ERG2016 Trieste

Local and Functional RG

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Outline of the talk

RG theory: what we know about the flow?

Fixed points and Wess-Zumino actions

Away from criticality

Weyl consistency conditions and Local RG

Exact RG flow for the c-functions

Some examples

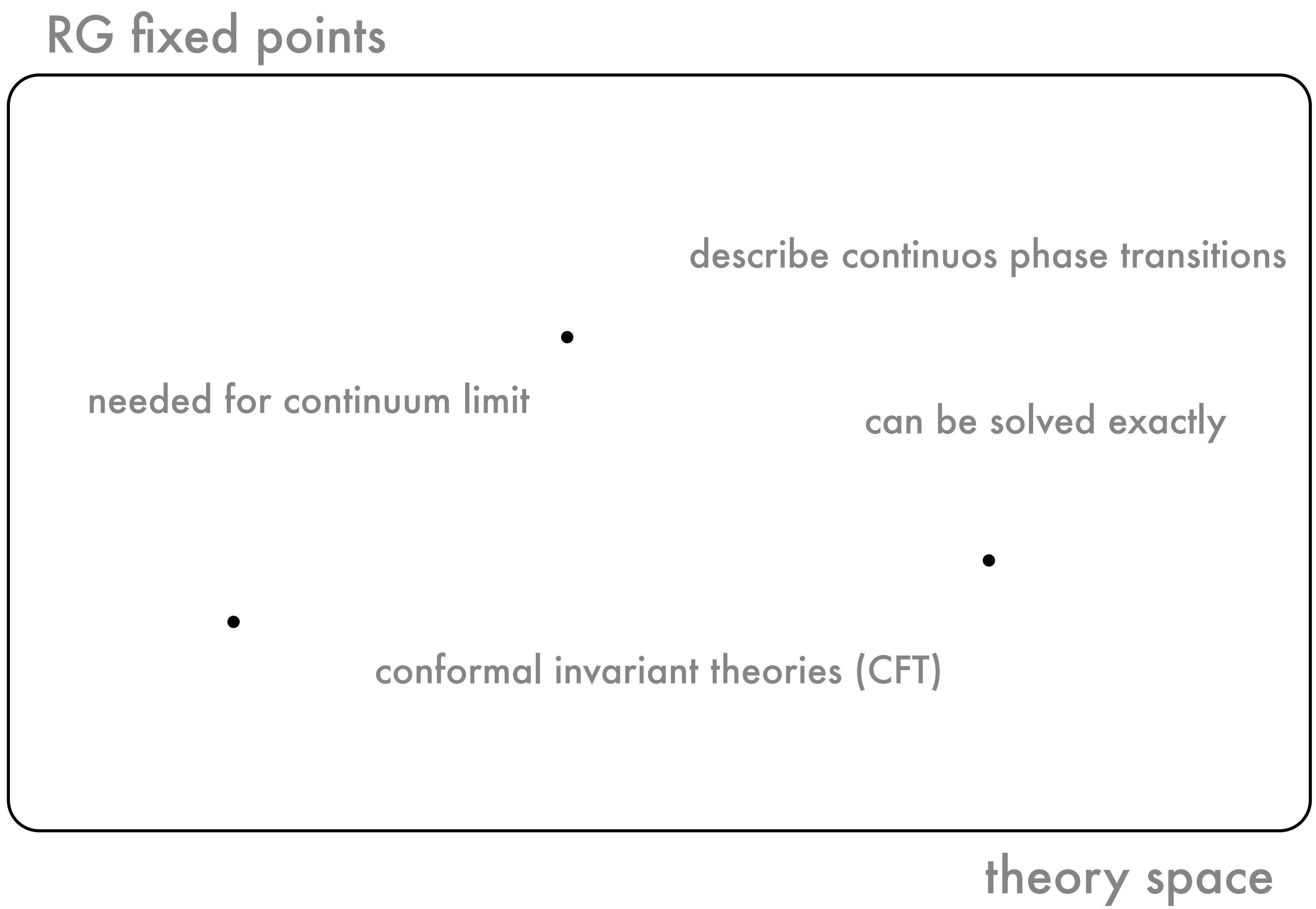
RG theory

RG flow

every theory consistent with the symmetries

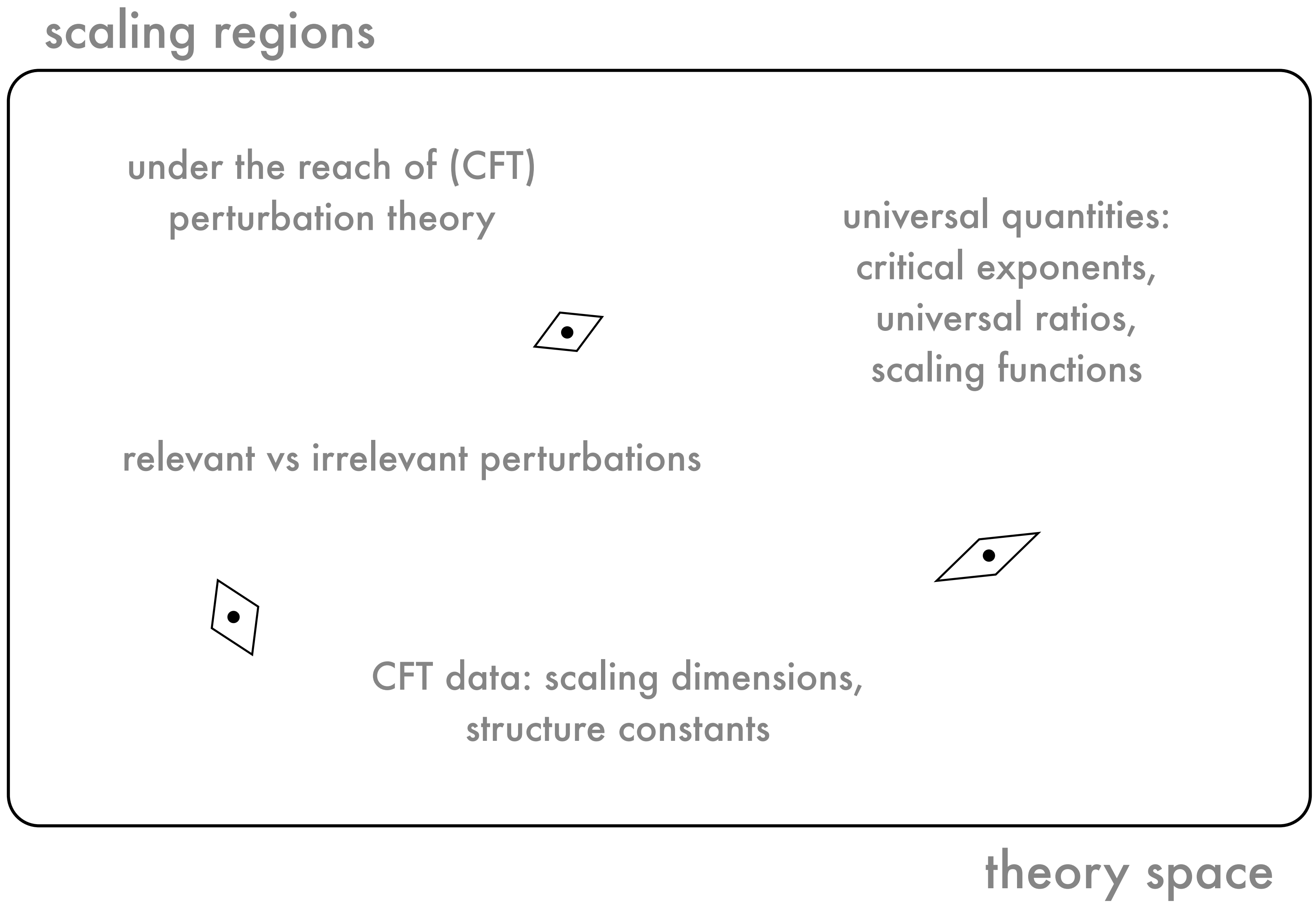
theory space

RG theory



RG fixed points
describe continuos phase transitions
needed for continuum limit
can be solved exactly
conformal invariant theories (CFT)
theory space

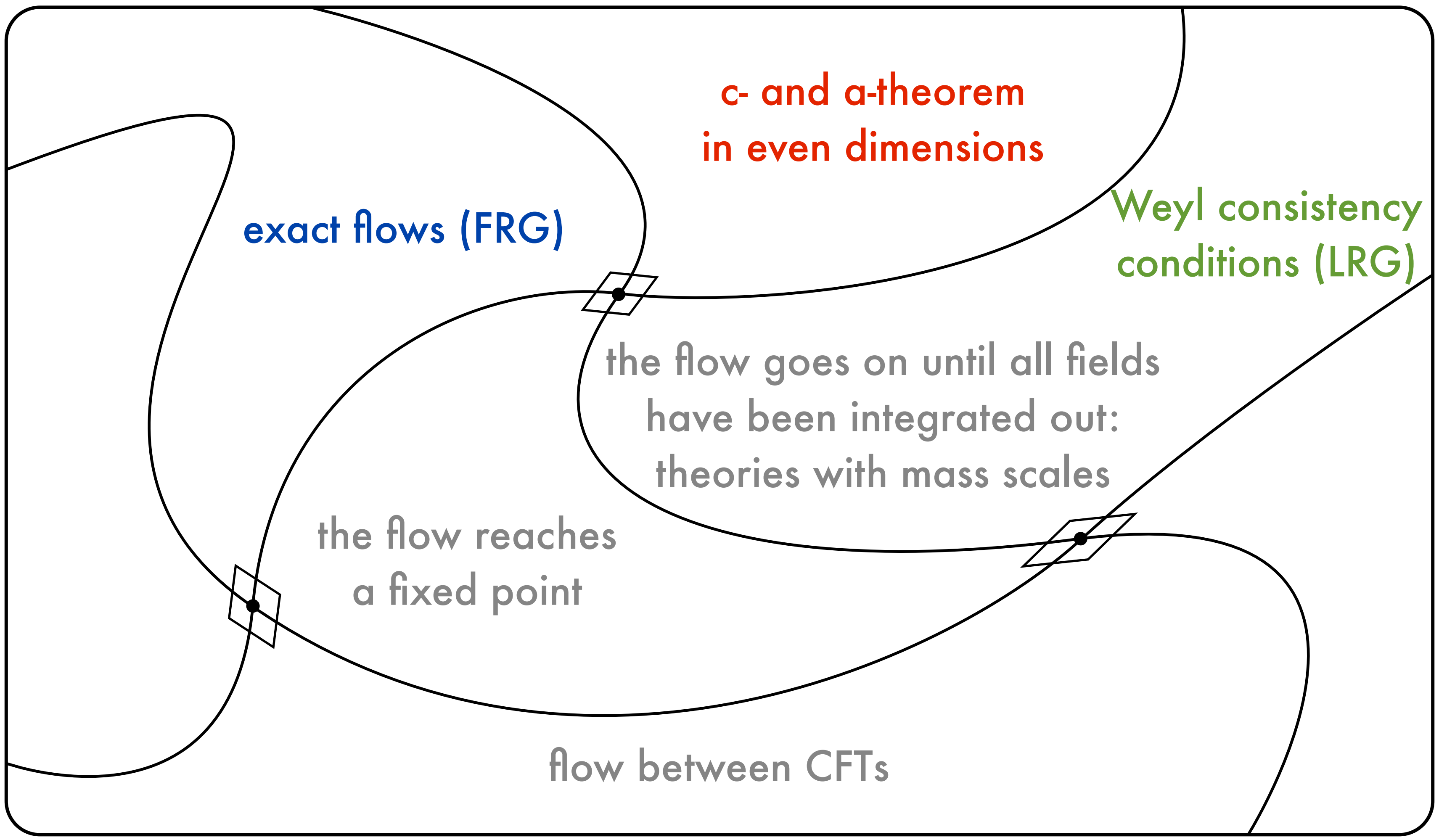
RG theory



scaling regions
under the reach of (CFT)
perturbation theory
universal quantities:
critical exponents,
universal ratios,
scaling functions
relevant vs irrelevant perturbations
CFT data: scaling dimensions,
structure constants
theory space

