Renormalisation and Observables in Quantum Gravity

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Introduction

- In quantum gravity we would like to compute observables:

$$\langle \mathcal{O} \rangle = \sum_{geometries} \mathcal{O}\ e^{iS}$$

- This formal expression needs to be regulated in order to obtain a meaningful result.

- Then the parameters of the theory should depend on a cutoff scale such that observables are renormalisation group (RG) invariants:

$$\Lambda \frac{d}{d\Lambda} \langle \mathcal{O} \rangle = 0$$

Introduction

- Typically beta functions are derived from the RG invariance of correlation functions:

$$\Lambda \frac{d}{d\Lambda} \langle \phi^{a_1} \phi^{a_2} ... \phi^{a_n} \rangle = 0$$

- These break diffeomorphism and re-parameterisation invariance and as consequence beta functions depend on the gauge fixing and the parameterisation of the fields.

- Instead I consider the RG invariance of diffeomorphism invariant observables directly:

$$\Lambda \frac{d}{d\Lambda} \langle \mathcal{O} \rangle = 0$$

Beta function for Newton's constant

- One loop beta function for Newton's constant (Weinberg '79):

$$\beta_G = (D-2)G - b\,G^2\,,$$

- The beta function depends on the gauge and parameterisation (talk by A. Pereira).

- Furthermore different beta functions are found if the Einstein-Hilbert or Gibbons-Hawking-York boundary term are considered. This breaks the required balance between the two terms (Gastmans, R. Kallosh, and C. Truffin 1978; Becker and Reuter 2012; Jacobson and Satz 2014).

$$S = \frac{1}{16\pi G}\left(\int d^d x\sqrt{-g}R + 2\int_{\Sigma} d^{D-1}y\sqrt{\gamma}K\right)$$

- These problems are acute when we consider asymptotic safety close to two spacetime dimensions i.e. simplest approximation that the continuum limit of Gravity can be studied.

Functional measure

- Here I consider Einstein theory within a semi-classical regime

$$S_\Lambda \approx S_{\mathrm{EH}} = -\frac{1}{16\pi G}\int \sqrt{g}(R-2\bar{\lambda}) + \ldots$$

with the ellipsis denoting required boundary terms.

- The functional measure should be the one obtained by canonical quantisation giving the functional integral:

$$\mathcal{Z} = \int d\mathcal{M}(\phi)\, e^{-S[\phi]}$$

- What is the field?

$$\phi^A = g_{\mu\nu}\,, \quad \phi^A = g^{\mu\nu}\,, \quad \phi^A = \sqrt{g}g^{\mu\nu} \quad \text{etc.}$$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \phi_{\mu\nu}\,, \qquad g_{\mu\nu} = \bar{g}_{\mu\rho}(e^{\phi})^{\rho}{}_{\nu}$$

- Choice should not affect the physics.

Functional measure

- The measure must be re-parameterisation invariant in order to manifestly preserve the invariance of the functional integral.

$$d\mathcal{M}(\phi) = V_{\text{diff}}^{-1} \prod_a \frac{d\phi^a}{(2\pi)^{1/2}} \sqrt{|\det C_{ab}(\phi)|}$$

- Involves a metric on the 'space of geometries' which provides the invariant volume element.

- Fields are just coordinates in the space of geometries.

- Invariant line element: $\delta l^2 = C_{ab}\delta\phi^a\delta\phi^b$

DeWitt notation:

e.g. $\phi^a = g_{\mu\nu}(x)$

Functional measure

- Correct form of the measure can be determined by BRST invariance (Fujikawa '83) or canonical quantisation (Fradkin and Vilkovisky '73, Toms '87).

- Use Fujikawa's measure which agrees with Toms. The metric is of the DeWitt type:

$$C_{ab}\delta\phi^a\delta\phi^b = \frac{\mu^2}{32\pi G}\frac{1}{2}\int d^D x\sqrt{g}(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma})\delta g_{\mu\nu}\delta g_{\rho\sigma}$$

Origin of gauge and parameterisation dependence

- Where does the gauge and parameterisation dependence come from?
- Standard approach: Faddeev-Popov functional integral with sources

$$e^{W[J]} = \int \prod_n \frac{d\varphi^n}{(2\pi)^{1/2}} \sqrt{|\mathrm{sdet}\, \mathcal{C}_{nm}(\varphi)|}\, e^{-\mathcal{S}[\varphi]+J_n\varphi^n}$$

