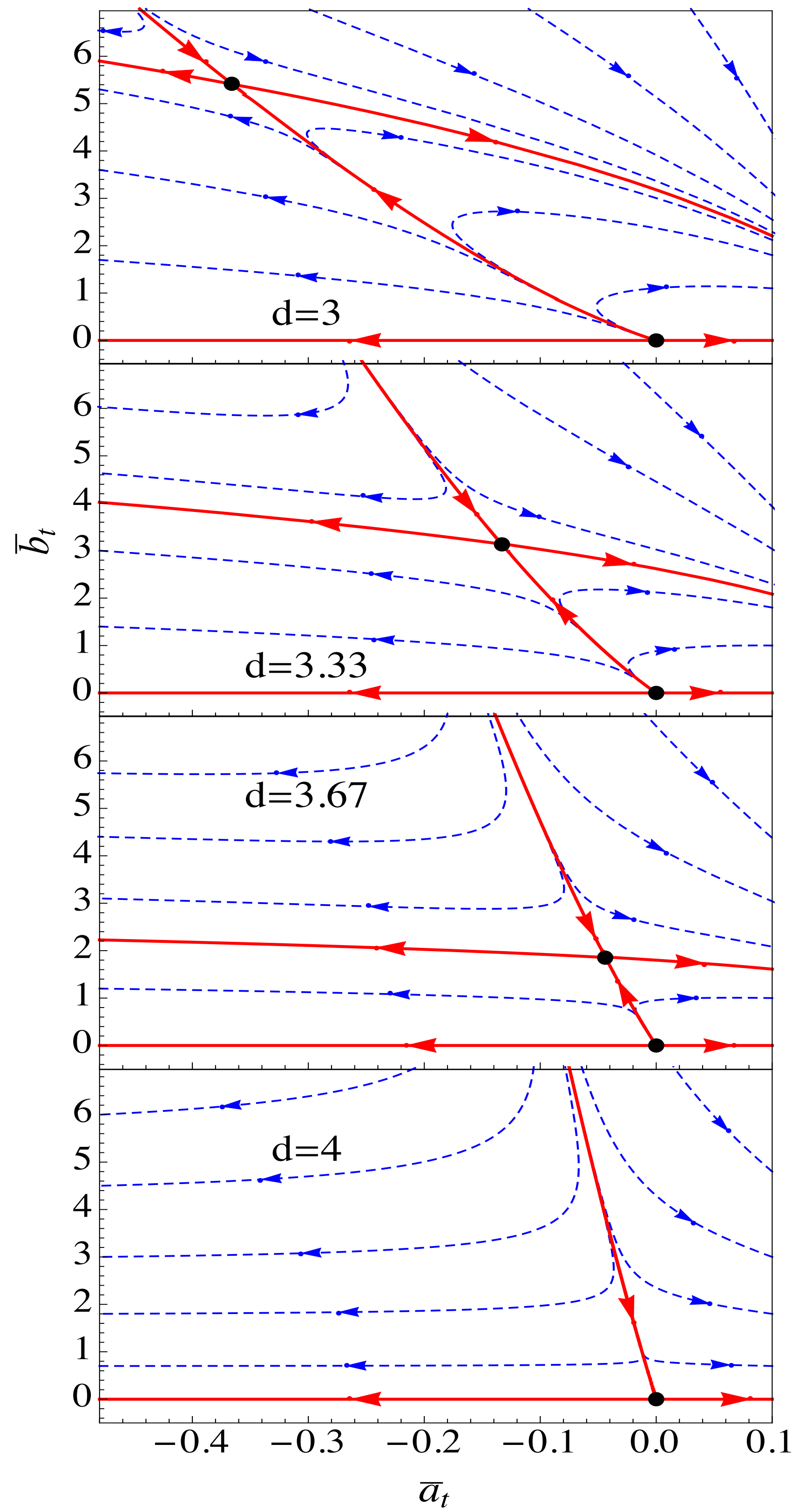
T-d relation

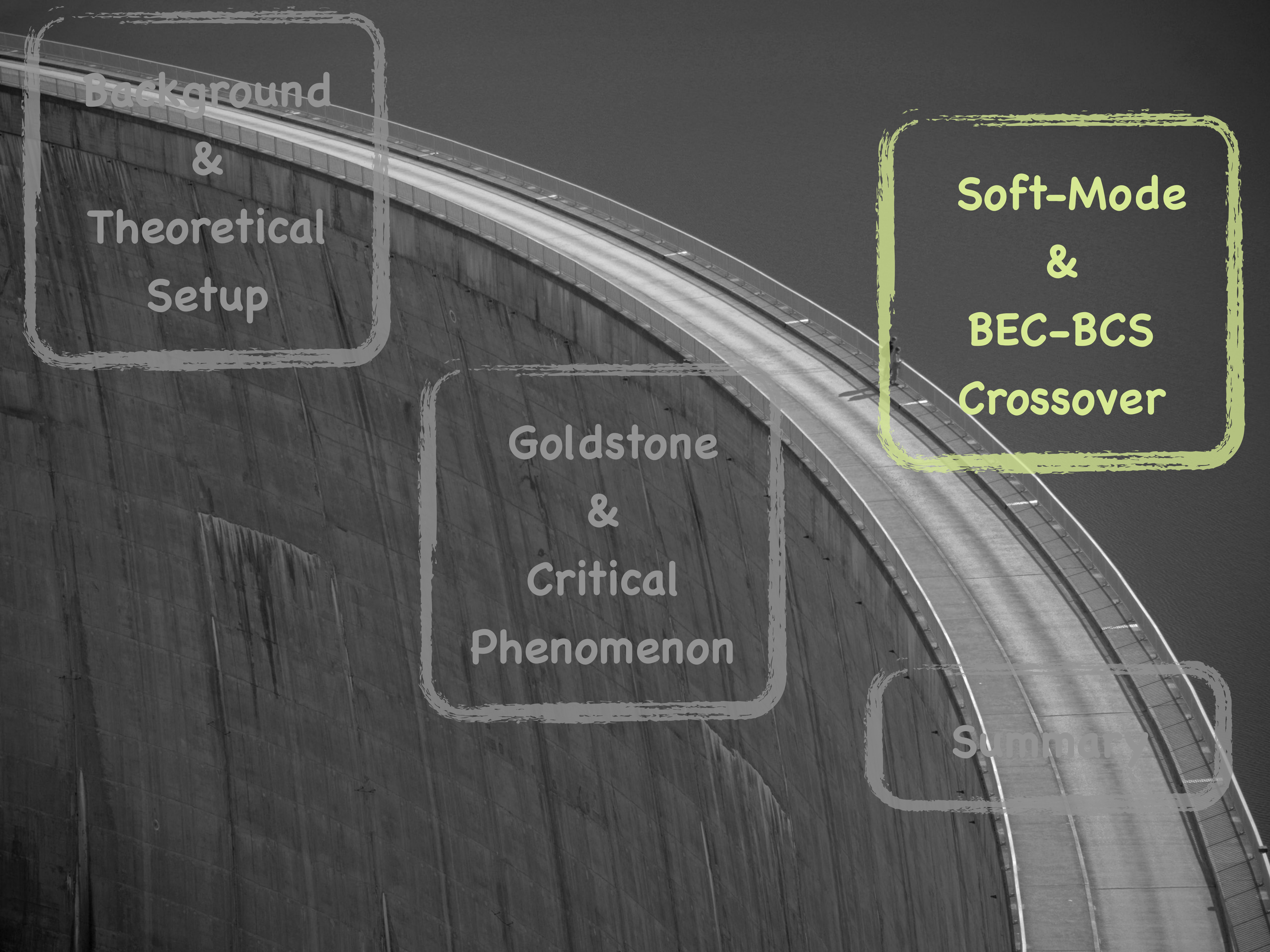
not universal

$T=0,\ d=4,\ \beta=0.5$

General O(2) MODEL

@ continuous d





Background & Theoretical Setup

Goldstone & Critical Phenomenon

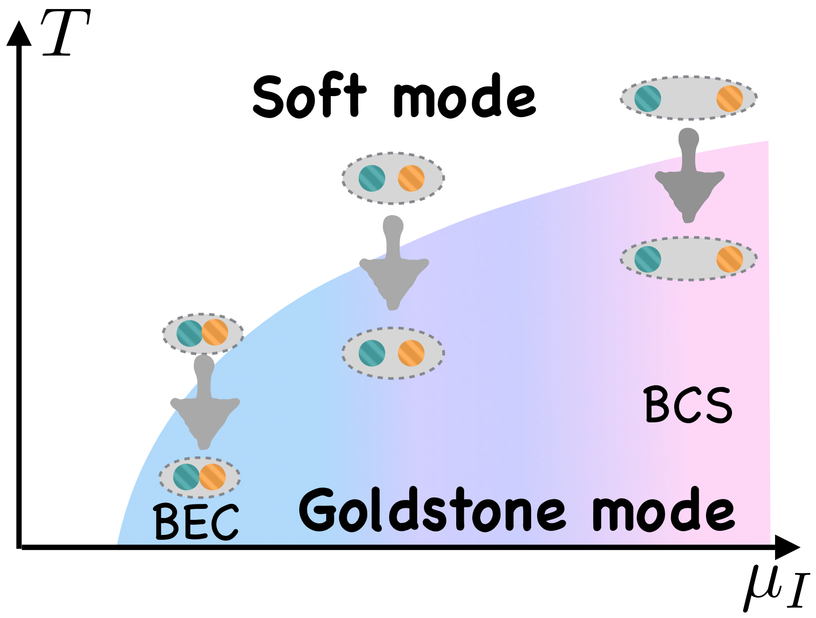
Soft-Mode & BEC-BCS Crossover

Summary

Soft-Mode BEC-BCS Crossover

- BEC-BCS Crossover: density & chiral symmetry restoration
- Soft mode @ T* > T > Tc

