Scaling, superscaling, emergent SUSY and the Lee-Yang model from the FRG

O. Zanusso
TPI Jena

Sept. 20$^{\mathrm{th}}$ 2016
ERG2016

An updated title:

Multicriticality and emergent SUSY

O. Zanusso
TPI Jena

Sept. 20$^{\rm th}$ 2016
ERG2016

Plan

- Introduction
- Unitary multicritical models and the ϵ-expansion
- Non-unitary multicritical models and the $\epsilon^{1/2}$-expansion
- Emergent SUSY

A brief introduction to the LPA

The derivative expansion

$$k\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left(\Gamma_k^{(2)}[\varphi] + \mathcal{R}_k\right)^{-1} k\partial_k \mathcal{R}_k$$

$$\Gamma_k[\varphi] = \int \mathrm{d}^d x \left\{ \frac{Z_k(\varphi)}{2}\partial_\mu\varphi\partial^\mu\varphi + V_k(\varphi)\right\}$$

$$k\partial_k V_k(\varphi) = \frac{1}{\mathrm{Vol}}\, k\partial_k\, \Gamma_k[\varphi]\big|_{\varphi=\mathrm{const.}}\ ,$$

$$k\partial_k Z_k(\varphi) = k\partial_k \frac{\partial}{\partial p^2}\ \frac{\delta^2\Gamma_k[\varphi]}{\delta\varphi_p\delta\varphi_{-p}}\bigg|_{\varphi=\mathrm{const.},\, p^2=0}$$

The local potential approximation(s)

$$\overline{\varphi} \equiv Z_{k,0}^{-1/2} k^{(2-d)/2} \varphi$$

$$v_k(\overline{\varphi}) \equiv k^{-d} V_k(\varphi)\,, \qquad z_k(\overline{\varphi}) \equiv Z_{k,0}^{-1} Z_k(\varphi)$$

$$\begin{aligned} \mathrm{LPA} \quad &\leftrightarrow \quad Z_k(\varphi) = 1 \\ \mathrm{LPA}' \quad &\leftrightarrow \quad Z_k(\varphi) \equiv Z_{k,0} = \text{const. in } \varphi \end{aligned}$$

$$0 = k\partial_k Z_k(\varphi) \quad \leftrightarrow \quad \eta = -Z_{k,0}^{-1} k\partial_k Z_{k,0}$$

Explicit cutoff dependence

$$k\partial_k V_k(\varphi)|_{Z_k=\mathrm{const.}} = \frac{1}{2(2\pi)^d}\int_q \mathcal{G}_k\, k\partial_k \mathcal{R}_k$$

$$k\partial_k Z_k(\varphi)|_{Z_k=\mathrm{const.}} = \frac{V^{(3)}(\varphi)^2}{(2\pi)^d}\int_q \left(\frac{\partial \mathcal{G}_k}{\partial q^2} + \frac{2q^2}{d}\frac{\partial^2 \mathcal{G}_k}{\partial q^2 \partial q^2}\right)\mathcal{G}_k^2\, k\partial_k \mathcal{R}_k$$

$$\begin{aligned} k\partial_k v(\varphi)|_{z=1} &= -dv(\varphi) + \frac{1}{2}(d-2+\eta)\varphi v'(\varphi) \\ &\quad + f_0 - f_1 v''(\varphi) + f_2 v''(\varphi)^2 - f_3 v''(\varphi)^3 + \ldots \\ k\partial_k z(\varphi)|_{z=1} &= -\eta + g_0 v'''(\varphi)^2 - g_1 v''(\varphi) v'''(\varphi)^2 + \ldots \end{aligned}$$

Unitary multicritical models and the ϵ-expansion

φ^{2n} critical model

$$v(\varphi) \sim \lambda_n \varphi^{2n}$$

$$d_n = \frac{2n}{n-1} = \left\{ \infty, 4, 3, \frac{8}{3}, \frac{5}{2}, \ldots, 2 \right\}$$

$$d = d_n - \epsilon$$

$$\epsilon \to 0 \quad \Longrightarrow \quad \lambda_n \to 0$$

An appropriate rescaling

$$\varphi = \frac{2f_1^{1/2}}{(d-2+\eta)^{1/2}}x\,, \quad u(x) \equiv v(\varphi)$$

$$\begin{aligned} k\partial_k v(\varphi)|_{z=1} &\propto \frac{2d}{d-2}u(x) - xu(x) + \frac{u''(x)}{2} \\ &\quad -f_2 u''(x)^2 + f_3 u''(x)^3 + \dots \end{aligned}$$

$$k\partial_k z(\varphi)|_{z=1} \propto -\eta + g_0 u'''(x)^2 - g_1 u''(x) u'''(x)^2 + \dots$$