RG theory

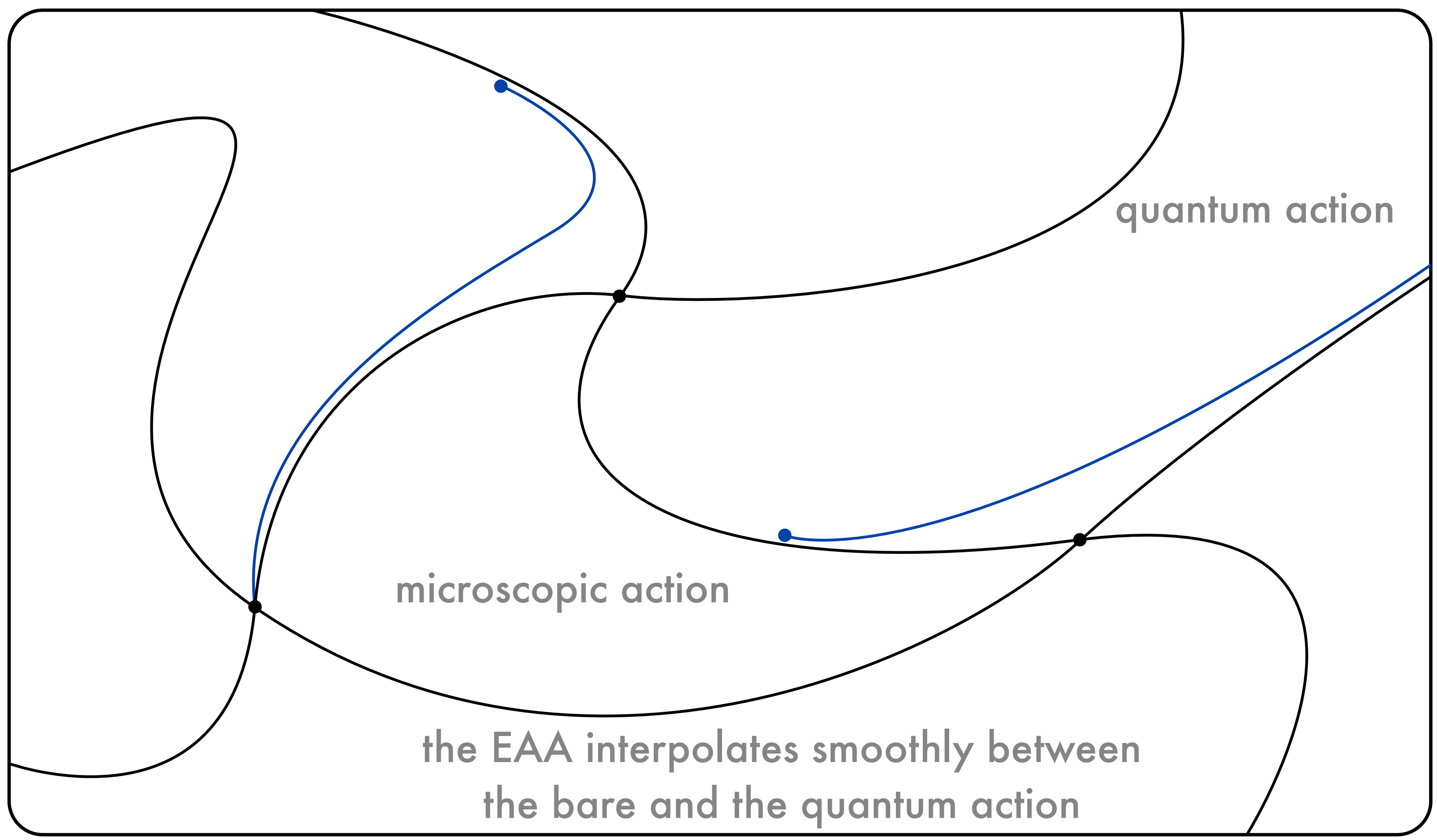
RG flow



theory space

Exact RG flows

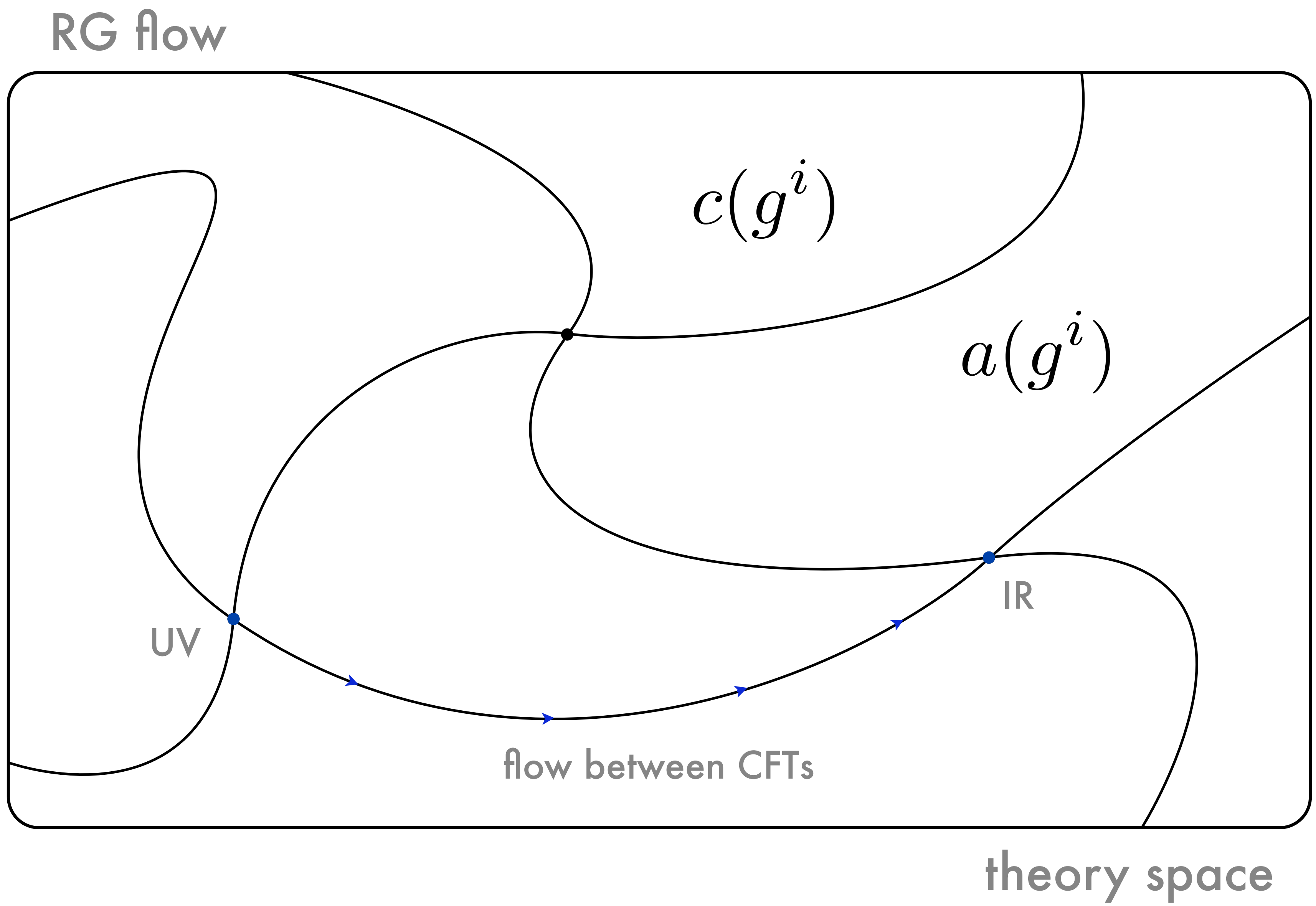
RG flow of the effective average action



theory space

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2} \mathrm{Tr} \left(\frac{\delta^2 \Gamma_k[\varphi]}{\delta\varphi\delta\varphi} + R_k \right)^{-1} \partial_t R_k$$

c- and a-theorem

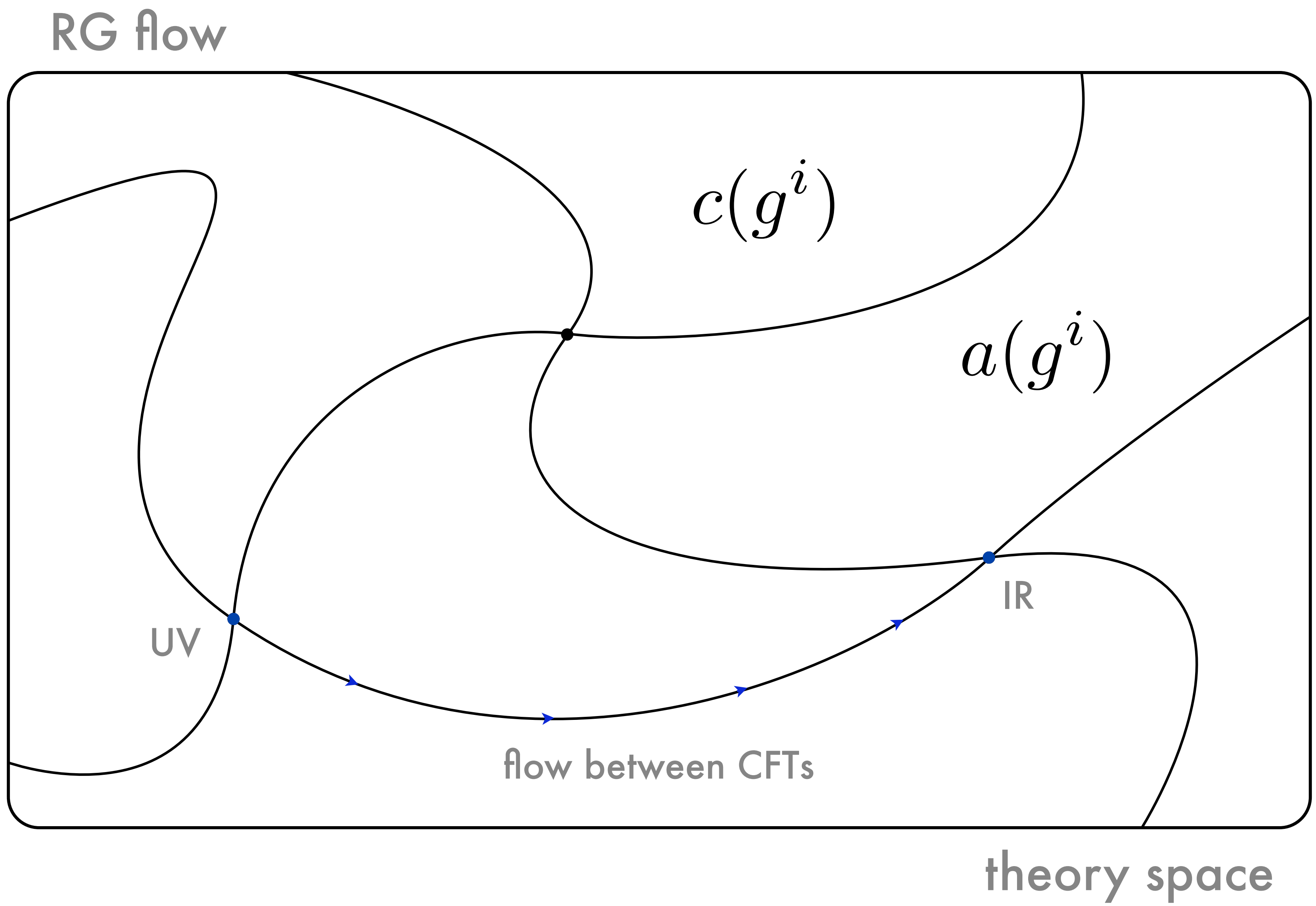


Δc and Δa are universal quantities depending on a whole trajectory!