tightly bound
thermal excitation



strongly overlapped
loosely bound
resonance

Soft-Mode BEC-BCS Crossover

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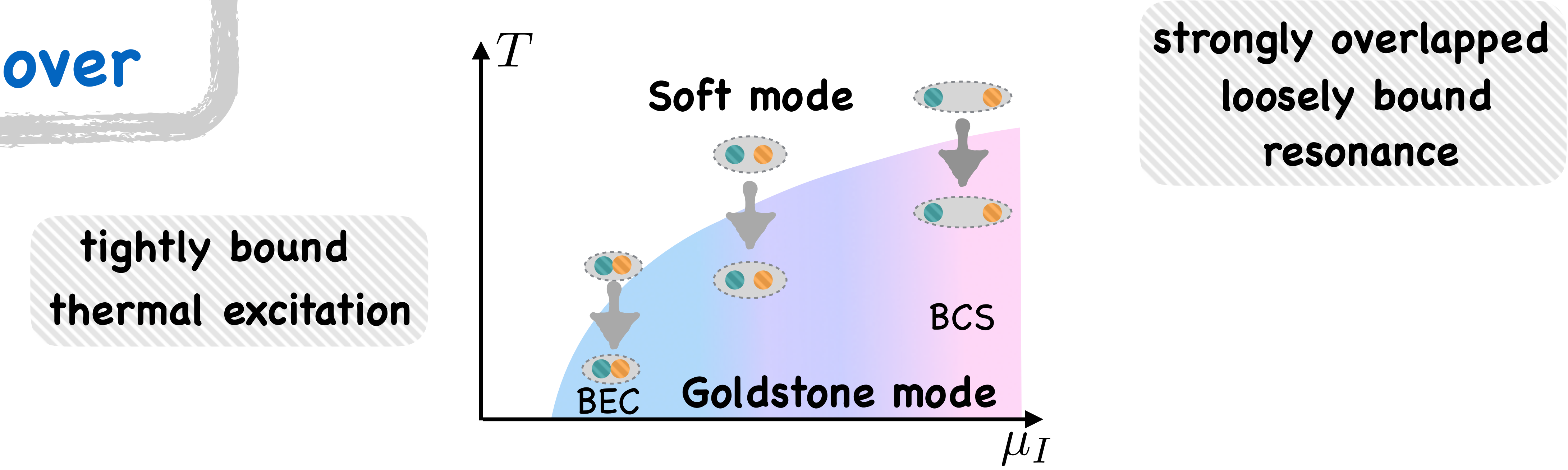

- Meson SPECTRAL function in QUARK MESON model @ T* > T > Tc

Free particle: pole ⟶ mass

Interaction: pole, branch-cut ⟶ mass, stability, decay channel

Soft-Mode BEC-BCS Crossover

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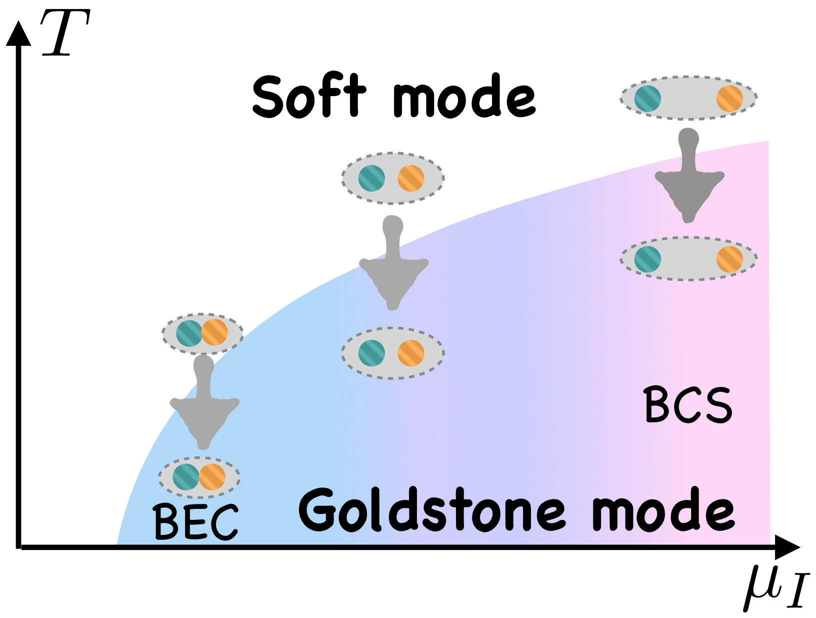

tightly bound
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- Meson SPECTRAL function in QUARK MESON model @ T* > T > Tc

Free particle: pole ⟶ mass

Interaction: pole, branch-cut ⟶ mass, stability, decay channel

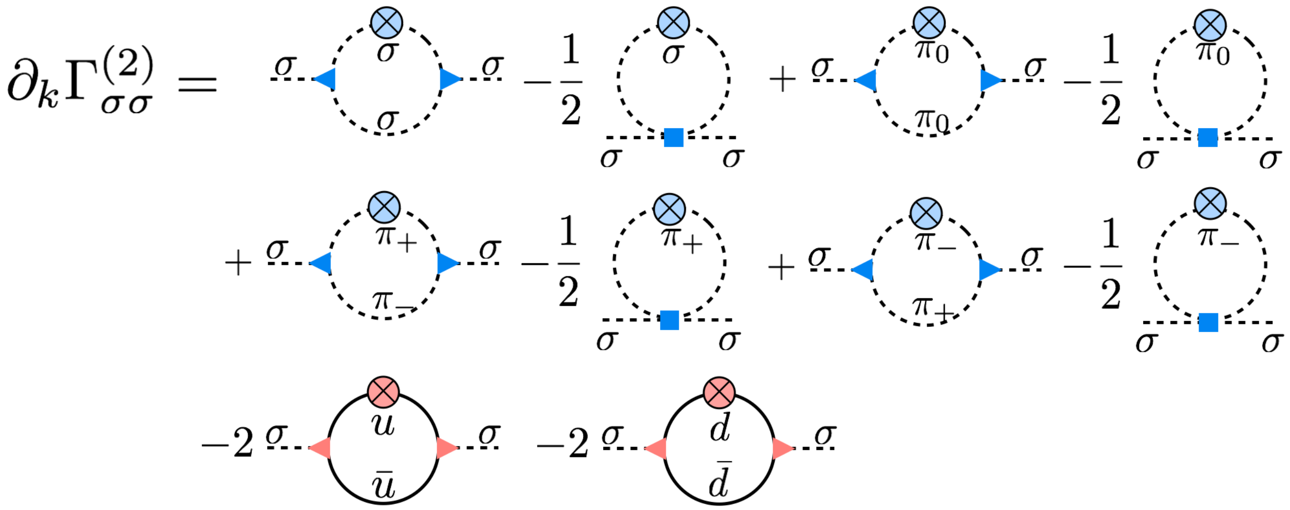
- Flow equation of meson 2-point function