Initial ansatz for the solution

$$\frac{2d}{d-2}u(x) - xu(x) + \frac{u''(x)}{2} - f_2 u''(x)^2 + f_3 u''(x)^3 + \cdots = 0$$

Define:

$$\mathcal{D}_{2n} \equiv \frac{1}{2}\partial_x^2 - x\partial_x + 2n$$

$$\frac{2d_n}{d_n - 2}u(x) - xu'(x) + \frac{u''(x)}{2} = \mathcal{D}_{2n}u(\varphi) = 0$$

$$\implies \quad u(x) \propto H_{2n}(x)$$

Ansatz:

$$u(x) = \epsilon c_n H_{2n}(x) + \dots$$

Advantages of using $\mathcal{D}_{2n}$ and Hermite polynomials

$\mathcal{D}_{2n}$ generates terms orthogonal to $H_{2n}(x)$ according to the norm:

$$\int \mathrm{d}x\, \mathrm{e}^{-x^2} H_n(x) H_m(x) = 2^n \pi^{1/2} \Gamma(n+1) \delta_{n,m}$$

$$\int \mathrm{d}x\, \mathrm{e}^{-x^2} H_n(x) H_m(x) H_l(x) \neq 0 \quad \text{for} \quad m, n, l \leq \frac{1}{2}(m+n+l) \in 2\mathbb{N}$$

Any desired order in ϵ corresponds to a finite sum:

$$\begin{aligned} u(x) &= \epsilon c_n H_{2n}(x) + \epsilon^2 c_n^2 \sum_{m=0}^{2n-2} a_{n,m} H_{2m}(x) + \mathcal{O}(\epsilon^3) \\ \eta &= \epsilon^2 c_n^2 \eta_2 + \mathcal{O}(\epsilon^3) \end{aligned}$$

Interesting exploit: the anomalous dimension

η is determined by projecting $k\partial_k z(\varphi)|_{z=1}$ onto $H_0(x) = 1$ which is effectively a new way of computing η in the LPA

$$\eta \;=\; \pi^{-1/2} \int \mathrm{d}x\, \mathrm{e}^{-x^2} \left\{ g_0 u'''(x)^2 - g_1 u''(x) u'''(x)^2 + \dots \right\}$$

- $\eta \neq 0$ even when $u'''(0) = 0$
- $\eta \sim \epsilon^2$ for $\epsilon \to 0$
- e^{-x^2} decays fast: global solutions are not needed
- admits a proper generalization to any d
  see **[Osborn & Twigg 2009]** for Polchinski's version

Iterative solution order by order in ϵ

For $p \neq n-1, n$

$$
\begin{aligned}
c_n &= \frac{(n-1)\Gamma(n)^3}{2^{n+4}\Gamma(2n)^2}\frac{1}{f_2}\,, \\
a_{p,n} &= \frac{2^{2n+3-p}n^2(2n-p-1)\Gamma(2n)^2}{(n-p)p^2\Gamma(2n-p)\Gamma(p)^2}f_2 \\
\eta_2 &= 4^{n+3}n^2(n-1)(2n-1)\Gamma(2n)g_0
\end{aligned}
$$

and a couple of ugly formulas for $a_{n-1,n}$ and $a_{n,n}$

see **[O'Dwyer & Osborn 2007]** for Polchinski's version

A criterion for cutoff optimization: ϵ^2 matching

LPA does not contain all operators generated at 1-loop, so it is not 2-loops exact.