Integrated (or weak) c- and a-theorems:

$$\Delta c > 0 \qquad \qquad \Delta a > 0$$

c- and a-theorem

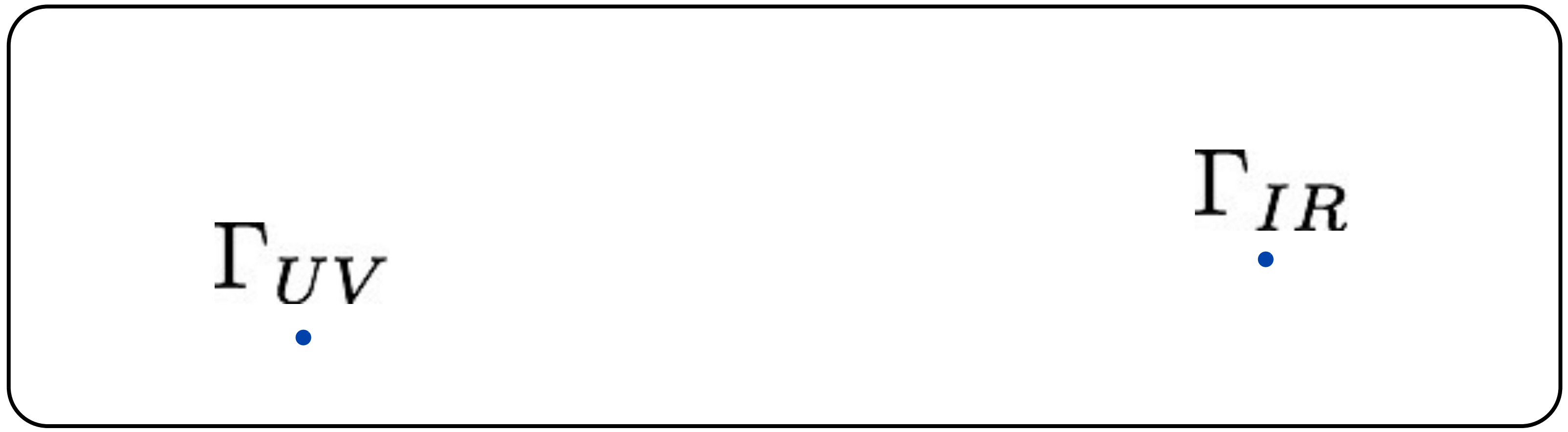


Δc and Δa are universal quantities depending on a whole trajectory!

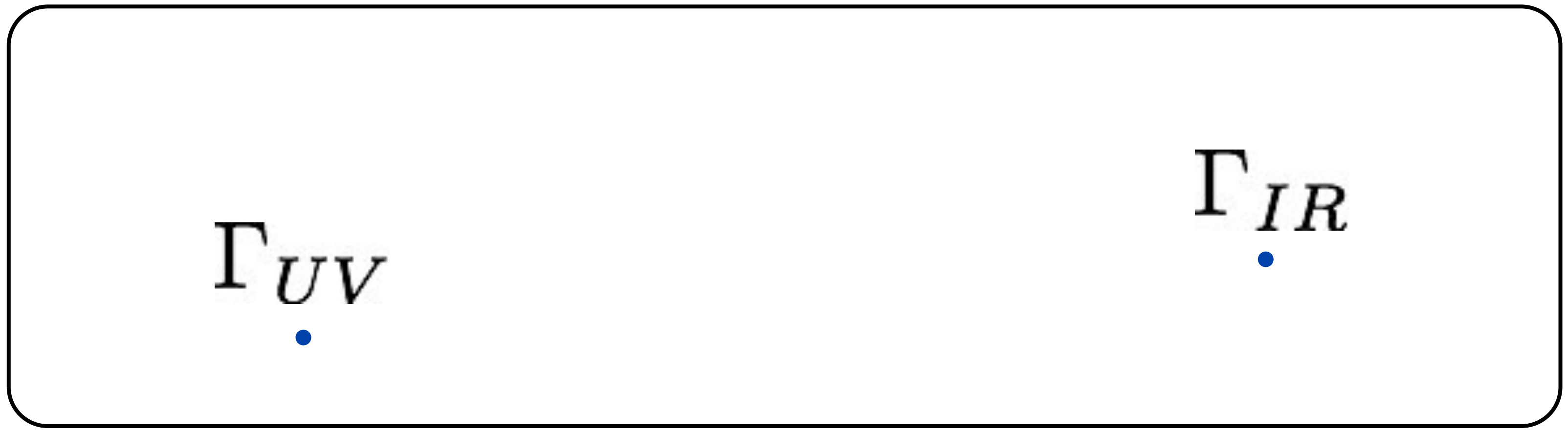
Strong c- and a-theorems:

$$\partial_t c > 0 \qquad\qquad \partial_t a > 0$$

Fixed point action



Fixed point action



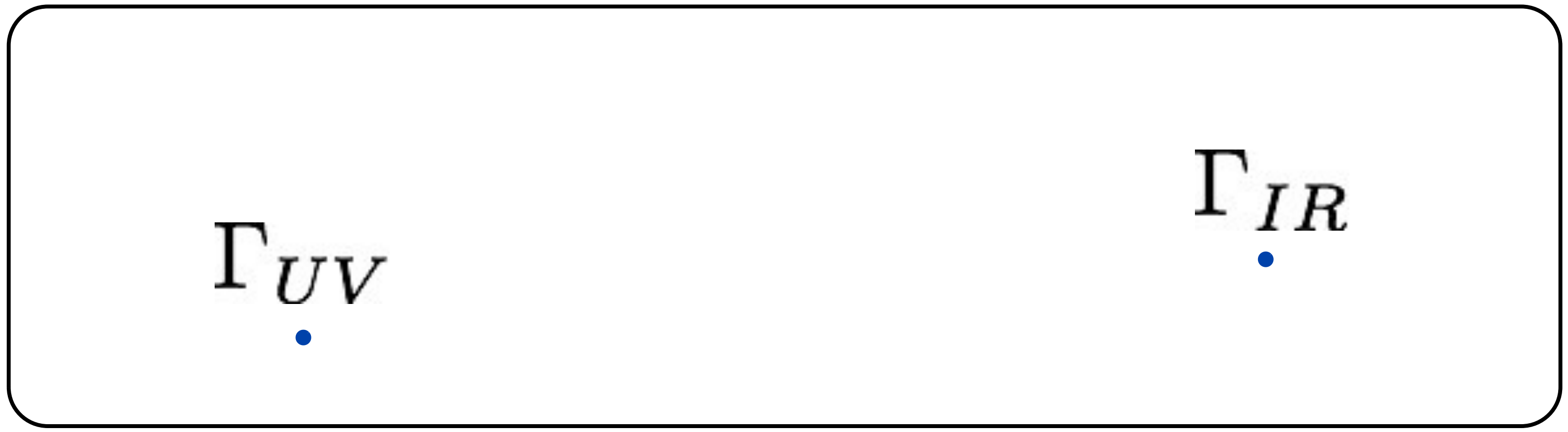
$$\Gamma_{UV/IR}[\varphi, g] = S_{CFT_{UV/IR}}[\varphi, g] + c_{UV/IR} S_P[g]$$

Weyl invariant
(covariantization of the CFT action)

Conformal anomaly
(Polyakov action)

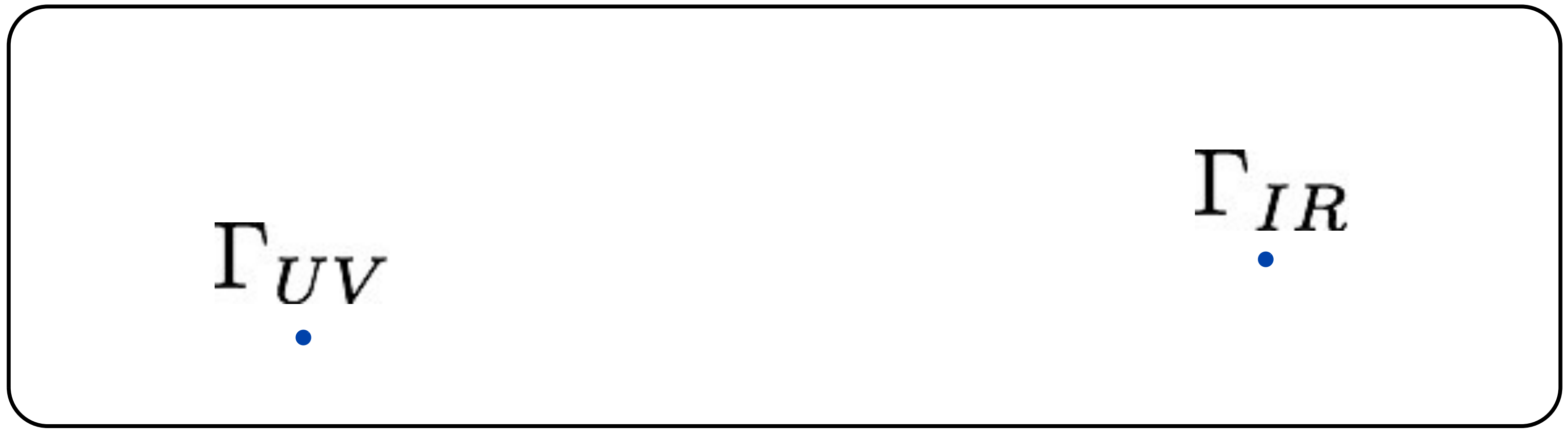
$$S_P[g] = -\frac{1}{96\pi} \int d^2x \sqrt{g} R \frac{1}{\Delta} R$$

Wess-Zumino action



$$\Gamma[e^{w\tau}\varphi, e^{2\tau}g] - \Gamma[\varphi, g] = \underbrace{S_{CFT}[e^{w\tau}\varphi, e^{2\tau}g] - S_{CFT}[\varphi, g]}_{=0} + c\,(\underbrace{S_R[e^{2\tau}g] - S_R[g]}_{\Gamma^{WZ}[\tau,g]})$$

Wess-Zumino action



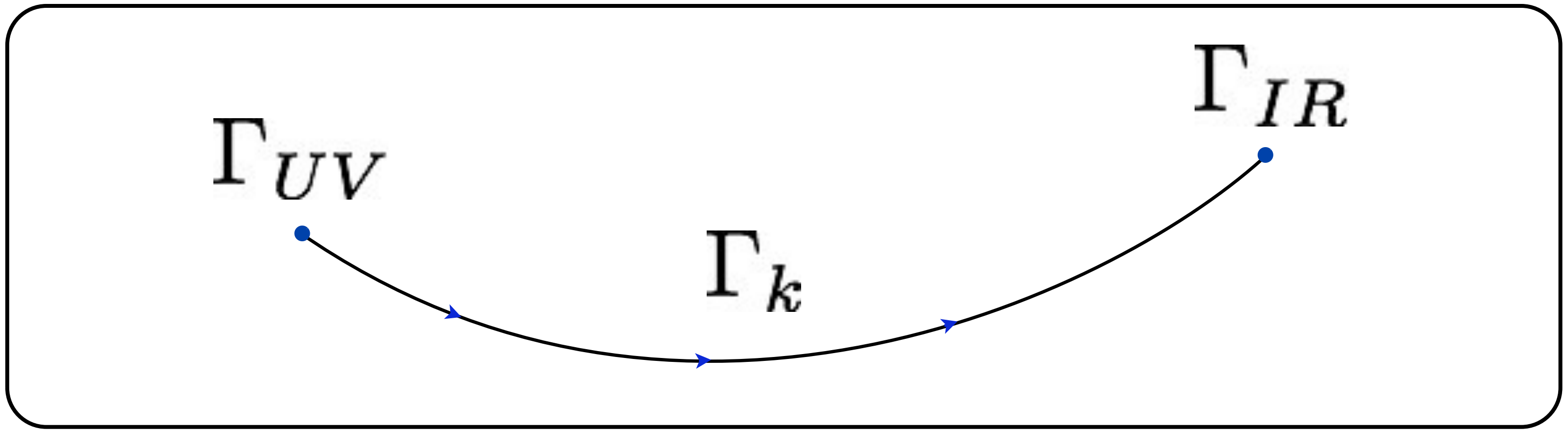
FP Wess-Zumino relation:

$$\Gamma[e^{w\tau}\varphi, e^{2\tau}g] - \Gamma[\varphi, g] = c\,\Gamma^{WZ}[\tau, g]$$