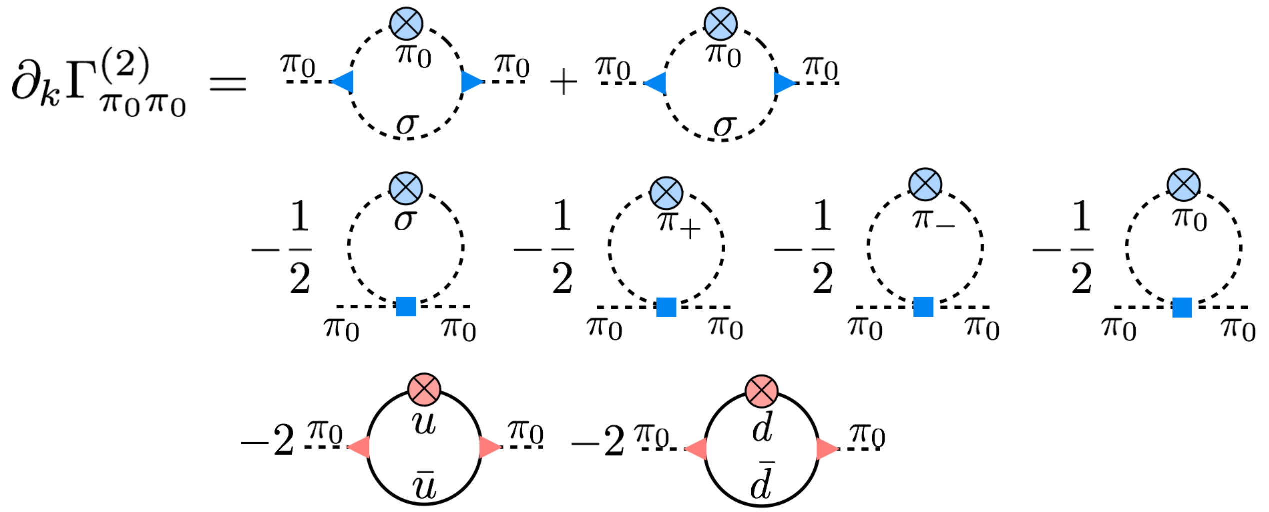
[K. Kamikado, et al., Eur. Phys. J. C 74, 2806 (2014)]
[R. Tripolt, et al, Phys. Rev. D 89, 034010 (2014)]
[R. Tripolt, et al, Phys. Rev. D 90, 074031 (2014)]

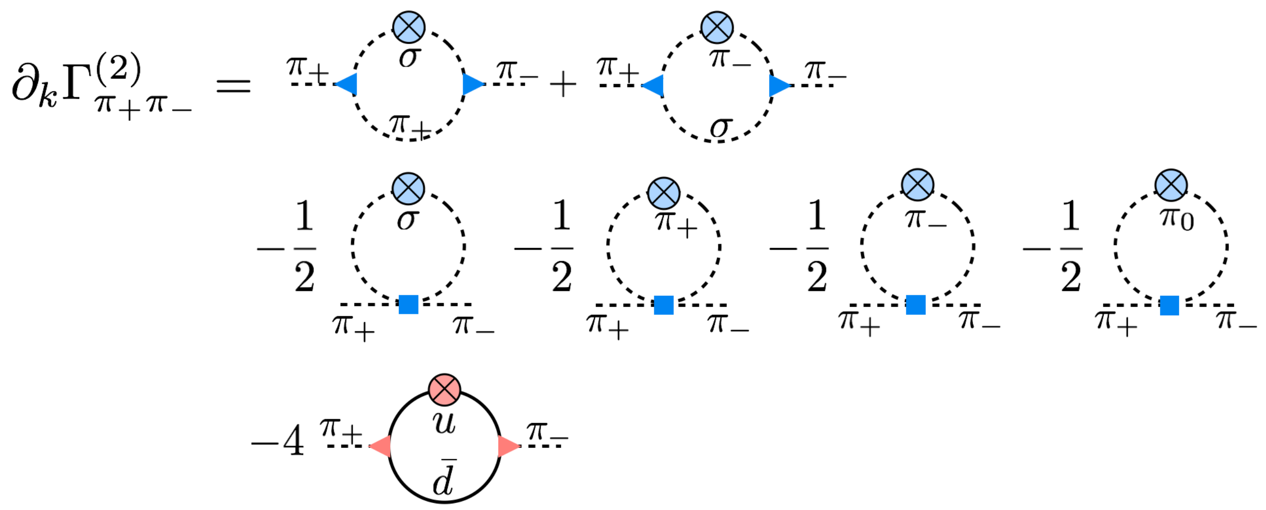
- Analytical continuation $\Gamma^{(2),R}(\omega,\vec{p}) = \lim_{\epsilon\to 0}\Gamma^{(2),E}\left(p_0 = -i(\omega + i\epsilon),\vec{p}\right)$
- Spectral function $\rho(\omega) = -\frac{1}{\pi}\frac{\mathrm{Im}\Gamma^{(2),R}(\omega)}{\left(\mathrm{Re}\Gamma^{(2),R}(\omega)\right)^2 + \left(\mathrm{Im}\Gamma^{(2),R}(\omega)\right)^2}$

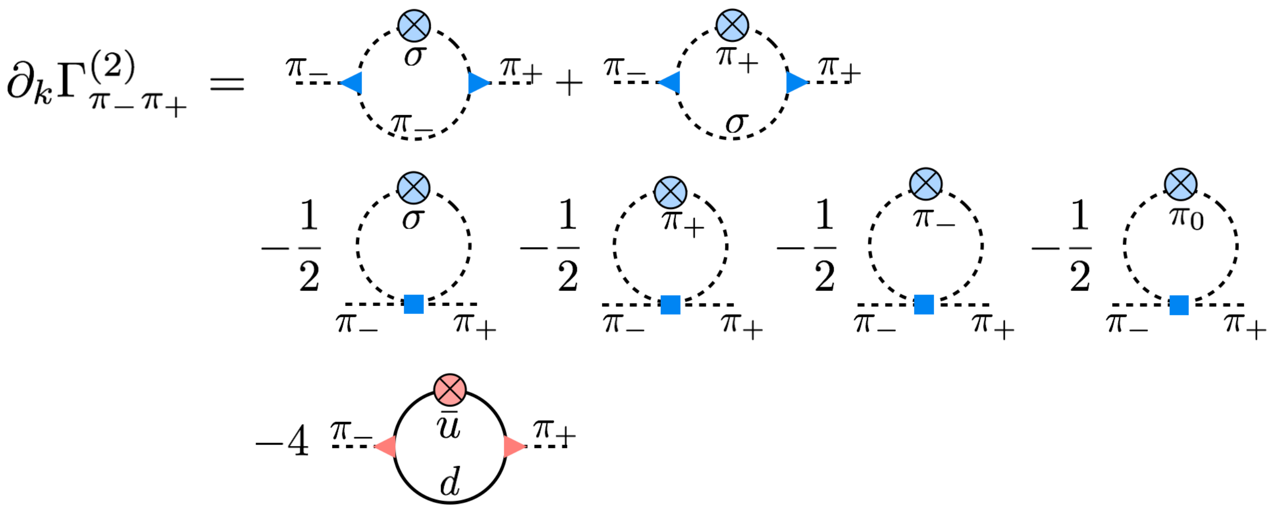
- Flow equation of meson 2-point function

