On gauge and field-parametrization dependence in quantum gravity

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Nobuyoshi Ohta (Kindai University) and Roberto Percacci (SISSA).
Gauges and functional measures in quantum gravity I: Einstein theory, JHEP **1606**, 115 (2016), [arXiv:1605.00454 [hep-th]].

Outline

- The core of the asymptotic safety scenario: existence of interacting fixed points in the RG flow;
- This requires the computation of beta functions within some truncation;
- Nevertheless, the standard QFT quantization of gravity is constructed upon several ambiguities *and* beta functions are, in general, off-shell quantities;
- Hence, it is expected that the resulting beta functions depend on these ambiguities;
- Immediate question: Can we play with these ambiguities in such a way that the fixed point disappears?

There two main sources of ambiguities that should be fixed:

- Gauge;
- Parametrization of quantum fluctuations.

To illustrate how different choices of gauges and/or parametrizations can lead to different beta functions and therefore different fixed points, let us consider the following simple example:

- We employ the background field method with the following split,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} + \omega h_{\mu\lambda} h^{\lambda}{}_{\nu} \,, \tag{1}$$

  with ω a free parameter;
- The standard gauge-fixing is employed, namely,

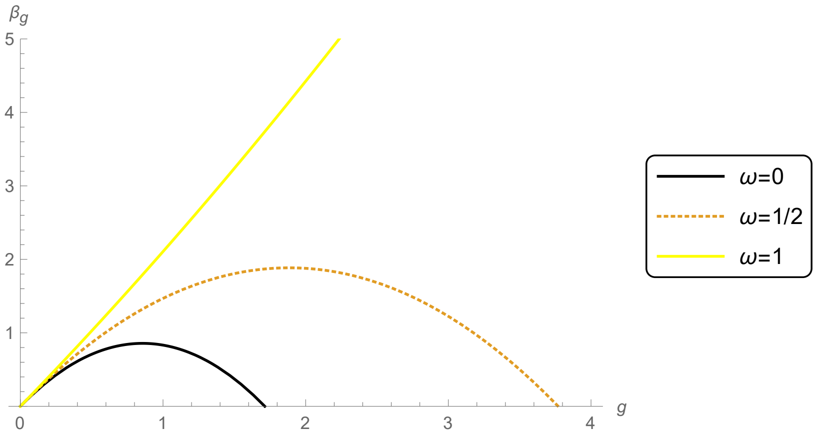
$$S_{gf} = \frac{Z}{2\alpha} \int d^d x \sqrt{\bar{g}} \left(\bar{\nabla}_{\lambda} h^{\lambda}{}_{\mu} - \frac{1+\beta}{d} \bar{\nabla}_{\mu} h \right)^2 \,, \tag{2}$$

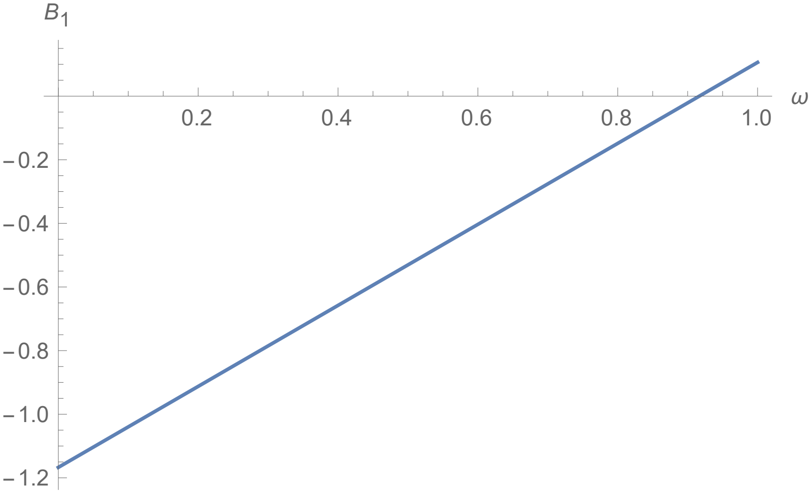
  and Feynman-de Donder gauge is chosen, $\alpha = 1$ and $\beta = d/2 - 1$;
- We put the cosmological constant to zero for simplicity;
- Compute the beta function for Newton's constant using the optmized cutoff at one-loop order.

The expression for the beta function for the dimensionless Newton's constant at one-loop order is written as

$$\beta_g = (d-2)g + B_1 g^2 \,. \tag{3}$$

If $B_1 < 0$ then we have a non-Gaussian fixed point for a positive value g^*. However, if $B_1 > 0$ then the non-trivial fixed point exists for a negative value of g.