FP Wess-Zumino action:

$$\Gamma^{WZ}[\tau, g] = -\frac{1}{24\pi}\int d^2x\sqrt{g}[\tau\Delta\tau + \tau R]$$

Away from the fixed point



Wess-Zumino relation away from criticality:

$$\Gamma_{ke^{-\tau}}[e^{w\tau}\varphi, e^{2\tau}g] - \Gamma_k[\varphi, g] = \Gamma_k^{WZ}[\tau, g]$$

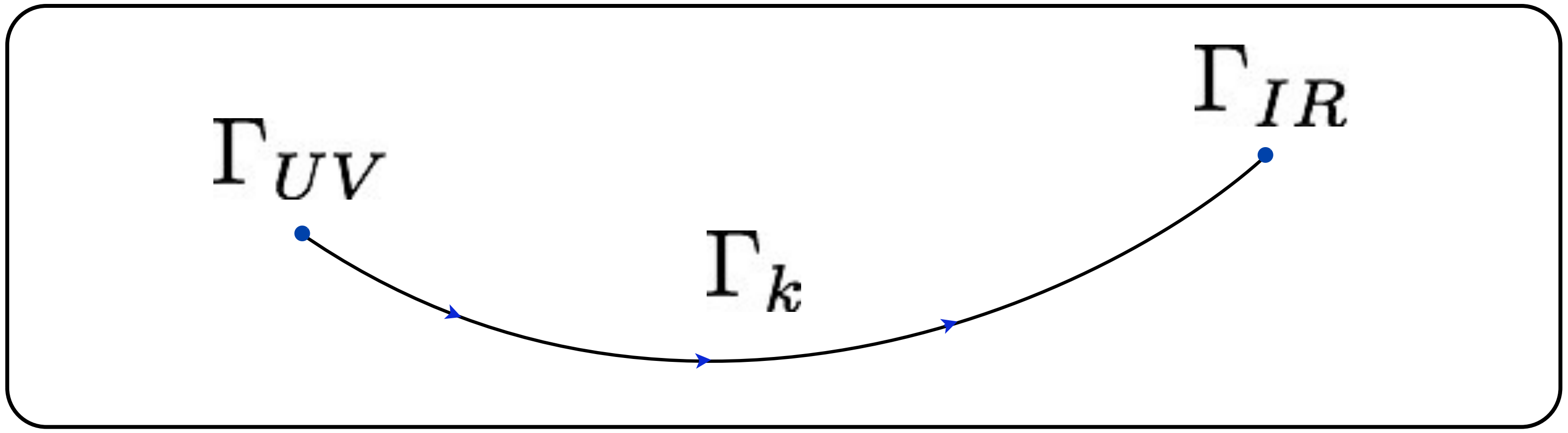
Running Wess-Zumino action:

$$\Gamma_k^{WZ}[\tau, g] = -\frac{1}{24\pi}\int \sqrt{g}\left[\tilde{\mathcal{C}}_k \tau \Delta \tau + \mathcal{C}_k \tau R\right] + \beta\text{–terms}$$

Running c-function(s)

Everything that vanishes at a FP...

Running WZ action



Identifying the running central charge

$$\Gamma_k^{WZ}[\tau, g] = -\frac{1}{24\pi}\int \sqrt{g}\left[\tilde{\mathcal{C}}_k \tau \Delta \tau + \mathcal{C}_k \tau R\right] + \beta\text{–terms}$$

Which is the form of the beta terms?

$$\mathcal{C}_k - \tilde{\mathcal{C}}_k = O(\beta) \qquad \Rightarrow \qquad O(\beta) = 24\pi\omega_i\beta^i + \ldots$$

$$\Gamma_k^{WZ}[\tau, g] = -\frac{1}{24\pi}\int \sqrt{g}\left[\left(\mathcal{C}_k + 24\pi\omega_i\beta^i\right)\tau\Delta\tau + \mathcal{C}_k \tau R\right] + \beta\text{–terms}$$

$$\boxed{c_k = \mathcal{C}_k + 24\pi\omega_i\beta^i}$$

Scale anomaly

Scale anomaly (classical + quantum):

$$\int \sqrt{g}\,\langle T^{\mu}_{\mu} \rangle = -\sum_i (\underbrace{\beta^i}_{\text{quantum}} - \underbrace{d_i g^i}_{\text{classical}}) \int \sqrt{g}\mathcal{O}_i$$

Dimensionless couplings and beta functions:

$$g^i = k^{d_i}\,\tilde{g}^i \qquad\qquad \beta^i - d_i g^i = k^{d_i}\,\tilde{\beta}^i$$

$$\beta\text{–terms} = -\sum_i k^{d_i}\,\tilde{\beta}^i \int \sqrt{g}\,\tau\mathcal{O}_i + \ldots$$

First interaction contribution to the flow of c!

Derivative expansion for the running WZ action

$$\Gamma_k^{WZ}[\tau, g] = \int \sqrt{g}\left[V_k(\tau) + Z_k(\tau)\partial_\mu\tau\partial^\mu\tau + F_k(\tau)R\right] + O(\partial^4)$$

We already know:

$$\begin{aligned} V_k(\tau) &= -\tau\beta^i\mathcal{O}_i + \ldots \\ Z_k(\tau) &= -\frac{\mathcal{C}_k}{24\pi} + \omega_i\beta^i + \ldots \\ F_k(\tau) &= -\frac{\mathcal{C}_k}{24\pi}\tau + \ldots . \end{aligned}$$

How do we determine the next terms?

Stuckelberg trick

Stuckelberg trick:

$$k \to e^{-\tau} k$$

Couplings become spacetime dependent!

$$g^i_k \to g^i_{ke^{-\tau}}$$

Natural way to introduce beta functions:

$$\begin{aligned} g^i_{ke^{-\tau}} &= g^i_{k(1-\tau+\ldots)} \\ &= g^i_k - \tau k \partial_k g^i_k + \ldots \\ &= g^i_k - \tau \beta^i_k + \ldots \end{aligned}$$

Apply to the c-function:

$$\mathcal{C}_{ke^{-\tau}} = \mathcal{C}_k - \tau \partial_t \mathcal{C}_k + O(\tau^2)$$

Stuckelberg trick

The CFT actions delete each others

Recovering the scale anomaly:

$$\begin{aligned}\Gamma_{e^{-\tau}k}[e^{w\tau}\varphi, e^{2\tau}g] - \Gamma_k[\varphi, g] &= \int \sqrt{g}\left[(g^i_k - \tau\beta^i)\mathcal{O}_i - g^i_k\mathcal{O}_i\right] + O(\tau^2) \\ &= -\int \sqrt{g}\,\tau\beta^i\mathcal{O}_i + O(\tau^2)\,.\end{aligned}$$

New higher order terms:

$$g^i_{ke^{-\tau}} = g^i_k - \tau\beta^i_k + \frac{1}{2}\tau^2\beta^j\partial_j\beta^i + O(\tau^3)$$

$$V_k(\tau) = \left[-\beta^i\tau + \frac{1}{2}\beta^j\partial_j\beta^i\tau^2\right]\mathcal{O}_i + O(\tau^3)$$

Stuckelberg trick

$$\beta\text{–terms} \;=\; \int \sqrt{g}\,\Big\{\Big[-\tau\beta^i + \frac{1}{2}\tau^2\beta^j\partial_j\beta^i + ...\Big]\mathcal{O}_i$$
$$+\left(-\omega_i\beta^i\right)\partial_\mu\tau\partial^\mu\tau + \left[\partial_t\left(\omega_i\beta^i\right) + ...\right]\tau\partial_\mu\tau\partial^\mu\tau\Big\} + O(\tau^4)$$

Non-trivial structure emerges:

$$V_k(\tau) \;=\; \left[-\beta^i\tau + \frac{1}{2}\beta^j\partial_j\beta^i\tau^2\right]\mathcal{O}_i + O(\tau^3)$$
$$Z_k(\tau) \;=\; -\frac{\mathcal{C}_k}{24\pi} - \omega_i\beta^i + \left[\partial_t\left(\frac{\mathcal{C}_k}{24\pi} + \omega_i\beta^i\right) + ...\right]\tau + O(\tau^2)$$
$$F_k(\tau) \;=\; -\frac{\mathcal{C}_k}{24\pi}\tau + \left[\partial_t\left(\frac{\mathcal{C}_k}{24\pi}\right) + ...\right]\tau^2 + O(\tau^3)\,.$$

... = missing terms that are not scale derivatives of lower order terms and define new RG quantities

Local RG

All possible
terms involving dilaton,
couplings and curvatures

Osborn's ansatz:

$$\Gamma_k^{WZ}[\tau, g] = \int \sqrt{g} \left[-\tau \beta^i \mathcal{O}_i + \chi_{ij} \partial_\mu g_k^i \partial^\mu g_k^j \tau + \omega_i \partial_\mu \tau \partial^\mu g_k^i - \frac{\mathcal{C}_k}{24\pi} \tau R \right] + O(\tau^2)$$

LRG: *insert Osborn's ansatz in WZ consistency conditions away from criticality to derive non-trivial RG relations*

Connection with the derivative expansion:

$$\begin{aligned} \partial_\mu g^i &= -\beta^i \partial_\mu \tau + O(\tau^2) \\ \chi_{ij} \partial_\mu g^i \partial^\mu g^j &= \chi_{ij} \beta^i \beta^j \partial_\mu \tau \partial^\mu \tau + O(\tau^3) \\ \omega_i \partial_\mu \tau \partial^\mu g^i &= -\omega_i \beta^i \partial_\mu \tau \partial^\mu \tau \\ &\quad + \partial_t \left(\omega_i \beta^i \right) \tau \partial_\mu \tau \partial^\mu \tau + O(\tau^4) \end{aligned}$$

Wess-Zumino consistency conditions

Weyl transformations are Abelian:

$$\Gamma^{WZ}[\tau_1, e^{2\tau_2}g] - \Gamma^{WZ}[\tau_1, g] = \Gamma^{WZ}[\tau_2, e^{2\tau_1}g] - \Gamma^{WZ}[\tau_2, g]$$

Infinitesimal FP WZ consistency conditions:

$$\Gamma^{WZ}[\tau_1, e^{2\tau_2}g] = \Gamma^{WZ}[\tau_1, g] + \delta_{\tau_2}\Gamma^{WZ}[\tau_1, g] + \ldots$$

$$\delta_\tau \equiv \int 2\tau g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}}$$

$$\boxed{\delta_{\tau_2}\Gamma^{WZ}[\tau_1, g] = \delta_{\tau_1}\Gamma^{WZ}[\tau_2, g]}$$

Wess-Zumino consistency conditions

The Wess–Zumino consistency conditions
are also valid away from criticality

$$\Gamma^{WZ}_{ke^{-\tau}}[\sigma, e^{2\tau} g] \quad = \Gamma^{WZ}_{k}[\sigma, g] + \Delta_\tau \Gamma^{WZ}_{k}[\sigma, g] + O\left(\tau^2\right)$$

$$\Delta_\tau \equiv \int d^2x \, \tau \left\{ 2 g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} - \beta^i \frac{\delta}{\delta g^i} \right\}$$

Infinitesimal WZ consistency conditions
away from a FP:

$$\boxed{\Delta_{\tau_2} \Gamma^{WZ}_{k}[\tau_1, g] = \Delta_{\tau_1} \Gamma^{WZ}_{k}[\tau_2, g]}$$

Local RG

Consistency condition deriving from the terms $\tau \partial_\mu \tau \partial^\mu \tau$

$$\partial_t \left(\frac{c_k}{24\pi} + \omega_i \beta^i \right) = \chi_{ij} \beta^i \beta^j$$

$$\boxed{\partial_t c_k = 24\pi \chi_{ij} \beta^i \beta^j}$$

Consistency conditions (LRG) don't tell us how to compute things...