[Demmel et al. 2013; several others]

We can however match $\eta = \eta_{\mathrm{PT}}$

$$\frac{n^3(n-1)^2\Gamma(n)^6\epsilon^2}{2\Gamma(2n)^3} = \frac{n^2(n-1)^3(2n-1)\Gamma(n)^6\epsilon^2}{2\Gamma(2n)^3}\frac{g_0}{f_2^2}$$

In the original (non-rescaled) quantities:

$$(d+2)f_1 g_0 = d f_2^2$$

f_i and g_i are (derivatives of) 1-loop integrals involving the IR regulator. Polchinski's eqn. does this automatically.

Spectrum

$$u(x) \to u(x) + \mathrm{e}^{-\lambda t} \delta u(x)$$

$$\mathcal{D}_{\hat{\lambda}} \delta u(x) - 2 f_2 u''(x) \delta u''(x) + 3 f_3 u''(x)^2 \delta u''(x) + \cdots = 0$$

$$\hat{\lambda} \equiv \frac{2(\lambda - d)}{d - 2}$$

Solve $\mathcal{D}_{\hat{\lambda}} \delta u(x) = 0$ with spectrum:

$$\delta u(x) = H_k(x) \,, \qquad \hat{\lambda} = k \in \mathbb{N}$$

and use QM perturbation theory to compute futher orders in ϵ.

Universality of the spectrum at order ϵ

$$\lambda = \frac{2n-k}{n-1} + \epsilon \left\{ \frac{k-2}{2} - \frac{(n-1)\Gamma(k+1)\Gamma(n)}{\Gamma(k-n+1)\Gamma(2n)} \right\} + \mathcal{O}(\epsilon^2)$$

It agrees with the scaling dimensions of $:\varphi^k:$
Lucky coincidence: simple mixing of φ^k and $\varphi^{k-2n}(\partial\varphi)^2$

Ising $n = 2$

$$\begin{aligned} \lambda &= 4 - k - \left(\frac{1}{6}k(k-4)+1\right)\epsilon \\ &= \left\{4-\epsilon, 3-\frac{\epsilon}{2}, 2-\frac{\epsilon}{3}, 1-\frac{\epsilon}{2}, -\epsilon, \ldots\right\} \end{aligned}$$

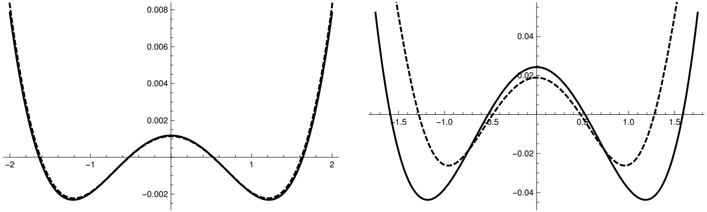
Tricritical $n = 3$

$$\lambda = 3 - \frac{k}{2} - \left(\frac{1}{30}k\left(k(k-3)-13\right)+1\right)\epsilon$$

Beyond the ϵ expansion

Comparison with the numerical method of the scaling solutions:

$\epsilon = 1/10$; $d = 3.9$ $\qquad\qquad$ $\epsilon = 1$; $d = 3$



The expansion fails at $\epsilon x^2 \sim 1$ and for $d \sim 2$

Numerics and CFT of $\mathcal{M}(n+1, n+2)$ are better suited for $d = 2$

Non-unitary multicritical models and the $\epsilon^{1/2}$-expansion

$i\varphi^{2n+1}$ critical model

The most general protected symmetry under reflection:

$$V_k(\varphi) = V_k^*(-\varphi)$$

Therefore:

$$V_k(\varphi) = S_k(\varphi) + iA_k(\varphi)\,, \quad S_k(-\varphi) = S_k(\varphi)\,, \quad A_k(-\varphi) = -A_k(\varphi)$$

Where are the critical models $v(\varphi) \sim i g_n \varphi^{2n+1}$?

$$d_n^A = 2\frac{2n+1}{2n-1} = \left\{6, \frac{10}{3}, \frac{14}{5}, \ldots, 2\right\}$$

New ansatz for the solution

At lowest order $\mathcal{D}_{2n+1} u(x) = 0$

A consistent expansion is:

$$\begin{aligned} u(x) &= \epsilon^{1/2} c_n H_{2n+1}(x) + \epsilon c_n^2 \sum_{m=0}^{2n-1} a_{n,m} H_{2m}(x) + \mathcal{O}(\epsilon^3) \\ \eta &= \epsilon c_n^2 \eta_1 + \mathcal{O}(\epsilon^2) \end{aligned}$$