- Fields now include ghosts and the diffeomorphisms are factored out

$$\text{e.g. } \varphi^a = \{g_{\mu\nu}(x), \bar{\eta}_\mu(x), \eta_\mu(x)\}$$

- Source term breaks re-parameterisation and diffeomorphism invariance.
- Effective action: $\Gamma[\bar{\varphi}] = -W[J] + \bar{\varphi}^n J_n \qquad \Gamma^{(1)}_n[\bar{\varphi}] = J_n$
- Gauge and parameterisation independence only realised by going on shell or computing an observable.
- Illustrative example: quantum corrections to the trajectory of a test particle (Dalvit and Mazzitelli '97; KF 2015).

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Origin of gauge and parameterisation dependence

- One-loop beta functions from the Legendre effective action effective action

$$\Gamma[g_{\mu\nu}] = S[g_{\mu\nu}] + \frac{1}{2}\mathrm{STr}\log\left(\mathcal{C}^{-1}\cdot\mathcal{S}^{(2)}\right)$$

- Contribution from the action and the measure

- Hessian has the form: $\mathcal{S}^{(2)}_{nm} = c_{no}(-\nabla^2\delta^o_m - E^o{}_m) \equiv c_{no}\Delta^o{}_m$

- Considering a ultra-local re-parameterisation: $\tilde{\mathcal{S}}^{(2)}_{nm} = \frac{\delta\varphi^o}{\delta\tilde{\varphi}^n}\mathcal{S}^{(2)}_{op}\frac{\delta\varphi^p}{\delta\tilde{\varphi}^m} + \frac{\delta\varphi^o}{\delta\tilde{\varphi}^n\delta\tilde{\varphi}^m}\mathcal{S}^{(1)}_o$

- The coefficient of the Laplacian transforms as a metric of the space of geometries:

$$\tilde{c}_{nm} = \frac{\delta\varphi^r}{\delta\tilde{\varphi}^n}c_{rs}\frac{\delta\varphi^s}{\delta\tilde{\varphi}^m}$$

Origin of gauge and parameterisation dependence

- Typically only the super-trace

$$\frac{1}{2}\mathrm{STr}\log\left(\Delta\right)$$

is regulated. Which leaves behind a divergent part:

$$\sim \mathrm{STr}\log(\mathcal{C}^{-1}\cdot c) = \delta(0)\int d^D x\, \mathrm{str}\log(\mathcal{C}^{-1}\cdot c)$$

- However for the correct BRST measure one has $\mathcal{C}_{nm} = c_{nm}$

- One either uses the correct measure or one has additional UV divergencies which are ignored in the effective average action approach.

Origin of gauge and parameterisation dependence

- Standard effective average action scheme (Reuter '96)

$$\Gamma[g_{\mu\nu}] = S[g_{\mu\nu}] + \frac{1}{2}\mathrm{STr}\log\left(\mathcal{C}^{-1}\cdot c\cdot(\Delta + \mathcal{R}_k(-\nabla^2))\right)$$

- Regardless of the measure we get the same flow equation:

$$k\partial_k\Gamma = \frac{1}{2}\mathrm{STr}\left[k\partial_k\mathcal{R}\cdot(\Delta + \mathcal{R}_k)^{-1}\right]$$

- The measure is not the origin of differences in beta functions for different parameterisations.

Origin of gauge and parameterisation dependence

- One-loop beta functions from the Legendre effective action effective action

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- Contribution from the action and the measure

- Hessian has the form: $\mathcal{S}^{(2)}_{nm} = c_{no}(-\nabla^2\delta^o_m - E^o{}_m) \equiv c_{no}\Delta^o{}_m$

- Considering a different parameterisation: $\tilde{\mathcal{S}}^{(2)}_{nm} = \frac{\delta\varphi^o}{\delta\tilde{\varphi}^n}\mathcal{S}^{(2)}_{op}\frac{\delta\varphi^p}{\delta\tilde{\varphi}^m} + \frac{\delta\varphi^o}{\delta\tilde{\varphi}^n\delta\tilde{\varphi}^m}\mathcal{S}^{(1)}_o$

- The second term is proportional to the equation of motion and is the origin of parameterisation dependence.

- Gauge dependence has the same origin since only the on shell hessian is guaranteed to be gauge invariant (Benedetti 2011; KF 2015) .