The flow of the c-function

$$\partial_t \Gamma_k^{WZ}[\tau, g] = \partial_t \Gamma_{e^{-\tau}k}[e^{w\tau}\varphi, e^{2\tau} g] - \partial_t \Gamma_k[\varphi, g]$$

$$\boxed{\partial_t c_k = -12\pi \ \mathrm{Tr} \left(\frac{\delta^2}{\delta\varphi\delta\varphi} \Gamma_k[e^{w\tau}\varphi, e^{2\tau}\delta] + R_k[\delta] \right)^{-1} \partial_t R_k[\delta] \Big|_{\int(\partial\tau)^2}}$$

Exact flow for the c-function!

$$\partial_t c_k = 12\pi \, \mathrm{Tr} \, \tilde{\partial}_t \left\{ G_k \frac{\delta^3 \Gamma_k^{WZ}}{\delta\tau\delta\varphi\delta\varphi} G_k \frac{\delta^3 \Gamma_k^{WZ}}{\delta\tau\delta\varphi\delta\varphi} \right\} \Big|_{p^2} - 12\pi \ \mathrm{Tr} \, \tilde{\partial}_t \left\{ G_k \frac{\delta^4 \Gamma_k^{WZ}}{\delta\tau\delta\tau\delta\varphi\delta\varphi} \right\} \Big|_{p^2}$$

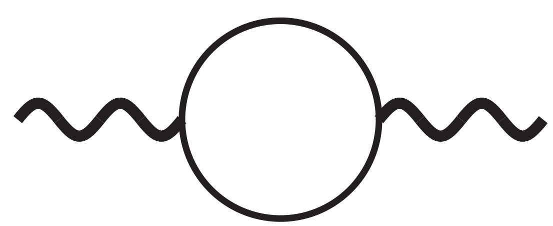
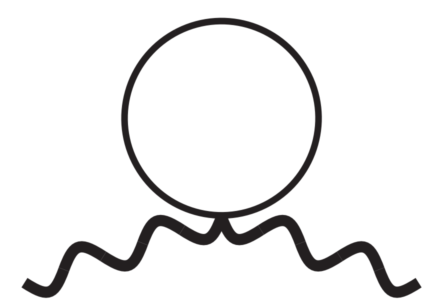
The flow is driven by matter-dilaton interactions...

The flow of the c-function

$$\partial_t \Gamma_k^{WZ}[\tau, g] = \partial_t \Gamma_{e^{-\tau}k}[e^{w\tau}\varphi, e^{2\tau} g] - \partial_t \Gamma_k[\varphi, g]$$

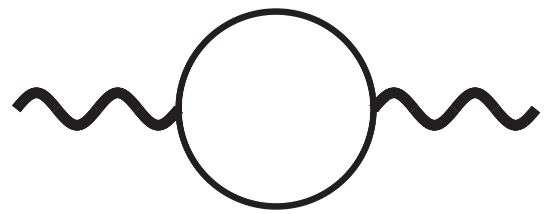
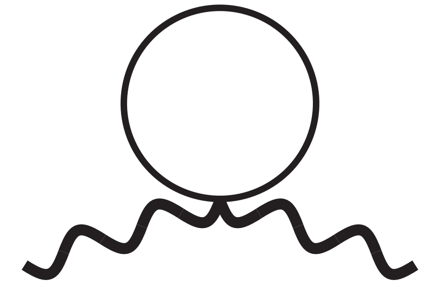
$$\boxed{\partial_t c_k = -12\pi \ \mathrm{Tr} \left(\frac{\delta^2}{\delta\varphi\delta\varphi} \Gamma_k[e^{w\tau}\varphi, e^{2\tau}\delta] + R_k[\delta] \right)^{-1} \partial_t R_k[\delta] \Big|_{\int(\partial\tau)^2}}$$

Exact flow for the c-function!

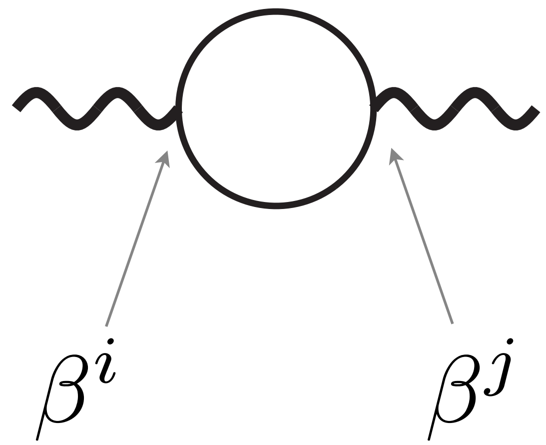
$\partial_t c_k \ = \ 12\pi$  -12π  $\Big|_{p^2}$

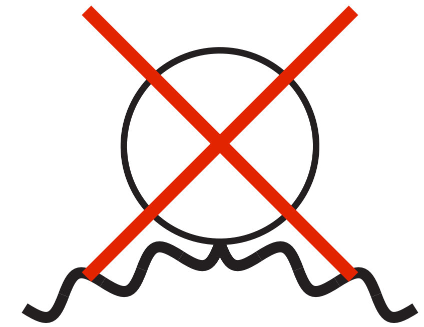
The flow is driven by matter-dilaton interactions...

Zamolodchikov's metric

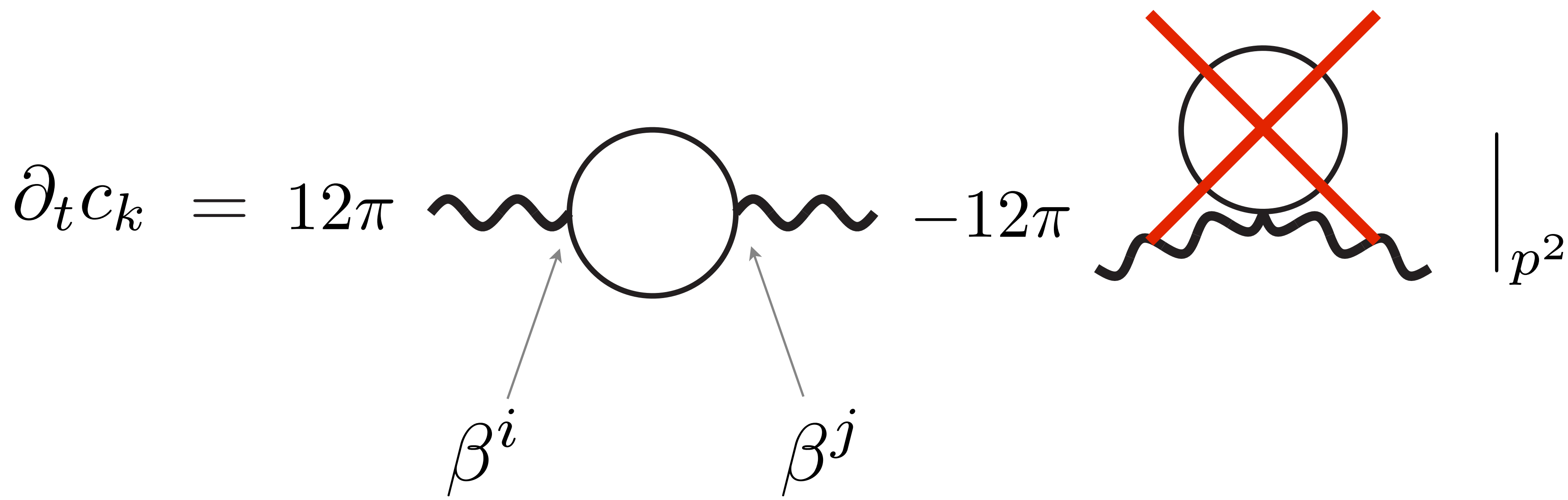
$\partial_t c_k \ = \ 12\pi$  -12π  $\Big|_{p^2}$

Zamolodchikov's metric

$\partial_t c_k \;=\; 12\pi$ 

-12π  $\Big|_{p^2}$

Zamolodchikov's metric



$$\boxed{\partial_t c_k = 24\pi \chi_{ij} \beta^i \beta^j}$$

$$\boxed{\chi_{ij} = \frac{1}{24\pi} \int \frac{d^2 q}{(2\pi)^2} \tilde{\partial}_t \left\{ G_k(q^2) G_k\left((q+p)^2\right) \right\} \mathcal{O}_i^{(2)}(q, q+p) \mathcal{O}_j^{(2)}(-q-p, -q)}$$

Explicit representation of RG quantities!

Massive deformation Gaussian FP

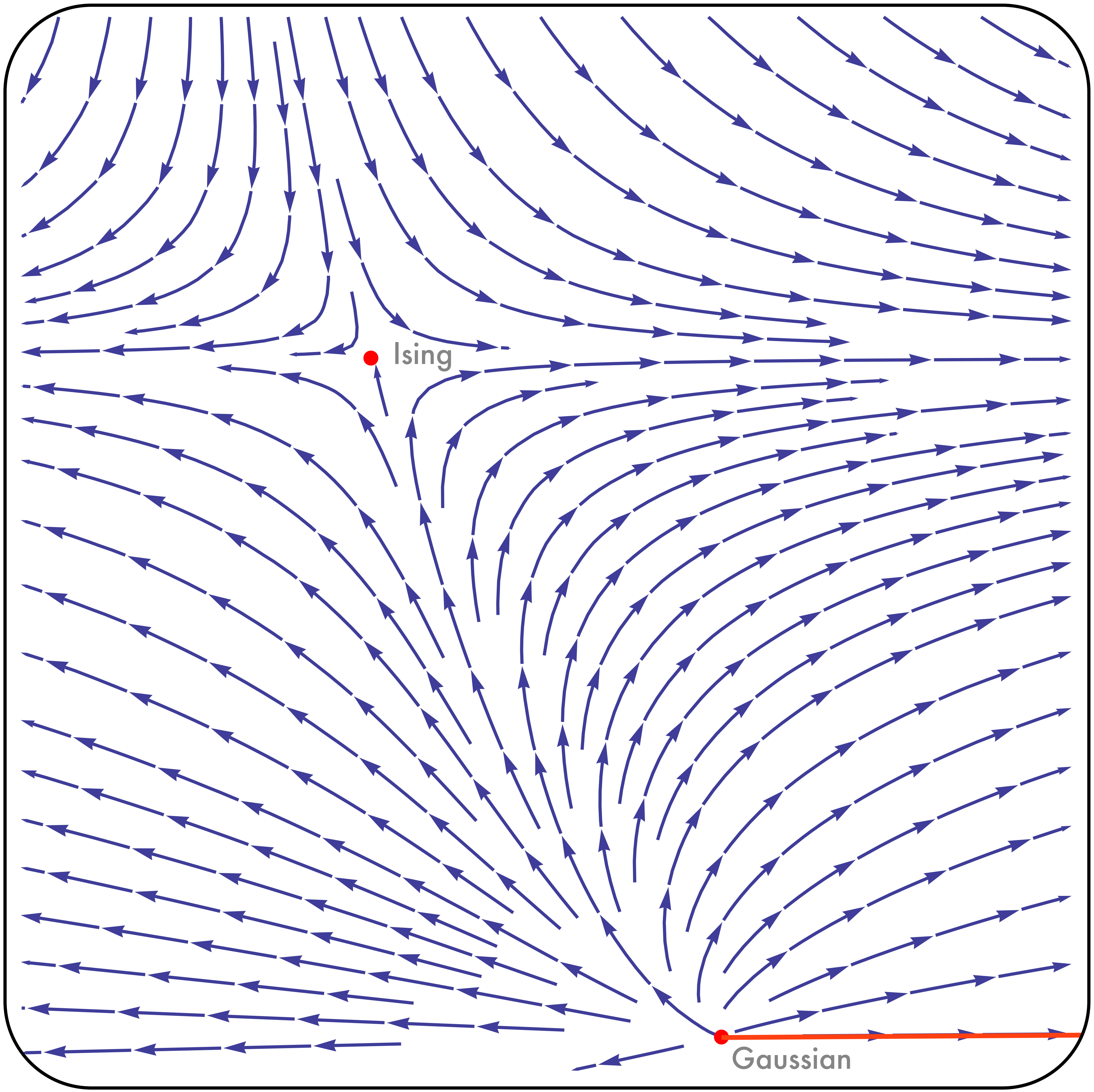
$$\Gamma_k[\phi, g] = \frac{1}{2}\int \sqrt{g}\phi\left(\Delta + m^2\right)\phi - \frac{c_k}{96\pi}\int \sqrt{g}R\frac{1}{\Delta}R$$