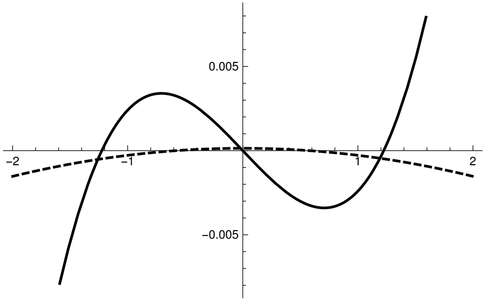
Physical quantities are analytic in ϵ, but $u(x)$ expands in $\epsilon^{1/2}$

Parity implies: $c_n \in i\mathbb{R}$, $a_{n,m} \in \mathbb{R}$

Lee-Yang model

The $n = 1$ solution is the Lee-Yang model

$\epsilon = 1/10;\ d = 5.9$



$$\eta = -\frac{\epsilon}{9}\,, \quad \lambda = \left\{6 - \epsilon, 4 - \frac{4}{9}\epsilon, 2 - \frac{5}{9}\epsilon, -\frac{5}{6}\epsilon, \ldots\right\}$$

A conjecture and the Blume-Capel model

Conjecture:
these solutions interpolate with the CFTs $\mathcal{M}(2, 2n+3)$

Corollary:
$n = 2$ is in the same universality class as the Blume-Capel model (spin chain) which has a non-trivial PT at imaginary magnetic field (non-Hermitian tricriticality)

[von Gehlen 1994]

Interesting observation:
Upper critical dim is $d_3^A = 10/3 \gtrsim 3$ so ϵ-expansion is expected to work well (more to come)

$\mathcal{N} = 1$ Wess-Zumino model in the LPA and emergent supersymmetry

$\mathcal{N} = 1$ **Wess-Zumino model**

Superfield:

$$\Phi = \varphi + (\bar{\theta}\psi) + \frac{1}{2}(\bar{\theta}\theta)F$$

SUSY transformation is linear:

$$\Phi \to \Phi + \delta_\epsilon \Phi = \Phi + \bar{\epsilon}\mathcal{Q}\Phi + \epsilon\overline{\mathcal{Q}}\Phi$$

Manifestly SUSY covariant formulation:

$$\begin{aligned}\Gamma_k^{\mathrm{WZ}} &= \int \mathrm{d}^d x \int \mathrm{d}\theta\, \mathrm{d}\bar{\theta} \Big(-\frac{1}{2} Z_k \Phi K \Phi + 2 W_k(\Phi) \Big) \\ &= \int \mathrm{d}^d x \Big(\frac{Z_k}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{i}{2} Z_k \bar{\psi} \partial\!\!\!/ \psi + \frac{Z_k}{2} F^2 - \frac{1}{2} W_k'' \bar{\psi}\psi + W_k' F \Big)\end{aligned}$$

Linearity $\Longrightarrow$ IR SUSY regulator very straightforward

On-shell action

Use EOM of F to unveil its "auxiliary" role:

$$F^{\mathrm{on\,sh.}} = W_k'(\varphi)/Z_k$$

On shell action:

$$\Gamma_k^{\mathrm{on\,sh.}} = \int \mathrm{d}^d x \Big(\frac{Z_k}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{W_k'^2(\varphi)}{2 Z_k} - \frac{i}{2} Z_k \bar{\psi} \partial\!\!\!/ \psi - \frac{1}{2} W_k''(\varphi) \bar{\psi} \psi \Big)$$

Compare for a moment a Yukawa model:

$$\Gamma_k^{\mathrm{Yukawa}} = \int \mathrm{d}^d x \Big(\frac{Z_k^\varphi}{2} \partial_\mu \varphi \partial^\mu \varphi + V(\varphi) - \frac{i}{2} Z_k^\psi \bar{\psi} \partial\!\!\!/ \psi - \lambda(\varphi) \bar{\psi} \psi \Big)$$