Generating function for observables

- Generating function:

$$e^{W(\lambda_J,\kappa_J)} = V^{-1}_{\mathrm{diff},\Lambda}\int\prod_a \frac{d\phi^a}{(2\pi)^{1/2}}\sqrt{|\det\, C^{\Lambda}_{ab}(\phi)|}\,\exp\left\{-(\lambda_J+\delta_\Lambda\lambda)\int d^Dx\sqrt{g}\right.$$
$$\left.+(\kappa_J+\delta_\Lambda\kappa)\int d^Dx\sqrt{g}R+\delta_\Lambda S[\phi]\right\}$$

- Observables obtained by taking derivatives with respect to couplings:

$$\left\langle\int d^Dx\sqrt{g}\right\rangle = -W^{(1,0)}(\lambda_0,\kappa_0)\,,\quad \left\langle\int d^Dx\sqrt{g}R\right\rangle = W^{(0,1)}(\lambda_0,\kappa_0)\ \ etc.$$

- Derive RG flow from: $$\frac{\partial}{\partial\Lambda}W(\lambda_J,\kappa_J)=0$$

$$\boxed{\kappa_0=\frac{1}{16\pi G_0}\qquad \lambda_0=\frac{\bar{\lambda}_0}{8\pi G_0}}$$

One-loop flow equation

- Perturbation theory around a saddle point: $\phi^a = \bar{\phi}^a(\kappa_J, \lambda_J) + \delta\phi^a$

- Saddle point geometry dependent on the couplings:

$$R_{\mu\nu}(\bar{\phi}) = g_{\mu\nu}(\bar{\phi}) \frac{1}{D-2} \frac{\lambda_J}{\kappa_J}$$

- Gauge and parameterisation independent

$$-W(\lambda_J, \kappa_J) = S_\Lambda[\bar{\phi}] + \frac{1}{2}\mathrm{Tr}_2 \log(\Delta_2/\mu^2) - \mathrm{Tr}_1' \log \Delta_1/\mu^2 + \log \Omega(\mu)$$

- Last term is the contribution of Killing vector diffeomorphisms which are left out of the vector trace (see e.g. Volkov and Wipf '00).

$$\Delta_1 \epsilon_\mu = \left(-\nabla^2 - \frac{R}{D}\right)\epsilon_\mu \,, \qquad \Delta_2 h_{\mu\nu} = -\nabla^2 h_{\mu\nu} - 2R_\mu{}^\rho{}_\nu{}^\sigma h_{\rho\sigma} \,.$$

One-loop flow equation

- Proper-time regulator implemented as a modification of the measure
- One-loop flow equation: $\Lambda\partial_\Lambda S_\Lambda = \mathrm{Tr}_2[e^{-\Delta_2/\Lambda^2}] - 2\mathrm{Tr}_1[e^{-\Delta_1/\Lambda^2}]$
- Heat kernel expansion:

$$\Lambda\partial_\Lambda S_\Lambda = \Lambda^D \frac{N_g}{(4\pi)^{\frac{D}{2}}} \int d^D x\sqrt{g} + \frac{1}{6}\frac{(N_g - 18)}{(4\pi)^{\frac{D}{2}}}\Lambda^{D-2}\int d^D x\sqrt{g}R + \ldots$$

$$N_g \equiv \frac{D(D-3)}{2}$$ Number polarisations of the graviton

- Beta function for the gravitational coupling:

$$\beta_G = (D-2)G - b\,G^2\,, \qquad b = \frac{1}{6}\frac{16\pi}{(4\pi)^{\frac{D}{2}}}(18 - N_g)$$

- Agrees with the previous gauge independent result (KF 2015)

Amplitudes

- On spacetime manifolds with boundaries we can consider amplitudes:

$$\langle \phi_1 | \phi_2 \rangle = \mathcal{Z}[\phi_1, \phi_2]$$

- We need to provide diffeomorphism invariant boundary conditions. Generically there is a lack of boundary conditions in quantum gravity which are diffeomorphism invariant and lead to a well defined heat kernel.

- On boundaries with extrinsic curvature: $K_{ij} = \frac{1}{D-1} K \, \gamma_{ij} \, , \qquad \partial_i K = 0 \, ,$

Moss and Silva '97 have found suitable boundary conditions.

$$\begin{aligned} h_{in} = 0 &= \epsilon_n \\ \dot{\epsilon}_i - K_i^j \epsilon_j &= 0 \\ \dot{h}_{nn} + K h_{nn} - 2K^{ij} h_{ij} &= 0 \\ \dot{h}_{ij} - K_{ij} h_{nn} &= 0 \end{aligned}$$

$$\delta\phi^a = h_{\mu\nu}(x)$$

Amplitudes

- Results can be generalised to manifolds with two disjoint boundaries with the addition of the Gibbons-Hawking-York term in the action.

$$e^{W(\lambda_J,\kappa_J,\lambda_J^{\Sigma_1},\lambda_J^{\Sigma_2})} = V^{-1}_{\mathrm{diff},\Lambda}\int\prod_a \frac{d\phi^a}{(2\pi)^{1/2}}\sqrt{\det C^{\Lambda}_{ab}(\phi)}\,\exp\Big\{-(\lambda_J+\delta_\Lambda\lambda)\int d^Dx\sqrt{g}$$
$$+(\kappa_J+\delta_\Lambda\kappa)\left(\int d^Dx\sqrt{g}R + 2\int_{\Sigma_1} d^{D-1}y\sqrt{\gamma}K + 2\int_{\Sigma_2} d^{D-1}y\sqrt{\gamma}K\right)$$
$$-(\lambda_J^{\Sigma_1}+\delta_\Lambda\lambda^{\Sigma_1})\int_{\Sigma_1} d^Dy\sqrt{\gamma} - (\lambda_J^{\Sigma_2}+\delta_\Lambda\lambda^{\Sigma_2})\int_{\Sigma_2} d^Dy\sqrt{\gamma} + \delta_\Lambda S[\phi]\Big\}$$

- This construction requires that there is only one Newton's constant for the bulk and boundary terms.

- Saddle point boundary geometry:

$$K_{\Sigma_{1,2}} = \frac{D-1}{D-2}\frac{\lambda_J^{\Sigma_{1,2}}}{2\kappa_J}$$

Amplitudes

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$$\boxed{+(\kappa_J+\delta_\Lambda\kappa)\left(\int d^Dx\sqrt{g}R+2\int_{\Sigma_1}d^{D-1}y\sqrt{\gamma}K+2\int_{\Sigma_2}d^{D-1}y\sqrt{\gamma}K\right)}$$
$$\left.-(\lambda_J^{\Sigma_1}+\delta_\Lambda\lambda^{\Sigma_1})\int_{\Sigma_1}d^Dy\sqrt{\gamma}-(\lambda_J^{\Sigma_2}+\delta_\Lambda\lambda^{\Sigma_2})\int_{\Sigma_2}d^Dy\sqrt{\gamma}+\delta_\Lambda S[\phi]\right\}$$

- This construction requires that there is only one Newton's constant for the bulk and boundary terms. Otherwise the action does not have a well defined variational principle (Hawking and Gibbons 1977) and the functional integral doesn't have the composition properties of an amplitude (Hawking 1980).

- All previous calculations have found this is not possible after renormalisation. However diffeomorphism invariance has been broken either by the action or the boundary conditions (or both). Jacobson and Satz (2014) showed the balance can be achieved on shell in four dimensions.

- Here we preserve diffeomorphism invariance…

Amplitudes

- Flow equation derived from:

$$\frac{\partial}{\partial \Lambda} W(\lambda_J, \kappa_J, \lambda_J^{\Sigma_1}, \lambda_J^{\Sigma_2}) = 0$$

- The one loop flow equation takes the same form but now the boundary terms are generated:

$$\Lambda \partial_\Lambda S^\Sigma_\Lambda = \frac{1}{(4\pi)^{\frac{D}{2}}} \int_\Sigma d^{D-1}y \sqrt{\gamma} \left(\frac{\sqrt{\pi}}{2} \frac{1}{2} (D-4)(D-3) \Lambda^{D-1} + \frac{1}{6} (N_g - 18) \Lambda^{D-2} \cdot 2\, K \right) + \ldots$$

- Bulk and boundary terms are renormalised preserving the required balance!

- Universal result near two dimensions:

$$\beta_G = \varepsilon G - \frac{38}{3} G^2 \qquad\qquad D = 2 + \varepsilon$$

Summary

- Gauge and parameterisation dependent beta functions come from looking at correlation functions (even if we take care of the measure).
- This prevents a direct physical interpretation of fixed points.
- We can avoid these problems by looking at observables.
- At one-loop three important problems are solved:
  - Gauge independence
  - Parameterisation independence
  - Bulk/boundary balance is preserved