$$\Gamma_k[\phi, e^{2\tau}\delta] = \frac{1}{2}\int \phi\left(\Delta + e^{2\tau}m^2\right)\phi - \frac{c_k}{24\pi}\int \tau\Delta\tau$$

$$\partial_t c_k = \frac{4ak^2m^4}{\left(ak^2+m^2\right)^3} \qquad R_k(z) = ak^2$$

$$c_k = 1 - \frac{m^4}{\left(ak^2+m^2\right)^2}$$

$$c_\infty = 1 \qquad c_0 = 0 \qquad \Delta c = 1$$



theory space

Massive deformation Wilson-Fisher FP

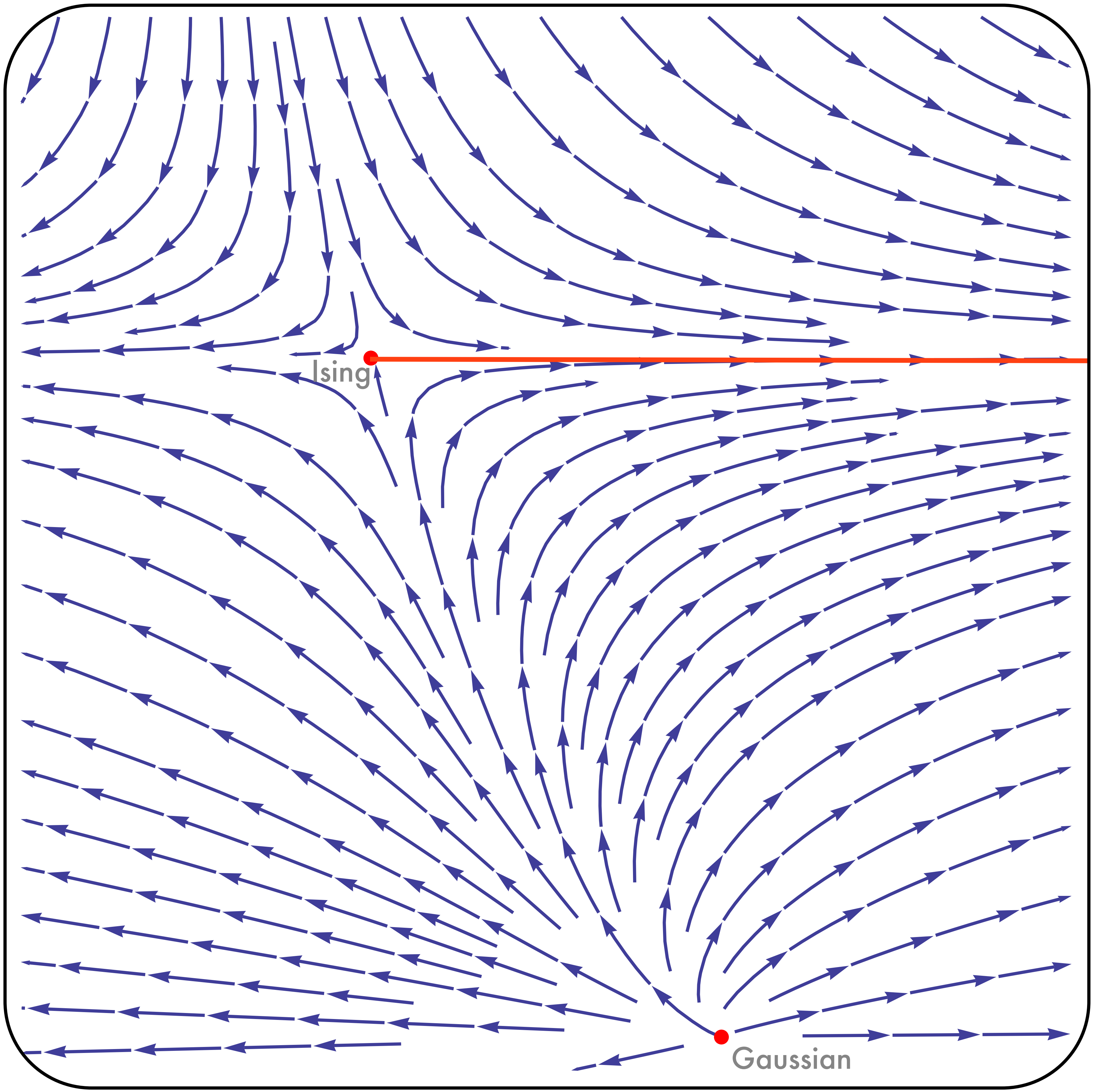
$$\Gamma_k[\bar{\psi}, \psi, g] = \int \sqrt{g}\, \bar{\psi}\, (\nabla + m)\, \psi - \frac{c_k}{96\pi} \int \sqrt{g} R \frac{1}{\Delta} R$$

$$\Gamma_k[e^{\tau/2}\bar{\psi}, e^{\tau/2}\psi, e^{2\tau}\delta] = \int \bar{\psi}\, (\nabla + e^{\sigma} m)\, \psi - \frac{c_k}{24\pi} \int \tau \Delta \tau$$

$$\partial_t c_k = \frac{akm^2}{(ak+m)^3}$$

$$c_k = \frac{1}{2} - \frac{m^2}{2\,(ak+m)^2}$$

$$c_\infty = \frac{1}{2} \qquad c_0 = 0 \qquad \Delta c = \frac{1}{2}$$



theory space

Conclusions & Outlook

Understanding of how to parametrize
the effective (average) action
away from criticality

Non-perturbative definition of the c- and a-functions

Framework to calculate approximated c- and a-functions

A proof of the strong
c- and a-theorems using the fRG?

Thank you

The *c*-function in the LPA

extend a given truncation:

$$\Gamma_k[\varphi] = \int \left[V_k(\varphi) + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + ... \right]$$

$$\downarrow$$

$$\Gamma_k[\varphi, g] = \int \sqrt{g} \left[V_k(\varphi) + \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + ... \right.$$
$$- \frac{1}{2}\partial_t V_k(\varphi)\frac{1}{\Delta}R + ...$$
$$\left. - \frac{c_k - c_\Lambda}{96\pi} R \frac{1}{\Delta} R + ... \right]$$

The c-function in the LPA

non-perturbative flow for the c-function:

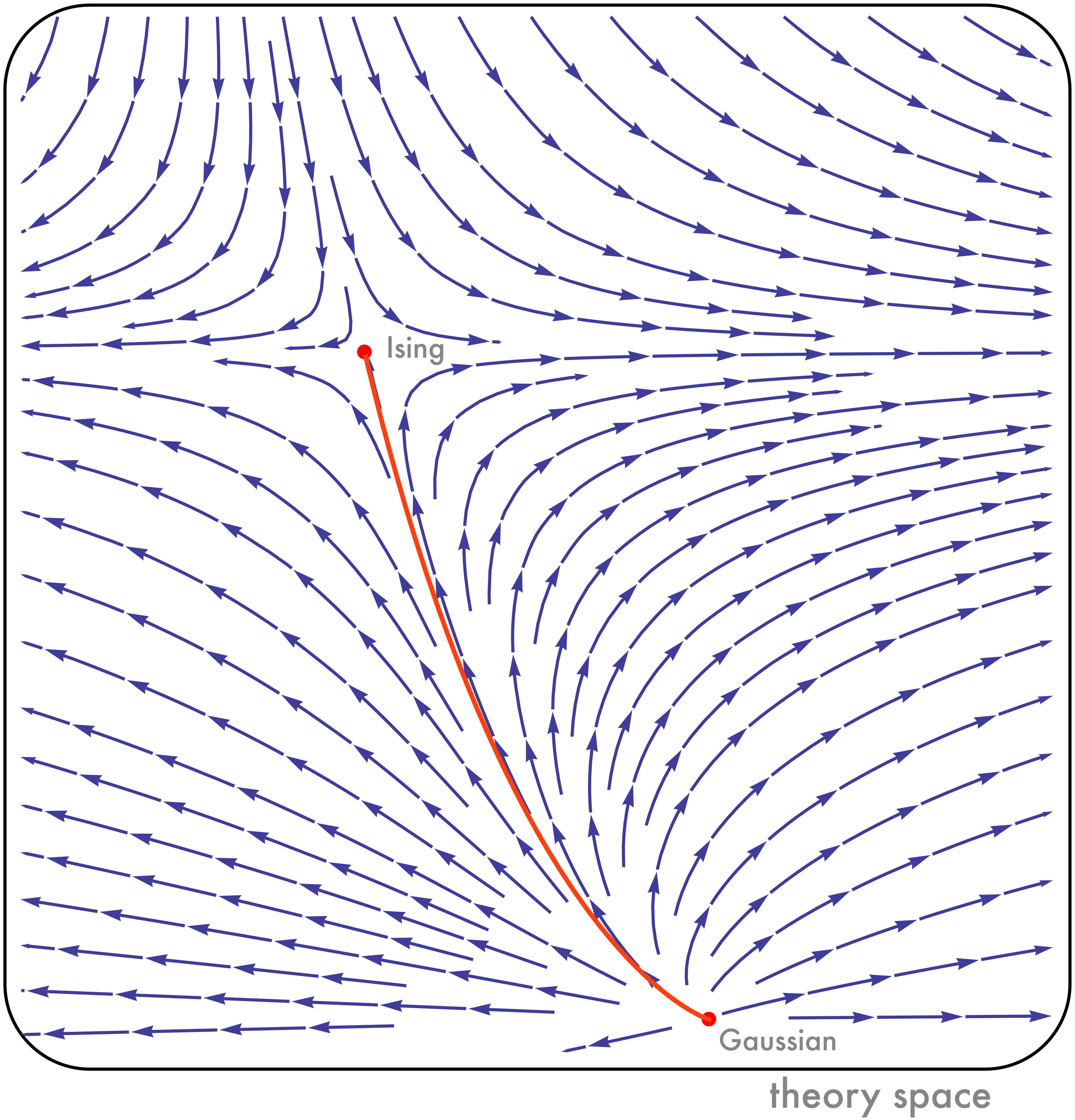
$$\boxed{\partial_t c_k = -24\pi\, \partial_t \Gamma_k[e^{-w\tau}\varphi, e^{2\tau} g]\Big|_{\int \tau\Delta\tau}}$$