  Truncation: LPA, neglect momentum dependence of $\Gamma^{(3)}, \Gamma^{(4)}$









- Flow equation of meson 2-point function

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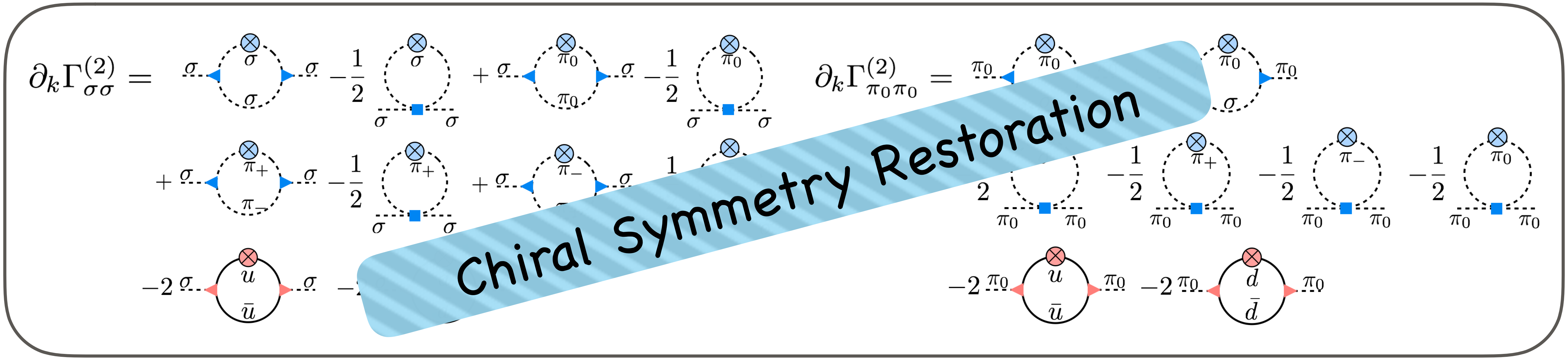
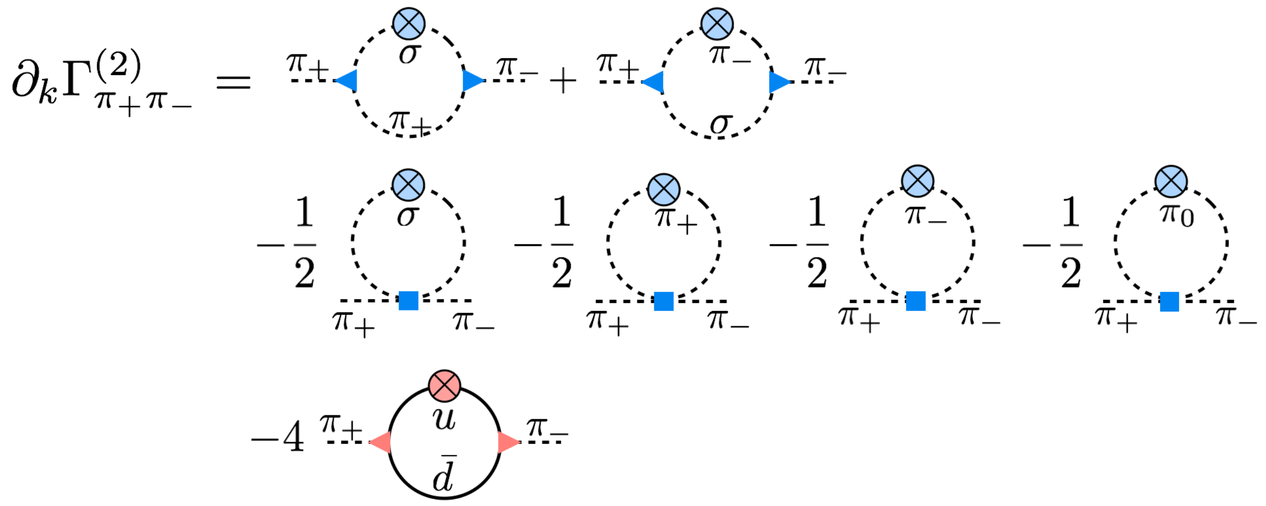
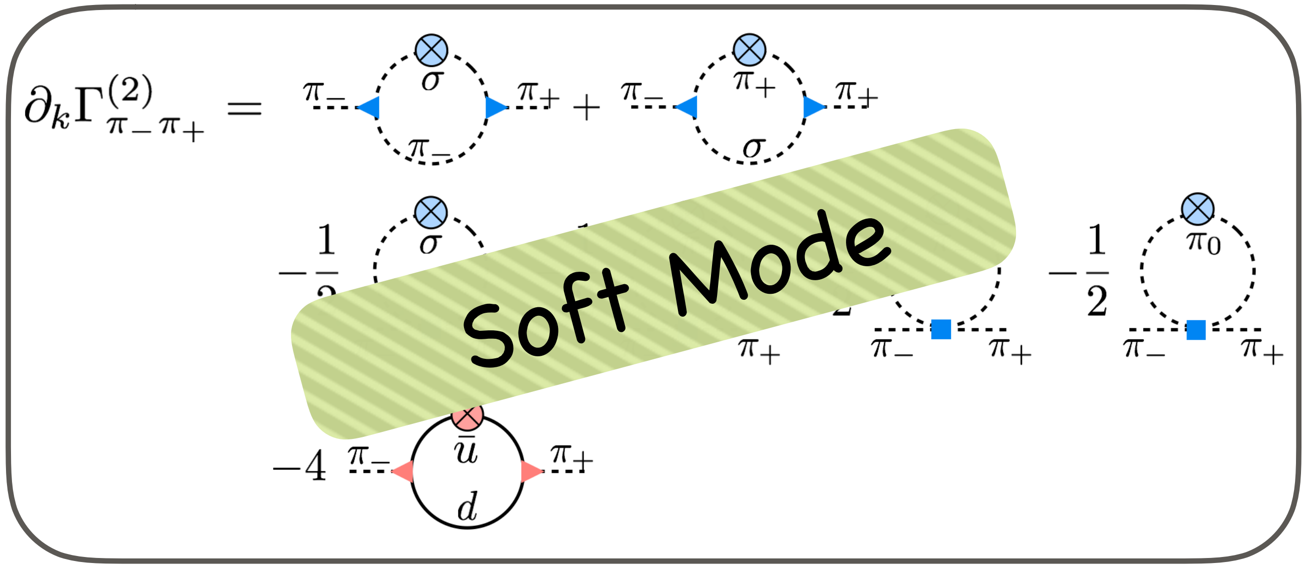






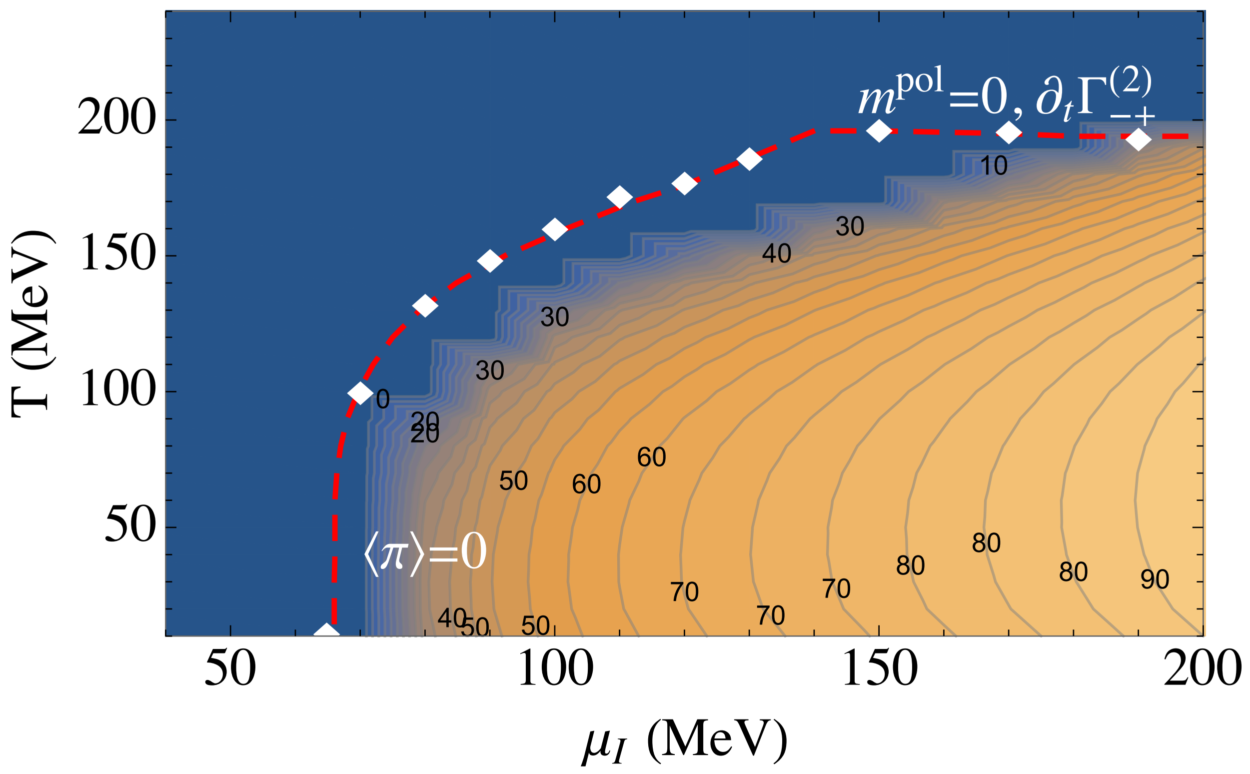
- Initial condition & Input

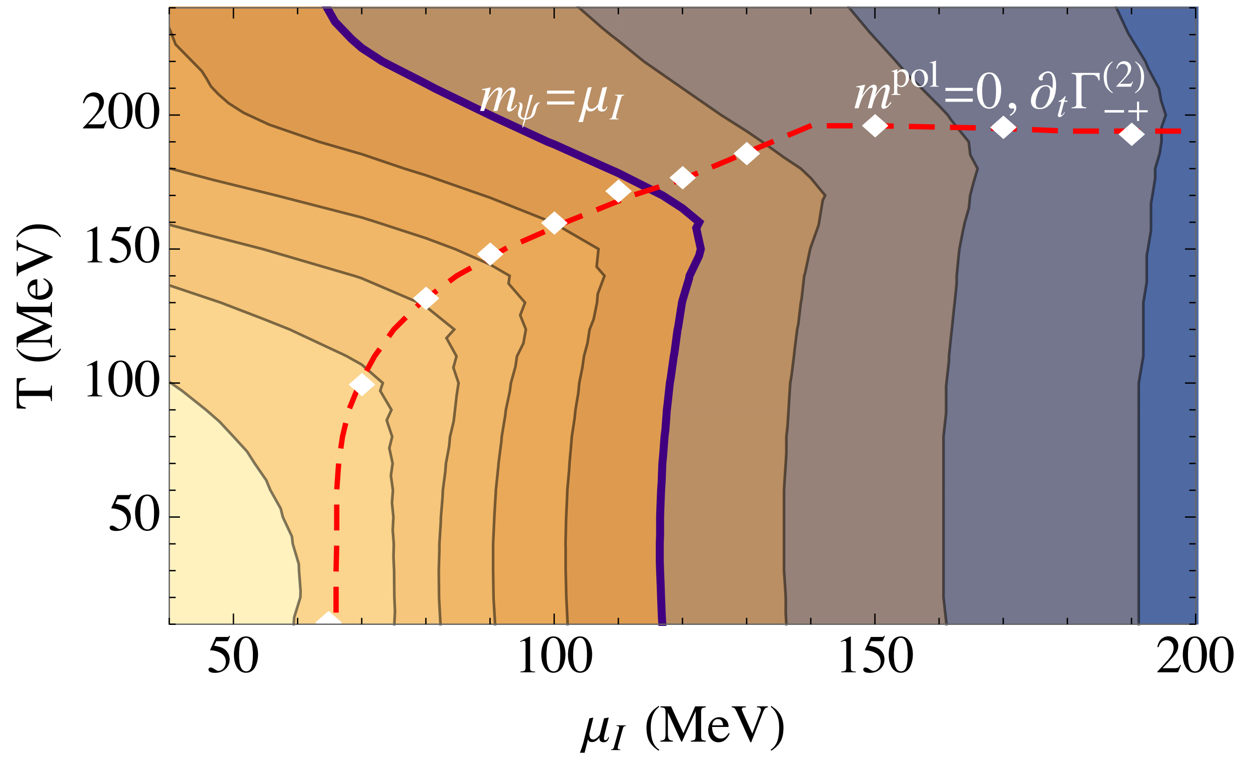
$$U_\Lambda(\rho) = \frac{1}{2}m_\Lambda^2\rho + \frac{1}{4}\lambda_\Lambda\rho^2$$