SUSY LPA

$$w_k(\overline{\varphi}_{\mathrm{R}}) \equiv k^{-d/2} W_k(\varphi)/Z_k$$

$$k\partial_k w(\varphi) \quad = \quad (1-d)w(\varphi) + \frac{d-2+\eta}{2}\varphi + \mathcal{F}(w'')$$

with

$$\mathcal{F}(-w'') = -\mathcal{F}(w'')$$

ϵ-expansion can be related to that of $i\varphi^{2n+1}$ in the limit $f_2 \to 0$

$d = 3$: $\mathcal{N} = 1$ generalization of Ising universality with spectrum $\eta \simeq 0.16$ and $\{3, 1.41 \pm 0.01, 0.58 \pm 0.01, -0.37 \pm 0.02, \dots\}$

The Yukawa model

Parametrize the Yukawa model as a soft breaking of WZ on shell (assume $Z_k^{\varphi} = Z_k^{\psi}$ for simplicity):

$$\Gamma_k^{\mathrm{Yukawa}}[\varphi,\psi] \;=\; \Gamma_k^{\mathrm{on\,sh.}}[\varphi,\psi] + \int \mathrm{d}^d x \Big(V_0 + h(\varphi)\overline{\psi}\psi \Big)$$

Introduce an auxiliary field F which completes SUSY off shell

$$\Gamma_k'^{\mathrm{Yukawa}}[\varphi,\psi,F] \;=\; \Gamma_k^{\mathrm{WZ}}[\varphi,\psi,F] + \int \mathrm{d}^d x \Big(V_0 + h(\varphi)\overline{\psi}\psi \Big)$$

Problem! RG step forces $V_0 \to V_0(\varphi)$

Counting of DOF

The RG spans:

$$\Gamma_k'^{\rm Yukawa}[\varphi,\psi,F] \;=\; \Gamma_k^{\rm WZ}[\varphi,\psi,F] + \int {\rm d}^d x \Big(V_0(\varphi) - h(\varphi)\overline{\psi}\psi \Big)$$

which is characterized by three “potentials”:

$$\{W(\varphi), V_0(\varphi), h(\varphi)\}$$

Only two “physical” interactions:

$$\text{Scalar potential} \qquad V(\varphi) = \frac{W'(\varphi)^2}{2} + V_0(\varphi)$$

$$\text{Yukawa interact.} \qquad \lambda(\varphi) = \frac{1}{2} W''(\varphi) + h(\varphi)$$

Field redefinition

Our suggested solution is to redefine $F \to F_k$ along the flow such that $V_0(\varphi) \to V_0 = \mathrm{const.}$

This can be done at the level of renormalized quantities and even maintaining the 1PI nature of the flow:

$$k\partial_k \Gamma_k \;=\; k\partial_k \Gamma_{k,\mathrm{old}} - \int \frac{\delta \Gamma_k}{\delta F_k} k\partial_k F_k + \int \mathcal{G}_k \cdot \frac{\delta}{\delta F_k} \left(\mathcal{R}_k k\partial_k F_k\right)$$

for the formalism see [Gies, Pawlowski 2007]

Useful insights for asymptotic safety and gravity?

Emergent SUSY 1.

For $d = 3$ the Yukawa spectrum differs by WZ only through irrelevant deformations

$$\theta_{1,\text{break}} \simeq -2.7\,, \quad \theta_{2,\text{break}} \simeq -5.1$$

Microscopic deformations from SUSY in the UV are suppressed in the IR for large scale separations $\Longrightarrow$ Same universality class!

Superscaling relations are expected to occur for the scaling dimensions at observable scales.

see also CFT and Bootstrap's literatures [Fei et al. 2016, ...]

Our work is new in that it does use a formalism that explicitly depends on an IR scale.

Emergent SUSY 2.

The idea of emergent SUSY in the IR is rather old...

[Iliopoulos et al. 1980]

but experimental setups have been recently suggested using superfluid He_3

[Grover et al. 2014]

For more details on all the above and more

[T. Hellwig's poster]

Conclusions

- Several features of Wetterich's flow and the LPA approach can be appreeciated by solving it with either ϵ- or $\epsilon^{1/2}$-expansions
- Approach suggests new ways to compute η from a local potential, and new criteria for optimization

- LPA approach admits a simple SUSY generalization
- The formalism can be used to evince that SUSY might emerge as a symmetry in systems with appropriate DOF