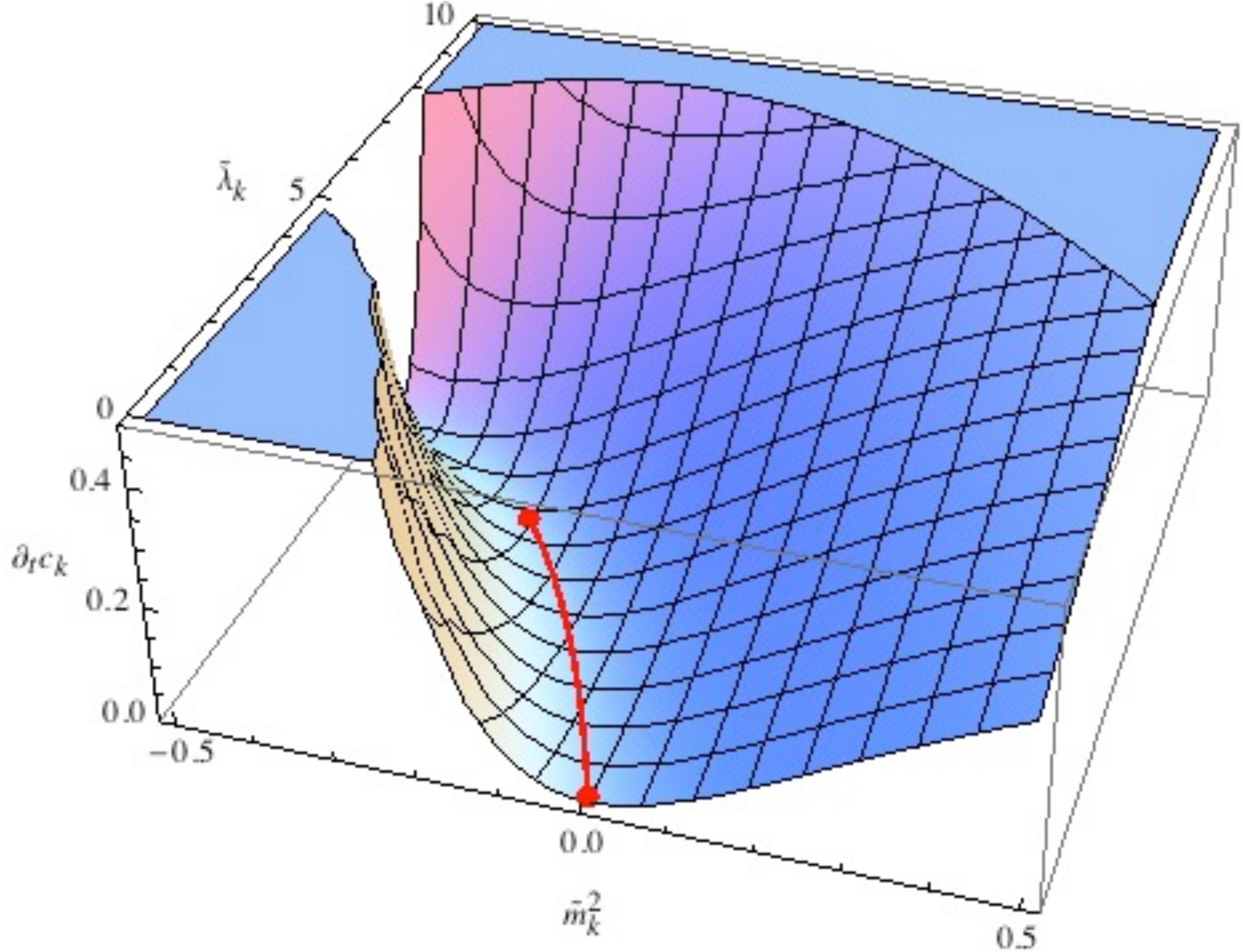
the c-function with in the LPA:

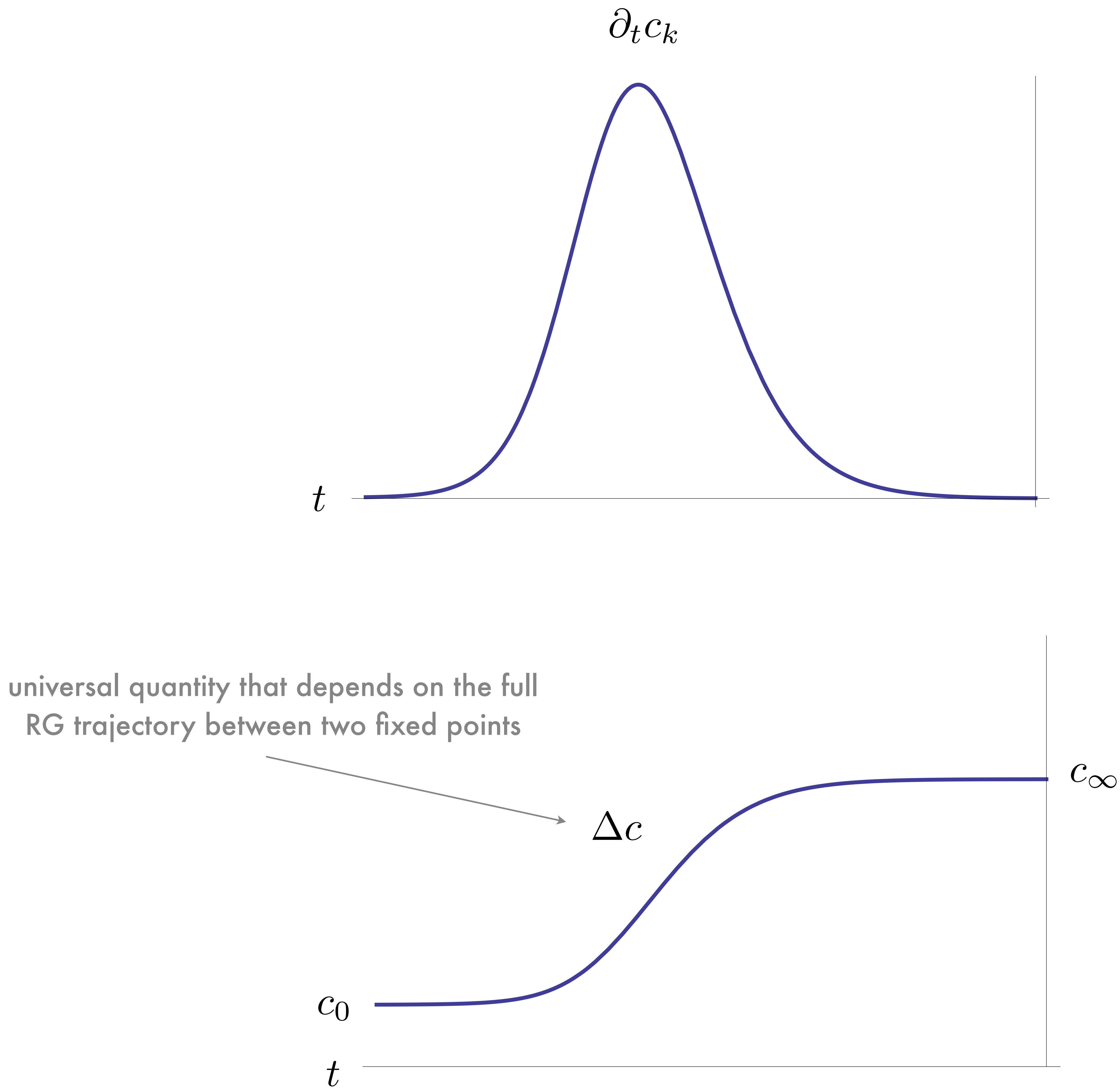
$$\begin{aligned}\partial_t c_k &= \frac{12}{(1+\tilde{m}_k^2)^4}\left(\tilde{\beta}_{m^2}\right)^2 \\ &= \frac{12}{(1+\tilde{m}_k^2)^4}\left(2\tilde{m}_k^2 + \frac{1}{4\pi}\frac{\tilde{\lambda}_k}{(1+\tilde{m}_k^2)^2}\right)^2\end{aligned}$$

the c-theorem is satisfied within our truncation!