$$\Gamma^{(2),R}_{\sigma\sigma,\Lambda} = -\omega^2 + 2U'_\Lambda + 4\phi^2 U''_\Lambda, \qquad \Gamma^{(2),R}_{\pi_0\pi_0,\Lambda} = -\omega^2 + 2U'_\Lambda$$

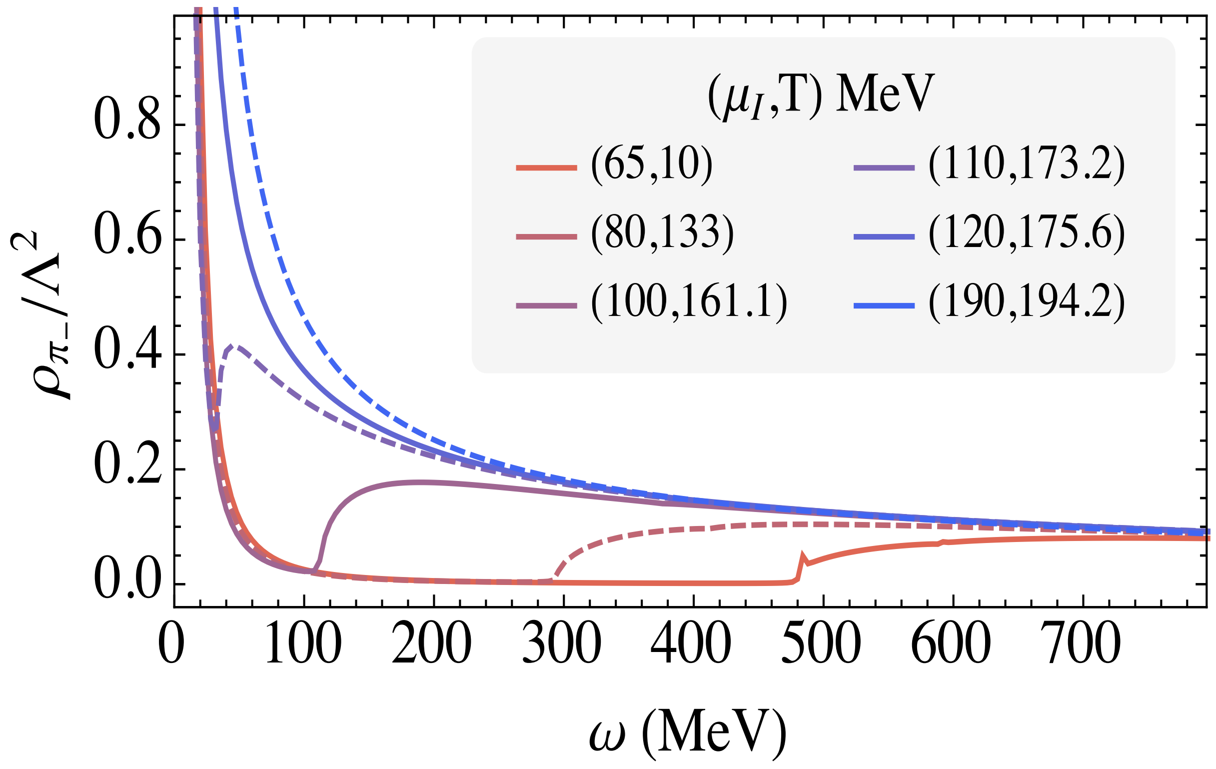
$$\Gamma^{(2),R}_{\pi_+\pi_-,\Lambda} = -(\omega - 2\mu_I)^2 + 2U'_\Lambda, \qquad \Gamma^{(2),R}_{\pi_-\pi_+,\Lambda} = -(\omega + 2\mu_I)^2 + 2U'_\Lambda$$

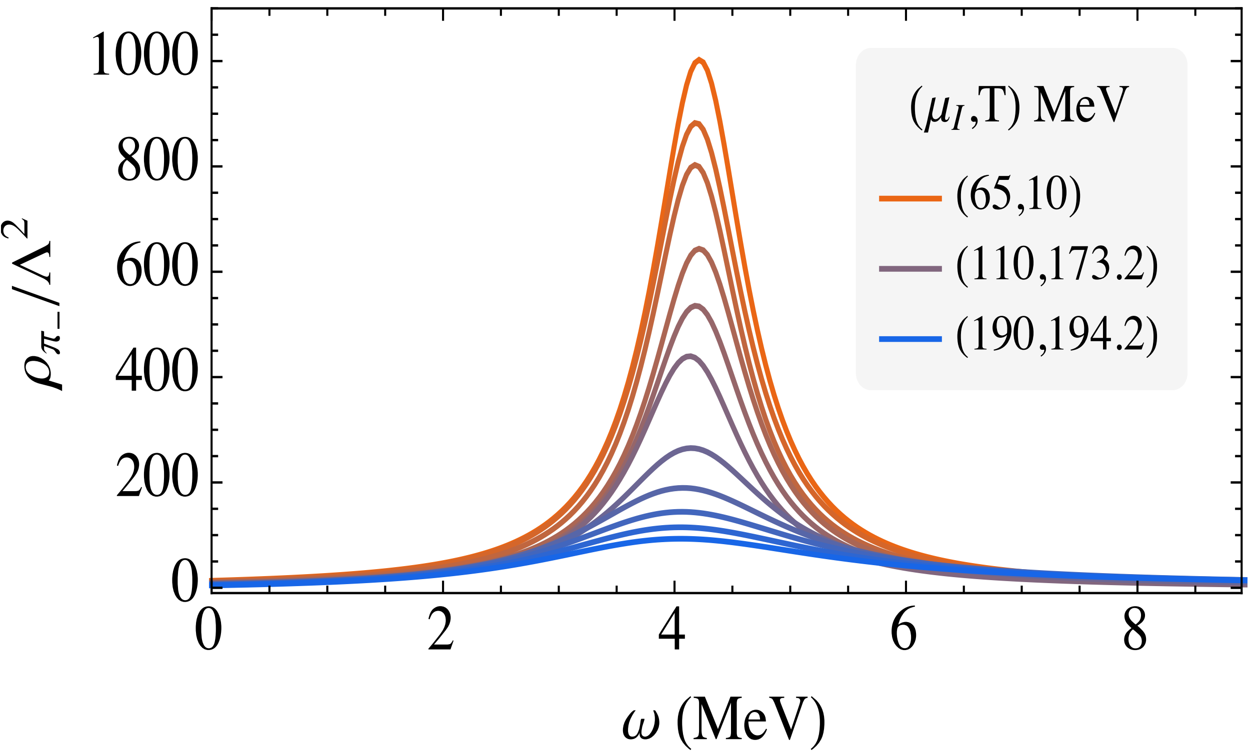
Phase diagram



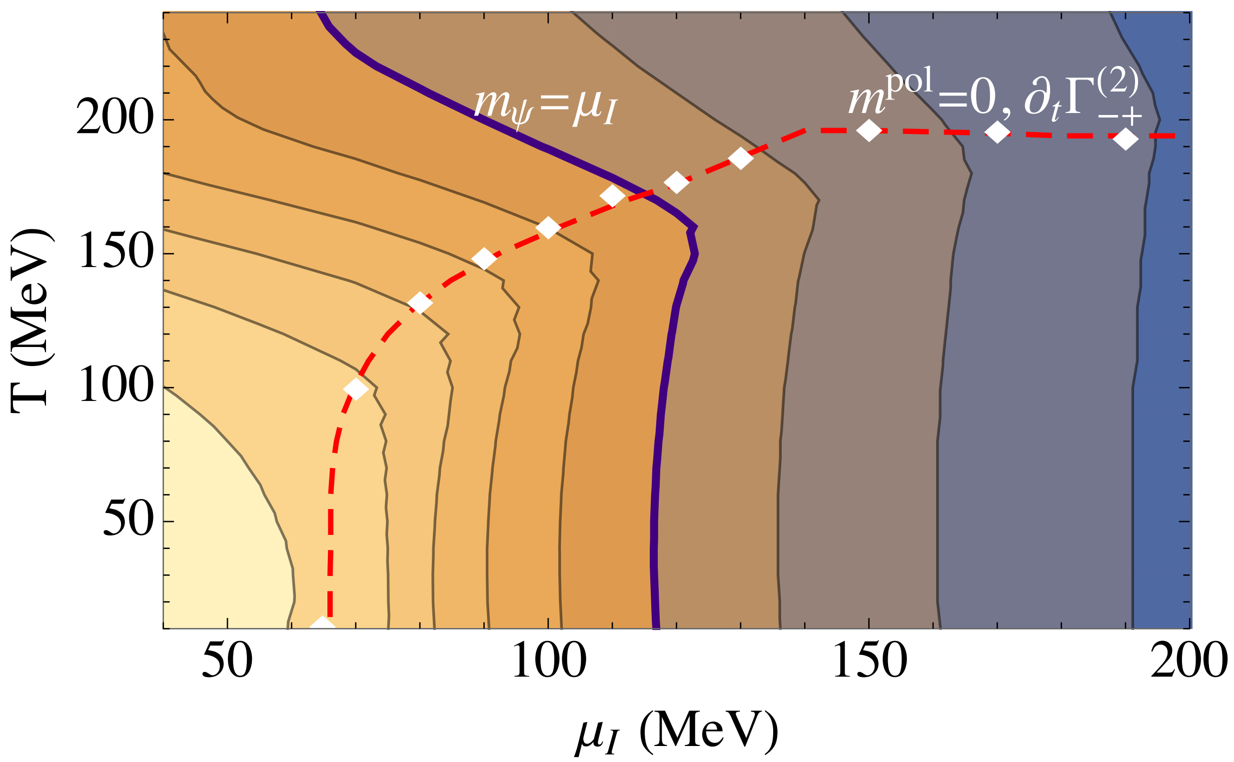


Spectral function of soft mode

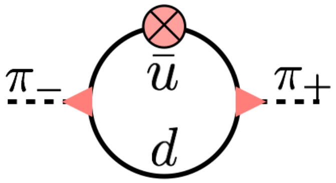




Phase diagram

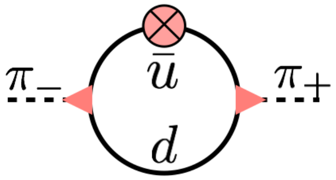


- Sharp peak in BEC limit



is suppressed

- Broad resonance in BCS limit

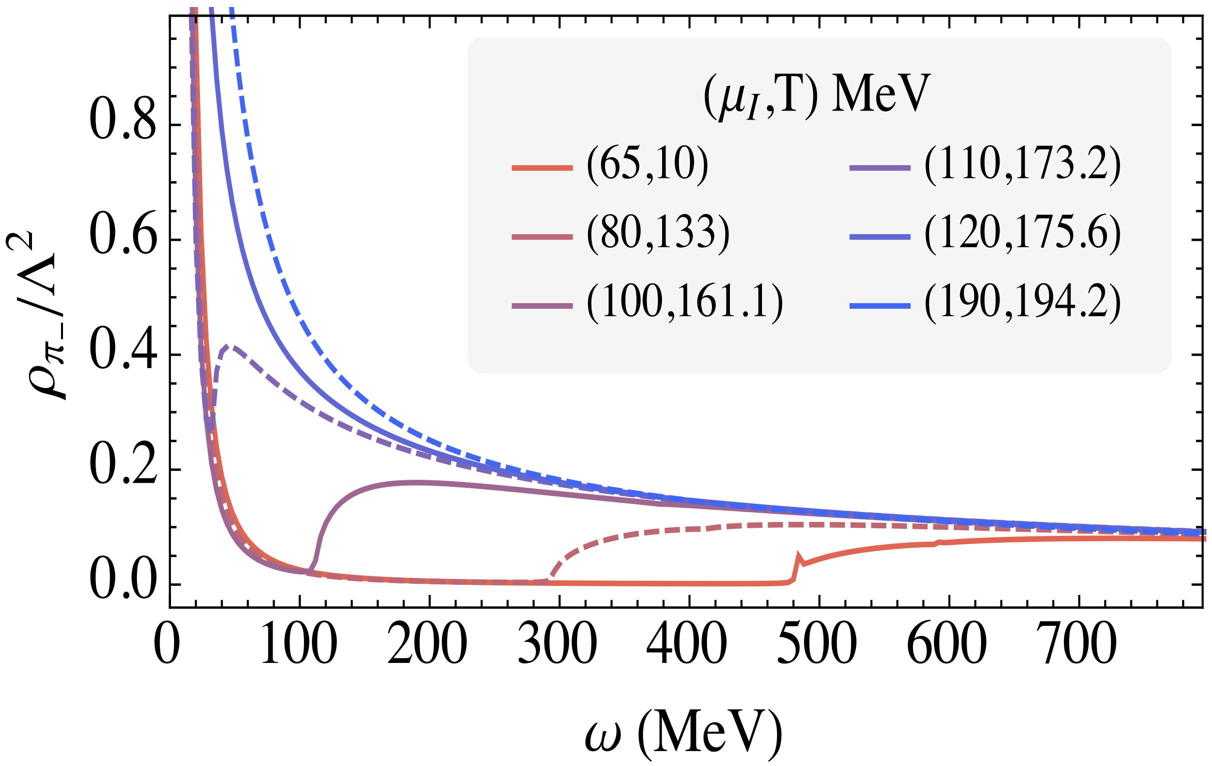


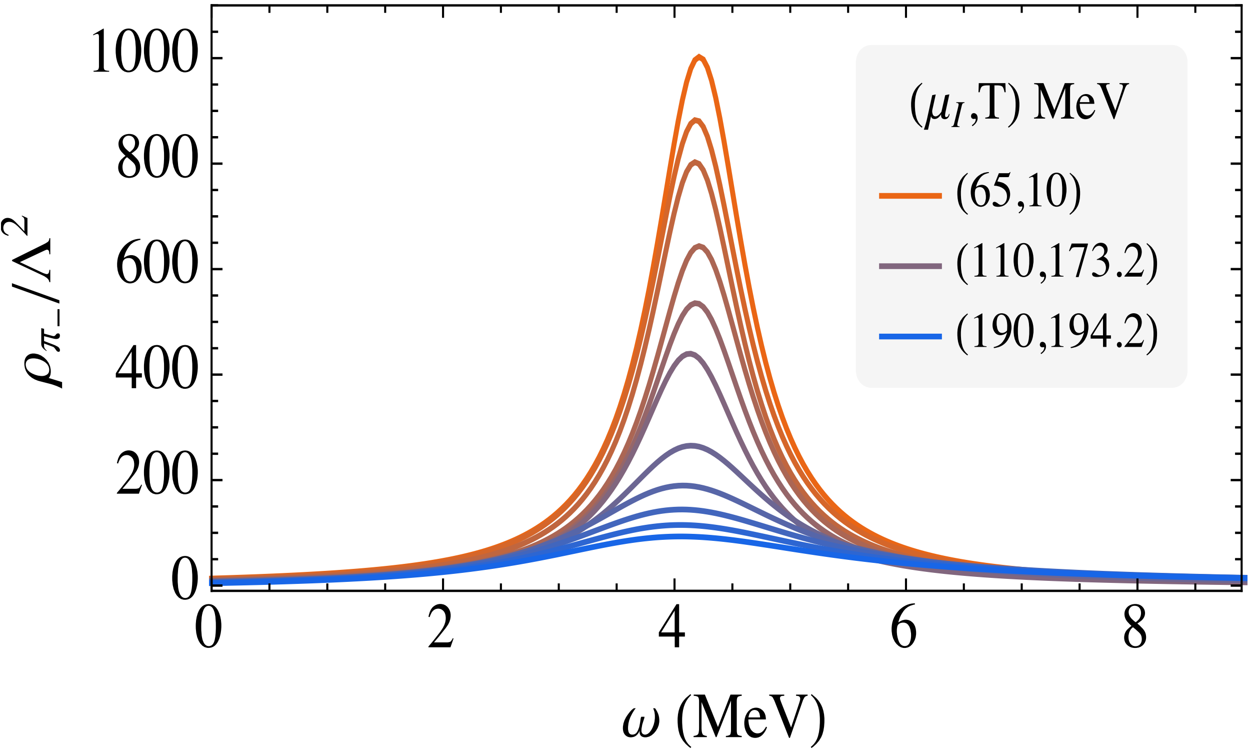
no longer suppressed

- 2-quark channel opened already at $\omega = 0$

$\omega > 2(m_q - \mu_I)$

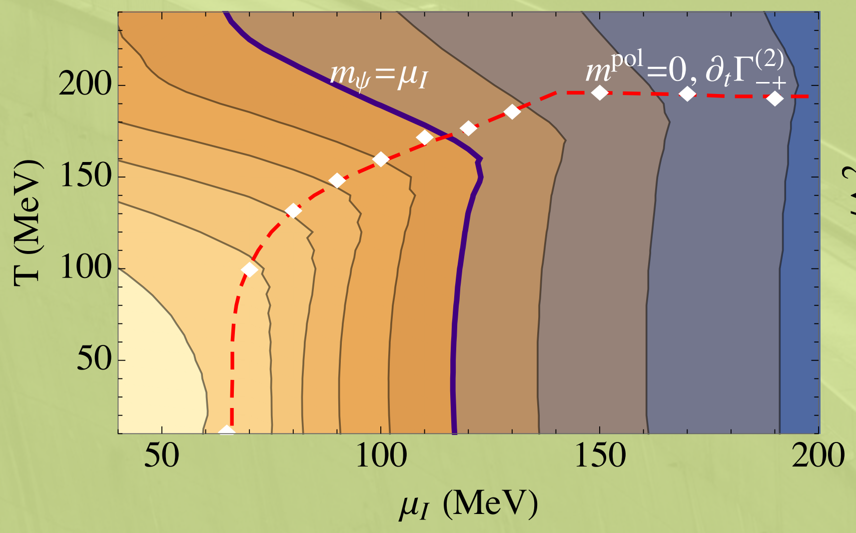
Spectral function of soft mode

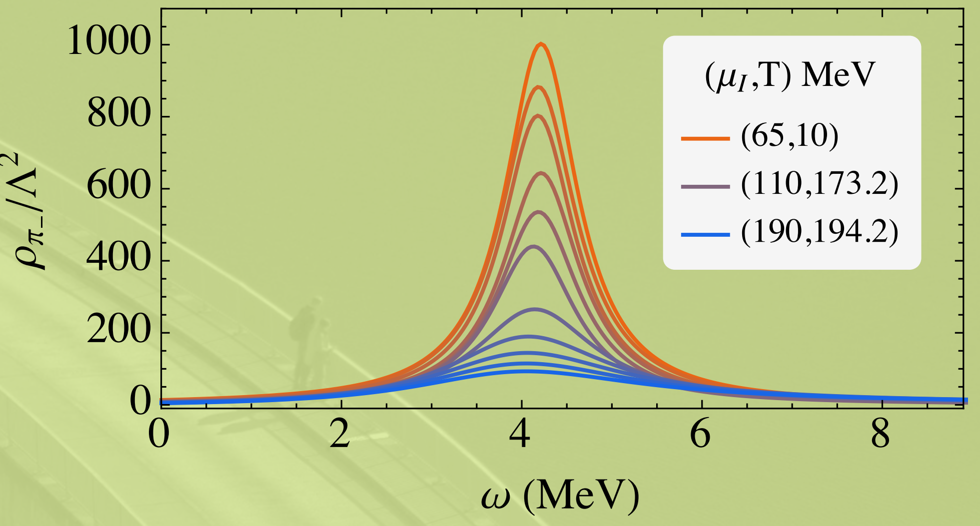




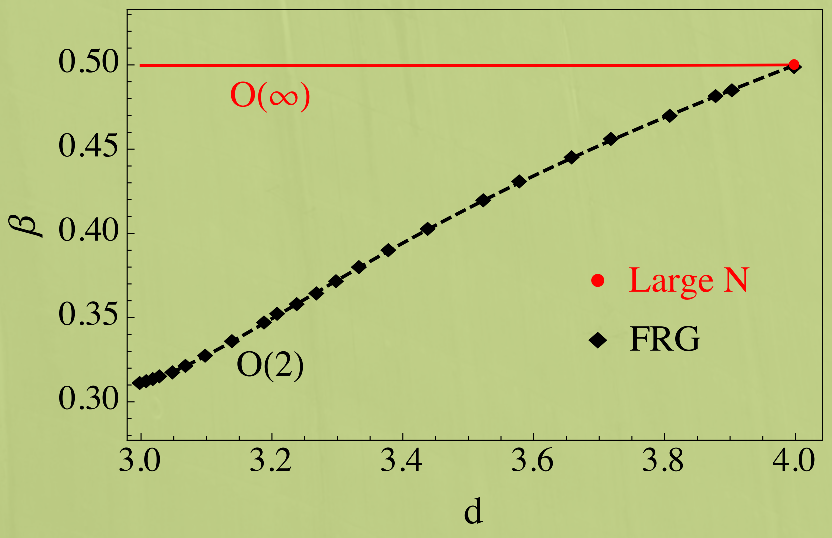
Summary!

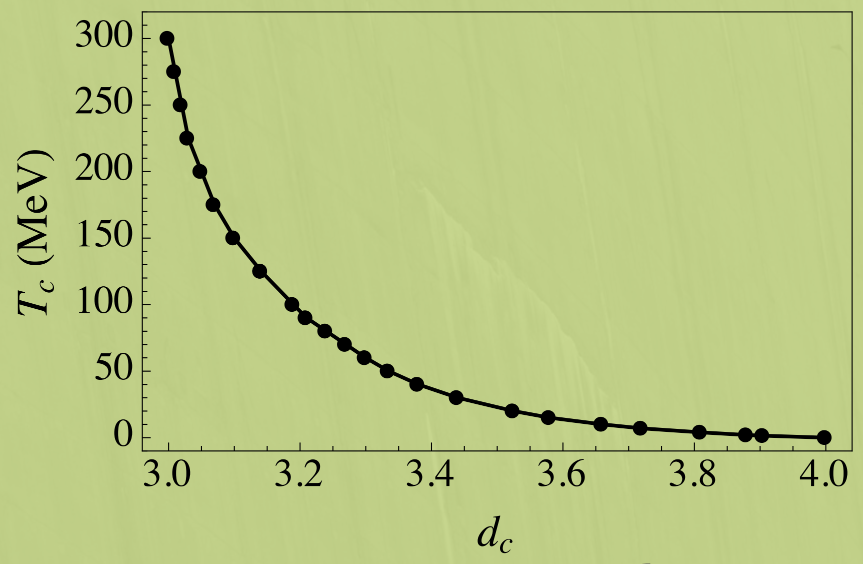
Soft Mode – BEC-BCS Crossover





Goldstone Mode – Critical Phenomenon





THANK YOU!

Backup I   Flow Equation of 2-point function

$$
\begin{aligned}
\partial_k \Gamma^{(2)}_{\sigma\sigma}(p_4) \;=\;& -\frac{1}{2}\Gamma^{(4)}_{\sigma\sigma\sigma\sigma} I^{(2)}_{\sigma} - \frac{1}{2}\Gamma^{(4)}_{\sigma\sigma\pi\pi} I^{(2)}_{\pi_0} - \frac{1}{2}\Gamma^{(4)}_{\sigma\sigma\pi_+\pi_-}\left(I^{(2)}_{\pi_+} + I^{(2)}_{\pi_-}\right) \\
& + (\Gamma^{(3)}_{\sigma\sigma\sigma})^2 J_{\sigma\sigma}(p_4) + (\Gamma^{(3)}_{\sigma\pi\pi})^2\Big(J_{\pi_0\pi_0}(p_4) + J_{++}(p_4) + J_{--}(p_4)\Big) \\
& -2N_c h^2\Big(F^{\sigma}_{++}(p_4,\mu_I) + F^{\sigma}_{--}(p_4,\mu_I)\Big) \\
\partial_k \Gamma^{(2)}_{\pi_0\pi_0}(p_4) \;=\;& -\frac{1}{2}\Gamma^{(4)}_{\sigma\sigma\pi\pi} I^{(2)}_{\sigma} - \frac{1}{2}\Gamma^{(4)}_{\pi_0\pi_0\pi_0\pi_0} I^{(2)}_{\pi_0} - \frac{1}{2}\Gamma^{(4)}_{\pi_0\pi_0\pi_+\pi_-}\left(I^{(2)}_{\pi_+} + I^{(2)}_{\pi_-}\right) \\
& + \frac{1}{2}(\Gamma^{(3)}_{\sigma\pi\pi})^2\Big(J_{\sigma\pi_0}(p_4) + J_{\pi_0\sigma}(p_4)\Big) \\
& -2N_c h^2\Big(F^{\pi}_{++}(p_4,\mu_I) + F^{\pi}_{--}(p_4,\mu_I)\Big) \\
\partial_k \Gamma^{(2)}_{\pi_+\pi_-}(p_4) \;=\;& -\frac{1}{2}\Gamma^{(4)}_{\sigma\sigma\pi_+\pi_-} I^{(2)}_{\sigma} - \frac{1}{2}\Gamma^{(4)}_{\pi_0\pi_0\pi_+\pi_-} I^{(2)}_{\pi_0} - \frac{1}{2}\Gamma^{(4)}_{\pi_+\pi_-\pi_+\pi_-}\left(I^{(2)}_{\pi_+} + I^{(2)}_{\pi_-}\right) \\
& + (\Gamma^{(3)}_{\sigma\pi\pi})^2\Big(J_{\sigma-}(p_4) + J_{+\sigma}(p_4)\Big) \\
& -4N_c h^2 F^{\pi}_{-+}(p_4,\mu_I) \\
\partial_k \Gamma^{(2)}_{\pi_-\pi_+}(p_4) \;=\;& -\frac{1}{2}\Gamma^{(4)}_{\sigma\sigma\pi_+\pi_-} I^{(2)}_{\sigma} - \frac{1}{2}\Gamma^{(4)}_{\pi_0\pi_0\pi_+\pi_-} I^{(2)}_{\pi_0} - \frac{1}{2}\Gamma^{(4)}_{\pi_+\pi_-\pi_+\pi_-}\left(I^{(2)}_{\pi_+} + I^{(2)}_{\pi_-}\right) \\
& + (\Gamma^{(3)}_{\sigma\pi\pi})^2\Big(J_{-\sigma}(p_4) + J_x\sigma+(p_4)\Big) \\
& -4N_c h^2 F^{\pi}_{+-}(p_4,\mu_I)
\end{aligned}
$$

Backup 2

Threshold Function-I

$$I_\alpha^{(2)}(\mu_I) \;=\; \frac{k^4}{3\pi^2}\frac{1}{4}\left\{\frac{1+n_B(E_\alpha-2\mu_I)+n_B(E_\alpha+2\mu_I)}{E_\alpha^3}-\frac{n'_B(E_\alpha-2\mu_I)+n'_B(E_\alpha+2\mu_I)}{E_\alpha^2}\right\}$$

$$\begin{aligned} J_{\alpha\alpha}(p_4,\mu_I) \;&=\; \frac{k^4}{3\pi^2}\frac{1}{4}\left\{\frac{12E_\alpha^2+p_4^2}{E_\alpha^3(4E_\alpha^2+p_4^2)^2}\left(1+n_B(E_\alpha+2\mu_I)+n_B(E_\alpha-2\mu_I)\right)\right. \\ &\left. -\frac{1}{E_\alpha^2(2E_\alpha-ip_4)ip_4}n'_B(E_\alpha-2\mu_I)+\frac{1}{E_\alpha^2(2E_\alpha+ip_4)ip_4}n'_B(E_\alpha+2\mu_I)\right\} \end{aligned}$$

$$\begin{aligned} J_{\alpha\beta}(p_4) \;&=\; \frac{k^4}{3\pi^2}\frac{1}{4}\left\{\frac{E_\beta^2-(E_\alpha-ip_4)(3E_\alpha-ip_4)}{E_\alpha^3((E_\alpha-ip_4)^2-E_\beta^2)^2}(1+n_B(E_\alpha))+\frac{E_\beta^2-(E_\alpha+ip_4)(3E_\alpha+ip_4)}{E_\alpha^3((E_\alpha+ip_4)^2)^2-E_\beta^2}n_B(E_\alpha)\right. \\ &+\frac{1}{E_\alpha^2((E_\alpha-ip_4)^2-E_\beta^2)}n'_B(E_\alpha)+\frac{1}{E_\alpha^2((E_\alpha+ip_4)^2-E_\beta^2)}n'_B(E_\alpha) \\ &\left.+\frac{2}{E_\beta((E_\beta-ip_4)^2-E_\alpha^2)^2}n_B(E_\beta)+\frac{2}{E_\beta((E_\beta+ip_4)^2-E_\alpha^2)^2}(1+n_B(E_\beta))\right\} \end{aligned}$$

Backup 2 Threshold Function-II

$$\begin{aligned} J_{+\sigma}(p_4,\mu_I) \;=\; & \frac{k^4}{3\pi^2}\frac{1}{4}\left\{\frac{E_\sigma^2-(E_\pi-ip_4-2\mu_I)(3E_\pi-ip_4-2\mu_I)}{E_\pi^3((E_\pi-ip_4-2\mu_I)^2-E_\sigma^2)^2}(1+n_B(E_\pi-2\mu_I))+\frac{n'_B(E_\pi-2\mu_I)}{E_\pi^2((E_\pi-ip_4-2\mu_I)^2}\right. \\ & +\frac{E_\sigma^2-(E_\pi+ip_4+2\mu_I)(3E_\pi+ip_4+2\mu_I)}{E_\pi^3((E_\pi+ip_4+2\mu_I)^2-E_\sigma^2)^2}n_B(E_\pi+2\mu_I)+\frac{n'_B(E_\pi+2\mu_I)}{E_\pi^2((E_\pi+ip_4+2\mu_I)^2-E_\sigma^2} \\ & \left.+\frac{2n_B(E_\sigma)}{E_\sigma((E_\sigma-ip_4-2\mu_I)^2-E_\pi^2)^2}+\frac{2(1+n_B(E_\sigma))}{E_\sigma((E_\sigma+ip_4+2\mu_I)^2-E_\pi^2)^2}\right\} \end{aligned}$$

$$\begin{aligned} J_{+\sigma}(p_4,\mu_I) \;=\; & \frac{k^4}{3\pi^2}\frac{1}{4}\left\{\frac{E_\sigma^2-(E_\pi-ip_4-2\mu_I)(3E_\pi-ip_4-2\mu_I)}{E_\pi^3((E_\pi-ip_4-2\mu_I)^2-E_\sigma^2)^2}(1+n_B(E_\pi-2\mu_I))+\frac{n'_B(E_\pi-2\mu_I)}{E_\pi^2((E_\pi-ip_4-2\mu_I)^2}\right. \\ & +\frac{E_\sigma^2-(E_\pi+ip_4+2\mu_I)(3E_\pi+ip_4+2\mu_I)}{E_\pi^3((E_\pi+ip_4+2\mu_I)^2-E_\sigma^2)^2}n_B(E_\pi+2\mu_I)+\frac{n'_B(E_\pi+2\mu_I)}{E_\pi^2((E_\pi+ip_4+2\mu_I)^2-E_\sigma^2} \\ & \left.+\frac{2n_B(E_\sigma)}{E_\sigma((E_\sigma-ip_4-2\mu_I)^2-E_\pi^2)^2}+\frac{2(1+n_B(E_\sigma))}{E_\sigma((E_\sigma+ip_4+2\mu_I)^2-E_\pi^2)^2}\right\} \end{aligned}$$

Backup 2 Threshold Function-III

$$
\begin{aligned}
F^0_{++}(p_4,\mu_I) &= \sum\!\!\!\!\!\!\!\int_q \big[-m^2-\vec{q}_r^{\,2}-(q_4+i\mu_I)(q_4+2p_4+i\mu_I)\big]D^2_+(q)D_+(q+p)\;4k\Theta(k^2-\vec{q}^{\,2})\\
&= \frac{4k^4}{6\pi^2}\Bigg\{\frac{4E_q^2-p_4^2}{E_q(4E_q^2+p_4^2)^2}\big(n_F(E_q+\mu_I)+n_F(E_q-\mu_I)-1\big)\\
&\qquad -\frac{1}{2E_q(2E_q-ip_4)}n_F'(E_q-\mu_I)-\frac{1}{2E_q(2E_q+ip_4)}n_F'(E_q+\mu_I)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
F^0_{++}(p_4,\mu_I) &= \sum\!\!\!\!\!\!\!\int_q D^2_+(q)D_+(q+p)\;4k\Theta(k^2-\vec{q}^{\,2})\\
&= \frac{4k^4}{6\pi^2}\Bigg\{\frac{12E_q^2+p_4^2}{4E_q^3(4E_q^2+p_4^2)^2}\big(1-n_F(E_q+\mu_I)-n_F(E_q-\mu_I)\big)\\
&\qquad +\frac{1}{4E_q^2(2E_q-ip_4)ip_4}n_F'(E_q-\mu_I)-\frac{1}{4E_q^2(2E_q+ip_4)ip_4}n_F'(E_q+\mu_I)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
F^\pi_{+-}(p_4,\mu_I) &= \sum\!\!\!\!\!\!\!\int_q \big[-m^2-\vec{q}_r^{\,2}-(q_4+i\mu_I)(q_4+2p_4-3i\mu_I)\big]D^2_+(q)D_-(q+p)\;4k\Theta(k^2-\vec{q}^{\,2})\\
&= \frac{4k^4}{6\pi^2}\Bigg\{\frac{1}{2E_q(2E_q+ip_4+2\mu_I)^2}\big(2n_F(E_q+\mu_I)-1\big)+\frac{1}{2E_q(2E_q-ip_4-2\mu_I)^2}\big(2n_F(E_q-\mu_I)-1\\
&\qquad -\frac{1}{2E_q(2E_q+ip_4+2\mu_I)}n_F'(E_q+\mu_I)-\frac{1}{2E_q(2E_q-ip_4-2\mu_I)}n_F'(E_q-\mu_I)\Bigg\}
\end{aligned}
$$