β_g
5
4
3
2
1
0
1
2
3
4
g
ω=0
ω=1/2
ω=1


B_1
0.2
0.4
0.6
0.8
1.0
ω
−0.2
−0.4
−0.6
−0.8
−1.0
−1.2

- Thus, it is evident from the previous plot that an appropriate fine tuning of ω renders a positive coefficient B_1.
- In particular, the choice $\omega = 1$ is of particular interest because at the order treated here, it corresponds to a linear split of the *inverse* metric, namely,

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} \,. \tag{4}$$

- This result suggests that controlling gauge and parametrization dependence of the beta functions is important in the task of establishing the existence of fixed points;
- Some results on this direction were already established in the context of asymptotic safety [Benedetti, Falls, Gies, Knorr, Lippoldt,...]
- Older investigations analyzed the gauge and parametrization dependence of the divergences coefficients [Pronin, Kazakov, Kallosh, Buchbinder, Shapiro, Odintsov...]

Recently, we analyzed the divergences coefficients for Einstein theory at one-loop order within a two-parameter family of field parametrization and a two-parameter family of gauges.

Let us consider a metric $g_{\mu\nu}$ and take as our fundamental variable the following "densitizied metric",

$$\gamma_{\mu\nu} = g_{\mu\nu}\left(\sqrt{\det g_{\mu\nu}}\right)^{w} . \tag{5}$$

This relation can be inverted leading to

$$g_{\mu\nu} = \gamma_{\mu\nu}\left(\det \gamma_{\mu\nu}\right)^{m} . \tag{6}$$

The inversion is possible provided that $m \neq -1/d$.

We want to evaluate the following object

$$\int [\mathcal{D}\gamma_{\mu\nu}]\, \mathrm{e}^{-S(g(\gamma))} , \tag{7}$$

at one-loop order. Of course, we could proceed by considering $\gamma^{\mu\nu}$ instead.

In order to employ the background field method, we have to split the quantum field $\gamma_{\mu\nu}$ (or $\gamma^{\mu\nu}$) as a background part plus a fluctuation. However, the way we parametrize the fluctuation is not unique. We take the following options,

$$\begin{aligned}
\gamma_{\mu\nu} &= \bar{\gamma}_{\mu\nu} + \hat{h}_{\mu\nu} \\
\gamma_{\mu\nu} &= \bar{\gamma}_{\mu\lambda}(\mathrm{e}^{\hat{h}})^{\lambda}{}_{\nu} = \bar{\gamma}_{\mu\nu} + \hat{h}_{\mu\nu} + \frac{1}{2}\hat{h}_{\mu\lambda}\hat{h}^{\lambda}_{\nu} + \ldots \\
\gamma^{\mu\nu} &= \bar{\gamma}^{\mu\nu} - \hat{h}^{\mu\nu} \\
\gamma^{\mu\nu} &= \bar{\gamma}^{\mu\lambda}(\mathrm{e}^{-\hat{h}})_{\lambda}{}^{\nu} = \bar{\gamma}^{\mu\nu} - \hat{h}^{\mu\nu} + \frac{1}{2}\hat{h}^{\mu\lambda}\hat{h}^{\nu}_{\lambda} + \ldots
\end{aligned} \tag{8}$$

In terms of the metric g,

$$\begin{aligned}
g_{\mu\nu} &= \bar{g}_{\mu\nu} + h_{\mu\nu} + m\bar{g}_{\mu\nu}h + mhh_{\mu\nu} + \frac{1}{2}m\bar{g}_{\mu\nu}(mh^2 - h^{\alpha\beta}h_{\alpha\beta}) \\
g_{\mu\nu} &= \bar{g}_{\mu\nu} + h_{\mu\nu} + m\bar{g}_{\mu\nu}h + \frac{1}{2}h_{\mu\rho}h^{\rho}{}_{\nu} + mhh_{\mu\nu} + \frac{1}{2}m\bar{g}_{\mu\nu}h^2 \\
g_{\mu\nu} &= \bar{g}_{\mu\nu} + h_{\mu\nu} + m\bar{g}_{\mu\nu}h + h_{\mu\rho}h^{\rho}{}_{\nu} + mhh_{\mu\nu} + \frac{1}{2}m\bar{g}_{\mu\nu}(mh^2 + h^{\alpha\beta}h_{\alpha\beta}) \\
g_{\mu\nu} &= \bar{g}_{\mu\nu} + h_{\mu\nu} + m\bar{g}_{\mu\nu}h + \frac{1}{2}h_{\mu\rho}h^{\rho}{}_{\nu} + mhh_{\mu\nu} + \frac{1}{2}m\bar{g}_{\mu\nu}h^2
\end{aligned}$$

All these possible parametrizations can be obtained out of a two-parameter family of parametrizations (interpolating parametrization) up to the quadratic order, namely,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g^{(1)}_{\mu\nu} + \delta g^{(2)}_{\mu\nu} + \dots , \tag{9}$$

with

$$\begin{aligned} \delta g^{(1)}_{\mu\nu} &= h_{\mu\nu} + m\bar{g}_{\mu\nu} h \\ \delta g^{(2)}_{\mu\nu} &= \omega h_{\mu\rho} h^{\rho}{}_{\nu} + mhh_{\mu\nu} + m\left(\omega - \frac{1}{2}\right)\bar{g}_{\mu\nu} h^{\alpha\beta} h_{\alpha\beta} + \frac{1}{2} m^2 \bar{g}_{\mu\nu} h^2 . \end{aligned}$$

In particular,
$\omega = 0$ linear expansion of metric
$\omega = 1/2$ exponential expansion
$\omega = 1$ linear expansion of inverse metric

We do not attribute any particular value to m for now.

For the gauge-fixing term, we employ the standard two-parameter generalized harmonic gauge,

$$S_{gf} = \frac{Z}{2a}\int d^dx\sqrt{\bar{g}}\left(\bar{\nabla}_\lambda h^\lambda{}_\mu - \frac{1+\bar{b}}{d}\bar{\nabla}_\mu h\right)^2 , \tag{10}$$

This entails the following Faddeev-Popov ghost term

$$\begin{aligned} S_{\rm gh} &= -\int d^dx\sqrt{\bar{g}}\;\bar{C}^\mu\frac{\partial F_\mu}{\partial\hat{h}_{\alpha\beta}}\mathcal{L}_C\gamma_{\alpha\beta} \\ &= -\int d^dx\sqrt{\bar{g}}\;\bar{C}^\mu\left[\delta^\nu_\mu\bar{\nabla}^2 + \left(1-2\frac{1+b}{d}\right)\bar{\nabla}_\mu\bar{\nabla}^\nu + \frac{\bar{R}}{d}\delta^\nu_\mu\right]C_\nu , \end{aligned} \tag{11}$$

with $\bar{b} = (1+md)b$.

We choose the Einstein-Hilbert action for explicit computations,

$$\begin{aligned} S_{EH}(g(\gamma)) &= Z_N \int d^d x \sqrt{g}(2\Lambda - g^{\mu\nu} R_{\mu\nu}(g)) \\ &= Z_N \int d^d x \, (\det\gamma)^{\frac{1+dm}{2}} \left(2\Lambda - (\det\gamma)^{-m} \gamma^{\mu\nu} R_{\mu\nu}(g(\gamma))\right) . \end{aligned}$$

with $Z_N = 1/(16\pi G)$.

The complete action is written as

$$\Sigma = S_{EH} + S_{gf} + S_{\mathrm{gh}} \, . \tag{12}$$

In order to compute the one-loop divergences, we expand Σ up to quadratic order in $h_{\mu\nu}$.

Hence, at the quadratic level,

$$
\begin{aligned}
\Sigma^{(2)} &= \frac{Z_N}{2}\int d^dx\,\sqrt{\bar{g}}\Bigg\{\frac{1}{2}h^{\mathrm{TT}}_{\mu\nu}\left[-\bar{\nabla}^2+\frac{2\bar{R}}{d(d-1)}-2(1+dm)(1-2\omega)\left(\Lambda-\frac{d-2}{2d}\bar{R}\right)\right]h^{\mathrm{TT}\mu\nu}\\
&+ \frac{1}{a}\hat{\xi}_\mu\left[-\bar{\nabla}^2-\frac{\bar{R}}{d}-2a(1+dm)(1-2\omega)\left(\Lambda-\frac{d-2}{2d}\bar{R}\right)\right]\hat{\xi}^\mu\\
&- \frac{d-1}{2d}\hat{\sigma}\left[\frac{a(d-2)-2(d-1)}{da}(-\bar{\nabla}^2)+\frac{2\bar{R}}{da}+2(1+dm)(1-2\omega)\left(\Lambda-\frac{d-2}{2d}\bar{R}\right)\right]\hat{\sigma}\\
&- \frac{(d-1)(1+dm)\big((d-2)a-2b\big)}{d^2a}\hat{\sigma}\sqrt{-\bar{\nabla}^2}\sqrt{-\bar{\nabla}^2-\frac{\bar{R}}{d-1}}h\\
&- h\frac{(1+dm)^2}{2d^2a}\Bigg[\big((d-1)(d-2)a-2b^2\big)(-\bar{\nabla}^2)-(d-2)a\bar{R}\\
&- da\left(d-2\frac{1-2\omega}{1+dm}\right)\left(\Lambda-\frac{d-2}{2d}\bar{R}\right)\Bigg]h\Bigg\}-\int d^dx\,\sqrt{\bar{g}}\Bigg[\bar{C}^{\mathrm{T}\mu}\left(\bar{\nabla}^2+\frac{\bar{R}}{d}\right)C^{\mathrm{T}}_\mu\\
&+ 2\frac{d-1-b}{d}\bar{C}'^L\left(\bar{\nabla}^2+\frac{\bar{R}}{d-1-b}\right)C'^L\Bigg]\,,
\end{aligned}
$$

where the York decomposition was employed and a maximally symmetric background was chosen.

- The path integral can be formally computed and on-shell it is independent of gauge parameters and field parametrization of course.
- One also notices that the exponential parametrization, namely, $\omega = 1/2$ renders an *almost* on-shell condition.
- This suggests that the exponential parametrization minimizes the gauge dependence of one-loop off-shell quantities.
- The sector which is not "completely" on-shell in the exponential parametrization is the $h - h$ part.
- Suggestion: Tune the gauge parameter b in such a way that $h = 0$. This corresponds to $b \to \infty$.

One-loop divergences

Now we compute the one-loop divergences in this two-parameter family of fluctuation parametrizations *and* two-parameter family of gauge-fixings.

- The one-loop effective action contains a divergent part

$$\Gamma_k = \int d^d x \sqrt{\bar{g}} \left[\frac{A_1}{16\pi d} k^d + \frac{B_1}{16\pi(d-2)} k^{d-2} \bar{R} + \frac{C_1}{d-4} k^{d-4} \bar{R}^2 + \dots \right] ,$$

  where k stands for a cutoff and we introduced a reference mass scale μ. In $d = 4$, the last term is replaced by $C_1 \log(k/\mu) \bar{R}^2$.

- The coefficients A_1, B_1, C_1 depend on d, m, ω, a, b and $\tilde{\Lambda} = k^{-2}\Lambda$.
- In the present computation, we have employed the optimized cutoff, namely, we introduced the regulator $R_k(\bar{\nabla}^2) = (k^2 + \bar{\nabla}^2)\theta(k^2 + \bar{\nabla}^2)$ through the replacement $-\bar{\nabla}^2 \to P_k(\bar{\nabla}^2) = -\bar{\nabla}^2 + R_k(\bar{\nabla}^2)$.

- In the general case, the coefficients A_1, B_1, C_1 are extremely complicated.
- We take some partial choices to draw some concrete conclusions.

Results

In order to simplify the analysis, we we mainly focus on $d = 4$ and set $\tilde{\Lambda} = 0$ by hand.

A1 coefficient

- In this case, setting $\tilde{\Lambda} = 0$ already entails an universal result for A_1 for general d, namely,

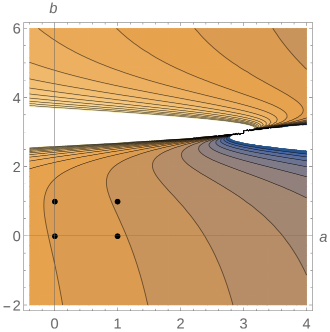
$$A_1 = \frac{16\pi(d-3)}{(4\pi)^{d/2}\Gamma(d/2)} \, .$$

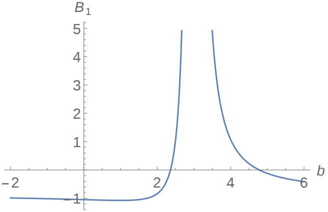
- This coefficient can be associated with the number of the degrees of freedom of the theory. [Falls]

$B1$ coefficient - Fixing the parametrization

To begin with, we choose $\omega = m = 0$. We obtain

$$B_1 = \frac{a\left(-6b^2 + 36b - 62\right) - 3\left(7b^2 - 50b + 79\right)}{8\pi(b-3)^2}\,.$$

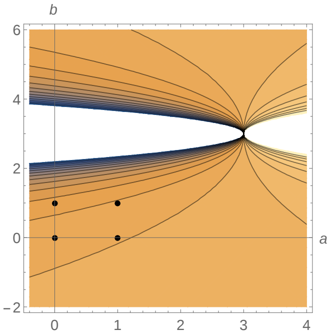


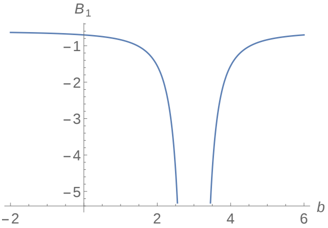


Now, we choose $\omega = 1/2$,

$$B_1 = -\frac{159 - 8a - 90b + 15b^2}{8\pi(b-3)^2}\,.$$

In this case, B_1 is independent of m.





It is valid to emphasize that the limit $b \to \infty$ automatically makes B_1 independent of a, namely,

$$B_1 = -\frac{15}{8\pi}\,.$$

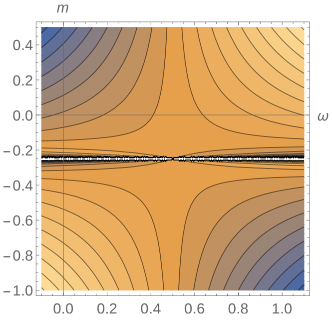
(These properties also hold for C_1).

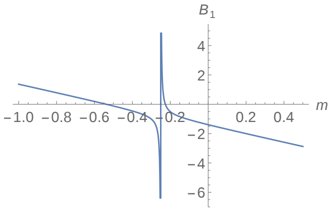
In fact, there is a more general statement: in the exponential parametrization and for the partial gauge-fixing $b \to \infty$, B_1 and C_1 become independent of a, m and $\tilde{\Lambda}$ for arbitrary d. In particular, B_1 is expressed as

$$B_1 = \frac{d^5 - 4d^4 - 9d^3 - 48d^2 + 60d + 24}{(4\pi)^{d/2-1}3(d-1)d^2\Gamma\left(\frac{d}{2}\right)}\,.$$

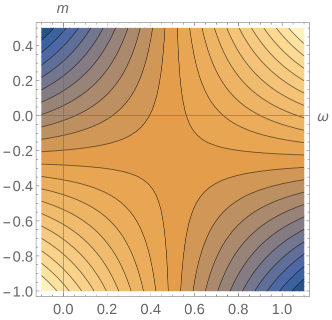
$B1$ coefficient - Fixing the gauge

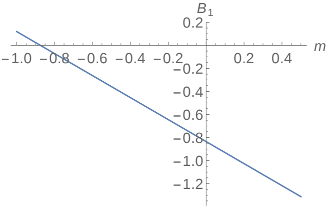
To cut the 4-parameter space in a different way, we fix the gauge parameters and see how B_1 depends on the parameters ω and m. In Feynman-de Donder gauge, $a = b = 1$, one obtains





Now, employing the (unimodular) gauge $b \to \infty$ and $a = 0$,





"Duality"

The plots of B_1 for a fixed gauge and free field parametrization parameters, exhibit a reflection symmetry. In fact, one can prove that this is not a coincidence of the particular gauges chosen here, but is a general property, namely,

$$B_1(\omega, m) = B_1\left(1-\omega, -m-\frac{2}{d}\right) .$$

This discrete symmetry also holds for A_1 and C_1.

- For the special particular case $\omega = 0$ and $m = 0$ (standard linear parametrization of the metric), the reflection symmetry is obtained by the simultaneous choice $\omega = 1$ and $m = -1/2$ (in $d = 4$), which corresponds to the expansion of a densitizied inverse metric in a linear way, where the density factor is $\sqrt{g}$.
- Hence, if we proceed with computations of the beta functions using the linear split of the metric, we will obtain the same results as doing the same computations with the linear split of $g^{\mu\nu}\sqrt{g}$.
- Using some absence of anomaly criteria, Fujikawa was able to specify a certain value of m for a given dimension d when one takes the metric as the fundamental variable of the theory or a value m' if the inverse metric is chosen as the fundamental variable instead. It is possible to show that m and m' are "duality" related.

Conclusions

- Field parametrization and gauge dependence are present in the computations of off-shell beta functions. We should try to control them in order to characterize unambiguously the fixed point.
- The exponential parametrization seems to be more suitable to reduce gauge dependence. (Plus its other advantages [Percacci, Vacca, Ohta, Falls, Nink, Demmel,...]
- The combination of the exponential parametrization and unimodular gauge (imposition of $h = 0$ strongly - $b \to \infty$) produces a decoupling of the cosmological constant of Newton's constant beta function.
- A given choice of parametrization for the metric has a dual description in terms of the inverse metric.
- The results here reported, in particular the duality, remain valid in higher derivative theories. [To appear]

Thank You!