$$\partial_t c_k \geq 0$$

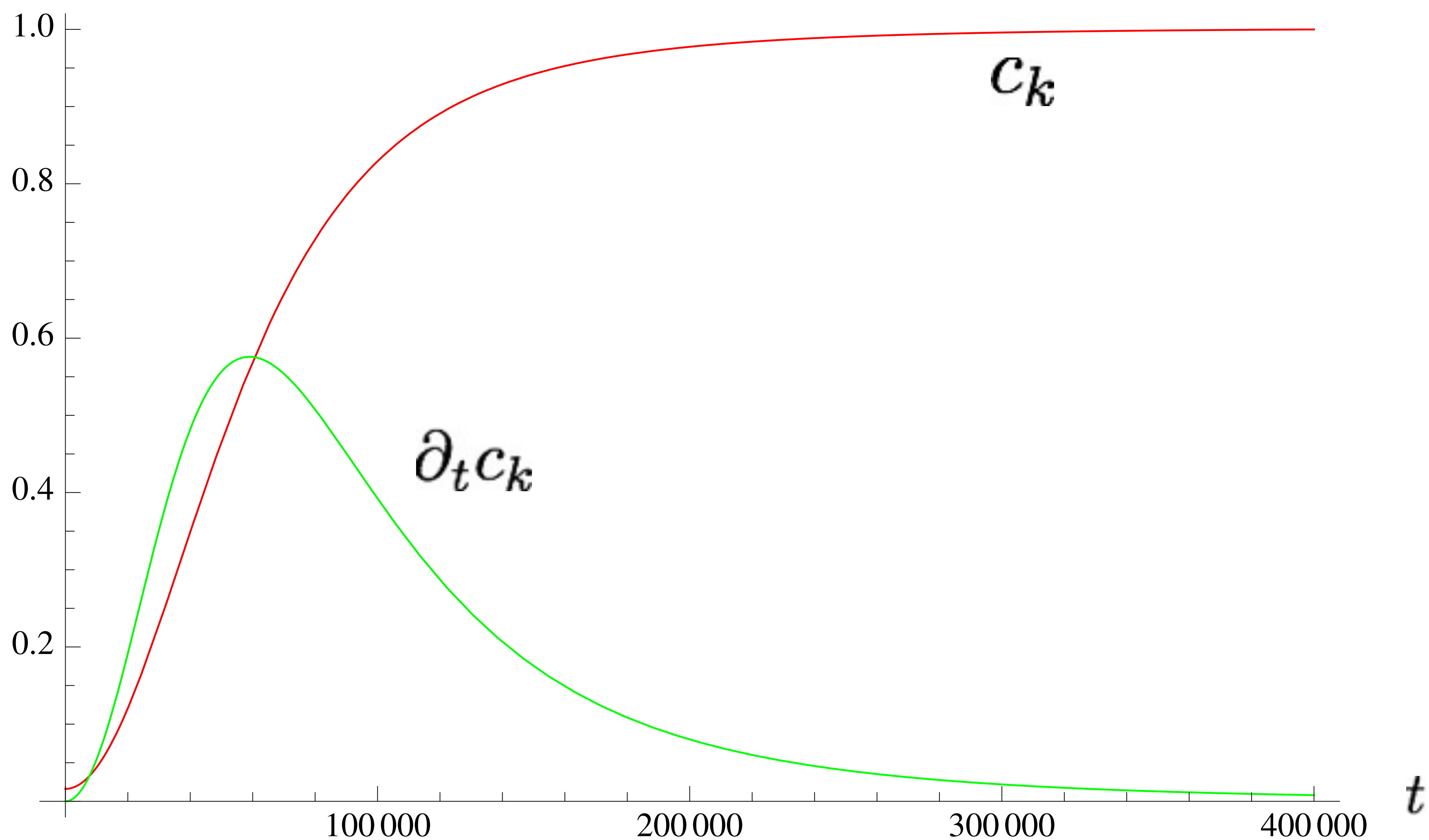

Ising
Gaussian
theory space


10
5
$\bar{\lambda}_k$
0
0.4
$\partial_t c_k$
0.2
0.0
−0.5
0.0
$\bar{m}_k^2$
0.5

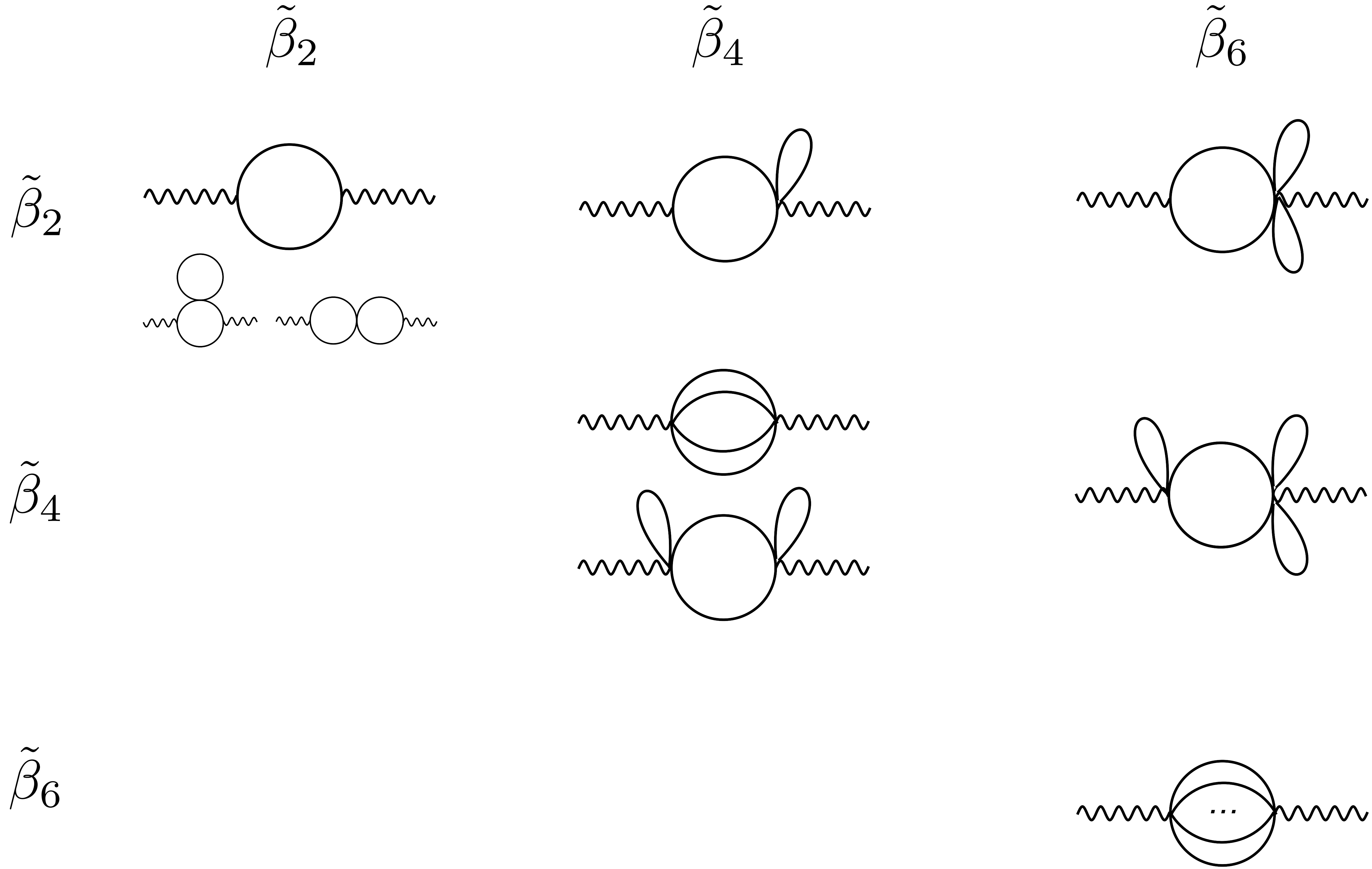

$\partial_t c_k$
t
universal quantity that depends on the full
RG trajectory between two fixed points
Δc
c_∞
c_0
t

Sine-Gordon model

$$S_{SG}[\phi] = \int \left[\frac{1}{2}\phi\Delta\phi - \frac{m^2}{\beta^2}\left(\cos\left(\beta\phi\right) - 1\right)\right]$$

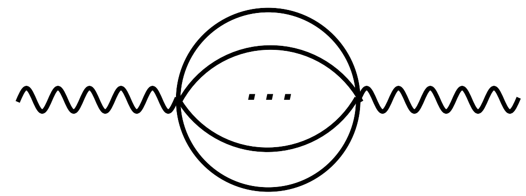


c- & a-functions in the loop expansion



c- & a-functions in the loop expansion

Diagonal contributions:



$$G_k\left(x-y\right)=\frac{1}{2\pi}K_0\left(|x-y|\sqrt{ak^2}\right)$$

$$\partial_t\Gamma_{L,k}=-\frac{1}{2(L+1)!}\,\tilde{\beta}_{L+1}^2\,k^4\int d^2x\int d^2y\,\tau_x\tau_y\,\tilde{\partial}_t\left[G_k\left(x-y\right)\right]^{L+1}$$

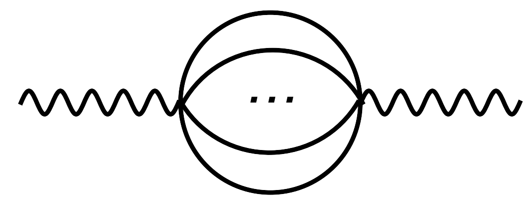
$$\partial_t\Gamma_{L,k}=\frac{k^4}{(L+1)!}\,\tilde{\beta}_{L+1}^2\int d^2x\,\tau_x\Delta\tau_x\int d^2y\,\frac{y^2}{2(2\pi)^{L+1}}\partial_a\left[K_0\left(|y|\sqrt{ak^2}\right)\right]^{L+1}\Bigg|_{a\to 1}$$

$$\partial_t c_{L,k}=\mathcal{A}_L\,\tilde{\beta}_{L+1}^2$$

$$\mathcal{A}_L\equiv\frac{3}{2^L\pi^{L-1}L!}\int_0^\infty dx\,x^4\left[K_0\left(x\right)\right]^L K_1\left(x\right)$$

c- & a-functions in the loop expansion

Diagonal contributions:



$$\mathcal{A}_L > 0$$

$$\partial_t c_k^{(diagonal)} = \sum_{i=1}^{\infty} \mathcal{A}_{2i-1}\, \tilde{\beta}_{2i}^2$$

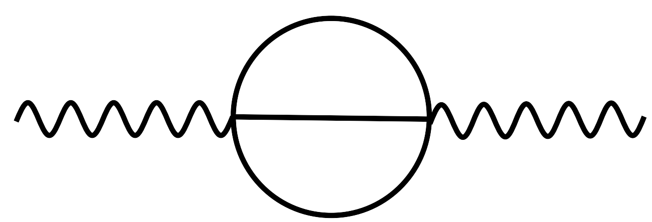
The c-theorem is satisfied by the diagonal contributions:

$$\partial_t c^{diagonal} > 0$$

c- & a-functions in the loop expansion

Non-unitary case:

$$S_{LY}[\phi] = \int d^2x \left[\frac{1}{2}\phi\Delta\phi + ig\phi^3\right]$$



$$\partial_t c_k = -\mathcal{A}_2\,\tilde{\beta}_3^2 < 0$$

$$\mathcal{A}_2 > 0$$

c- & a-functions in the loop expansion

d = 4

$$\partial_t a_k^{(diagonal)} = \mathcal{A}_3 \tilde{\beta}_4^2 + ...$$

$$\mathcal{A}_3 = \frac{1}{2^{12}\pi^6 \left(4!\right)^2}$$

Scheme independent!

$$\partial_t a_k^{diagonal} > 0$$

The a-theorem is valid in the loop expansion

Switch on gravity!

$$\mathcal{O} = R$$

$$\Gamma_k[g] = \int \sqrt{g}\left[-\frac{1}{16\pi G_k}R + \ldots \right.$$
$$\left. -\frac{1}{4}\partial_t\left(-\frac{1}{16\pi G_k}\right)R\frac{1}{\Delta}R + \ldots\right]$$

$$= \int \sqrt{g}\left[-\frac{1}{16\pi G_k}R + \ldots \right.$$
$$\left. -\frac{c_k - c_\Lambda}{96\pi}R\frac{1}{\Delta}R + \ldots\right]$$

$$\downarrow$$

$$\boxed{\partial_t\left(-\frac{1}{16\pi G_k}\right) = \frac{c_k - c_\Lambda}{24\pi}}$$

Switch on gravity!

$$\boxed{\partial_t c_k = \frac{3}{2G_k^2}\left(\partial_t \beta_{G_k} - 2\frac{\beta_{G_k}^2}{G_k}\right)}$$

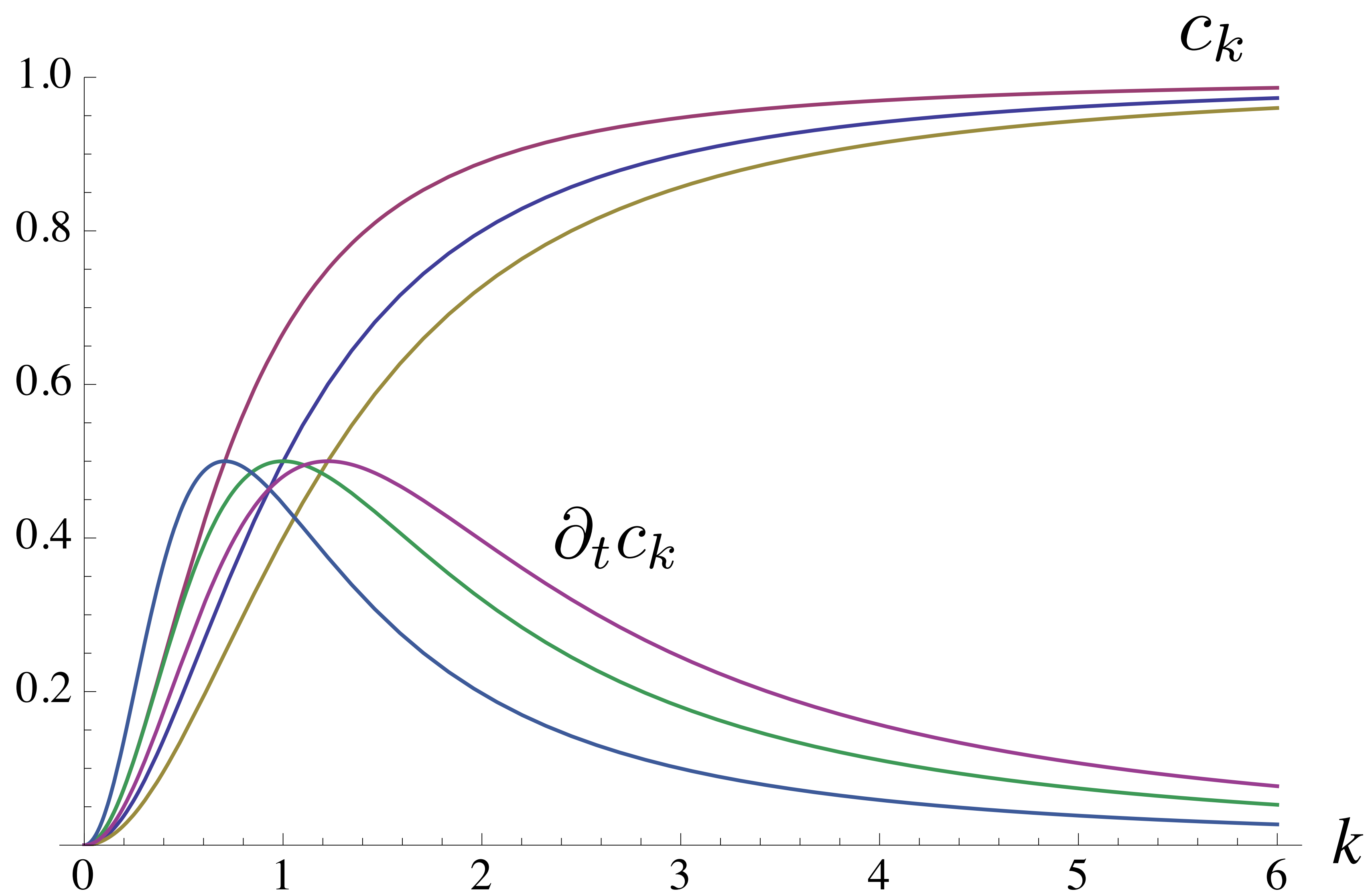
minimally coupled scalar:

$$c_k = \frac{ak^2}{ak^2 + bm^2} \qquad \partial_t c_k = \frac{2abk^2m^2}{\left(ak^2 + bm^2\right)^2}$$

$$R_k(z) = \frac{az}{e^{bz/k^2} - 1}$$

Switch on gravity!

minimally coupled scalar:



Switch on gravity!

interacting scalar:

$$c_k = \frac{ak^2}{ak^2 + bV_k''(\varphi_0)}$$

$$\partial_t c_k = -\frac{abk^2\left(\partial_t V_k''(\varphi_0) - 2V_k''(\varphi_0)\right)}{\left(ak^2 + bV_k''(\varphi_0)\right)^2}$$

$$\partial_t c_k = \begin{cases} -\frac{ab\,\partial_t \tilde{m}_k^2}{(a+b\,\tilde{m}_k^2)^2} & \text{ordered phase} \\ \\ \frac{2ab\,\partial_t \tilde{m}_k^2}{(a-2b\,\tilde{m}_k^2)^2} & \text{broken phase}\,. \end{cases}$$

Switch on gravity!

interacting scalar: