Background independence

and covariant RG flow

Gian Paolo Vacca

INFN - Bologna

Istituto Nazionale di Fisica Nucleare

ERG 2016 ICTP

September 21, 2016 , Miramare, TS

Outline

- Introduction
- Splitting Ward Identities
- Standard QFT
- EAA and RG flow
- Conclusions

Introduction

Background fields are often introduced in QFT computations with effective actions for several reasons

- Gauge theories
    - For gauge fixing and to implement the background field method which permits to have gauge invariance
- Gravity
    - Apart because is a gauge theory, background is also needed also to define the notion of a scale
- Non linear sigma models
    - To have a covariant (under field reparameterization) geometrical approach.

Gauge theories

The construction of an RG flow in presence of a background field in the above problems induces a double field dependence

$$\Gamma_k[\varphi, \bar{\xi}]$$

This has been encoutered in gauge theories using the background field method

The RG flowing effective average action depends on two gauge fields

- Slavnov-Taylor identities are modified because of the IR regulator.
- Dependence on the background field: (modified) Splitting Ward Identities
  Very complicated! They encode the fact that the dependence is just in one field.

Wetterich, Reuter, Freire, Pawlowski, Liitm, Morris …

The two identities are related: formally gauge degrees of freedom could be disantangled by non local field reparameterizations

Gravity: asymptotic safety

Quantum gravity: the presence of the background dependence is a serious obstacle for the asymptotic safety program

$$\Gamma_k[\bar{g}, g] \qquad\qquad g = \langle h \rangle$$

- Physics should not depend on the background employed for the computation!
- Double gauge field dependence complicates enormously the analysis!
  New relevant non physical operators can appear.
  Difficulty in constructing reliable truncations!

$$\Gamma[\bar{g}, g] = \Gamma[\bar{g}, g]_{g=\bar{g}} + \Delta\Gamma[\bar{g}, g]$$

- Single metric truncation (background)

- Expansion in average fluctuations: vertex expansion

Asymptotic safety looks to be there, but several disagreements.
Approximation is not under control!

Non linear sigma models

They are models where the field configurations belong to a non trivial manifold

One can compute S-matrix elements in perturbation theory as an effective theory using any parameterization of the fields

This means that one can pick up a point on the manifold (background) and study fluctuations around it.

Some analysis (off-shell) for NLSM have been based on geometric approach, actually mainly at background level, but with an intrinsically double field dependence in the effective action.

Honerkamp,Vilkovisky, DeWitt, Howe, Stelle…
Codello, Percacci, Zanusso, Wipf,…

- Question: can one find a covariant and background independent approach?

Next slides…

Essentially avoid the real complicated problem

No gauge theories

No complicated background-fluctuation splitting

Point of view

Are there special solution to the modified splitting Ward Identities
if for any reason we introduce a background?

The off-shell effective action is a point in theory space: it would be natural to have also a geometric description, i.e. be also a scalar under field reparameterization. In practice a recipe to compute it in any field frame consistently.

We shall then consider this problem for a generic scalar QFT, for both standard effective action and for the flowing effective average action.

Based on arXiv:1607.03053, arXiv:1607.07074

M. Safari, G.P.V.

(Modified) Splitting Ward Identities

Functional integral definition for the Effective (average) action

Total field ϕ splitted in background φ and quantum fluctuation ξ : $\phi(\varphi, \xi)$

Covariance under field reparameterizaion: quantum fluctuation should be a vector

$$e^{-\Gamma_k} = \int D\phi\, \mu(\phi)\, e^{-S[\phi]+\Gamma_{k;i}(\xi-\bar{\xi})^i - S_k[\varphi, \xi-\bar{\xi}]}$$

$$\bar{\xi} = \langle \xi \rangle$$

IR regulator

$$S_k[\varphi, \xi] = \tfrac{1}{2}\, \xi \cdot R_k(\varphi) \cdot \xi$$

Background independence: mspWI for $\Gamma_k[\varphi, \bar{\xi}]$

$$0 = \Gamma_{,i} + \Gamma_{;j} \langle \xi^j_{,i} \rangle - \tfrac{1}{2} \langle \left[(\xi-\bar{\xi})^m R_{mn} (\xi-\bar{\xi})^n \right]_{,i} \rangle = \Gamma_{,i} + \Gamma_{;j} \langle \xi^j_{,i} \rangle - \tfrac{1}{2} G^{mn} (R_{nm})_{,i} - G^{np} R_{pm} \langle \xi^m_{;i} \rangle_{;n}$$

Compact Notation: $\frac{\delta X}{\delta \varphi^i} \equiv X_{,i}$ $\quad \frac{\delta X}{\delta \bar{\xi}^i} \equiv X_{;i}$ $\quad G_{mn} = \Gamma_{;mn} + R_{mn}$ $\quad G^{mn} G_{nl} = \delta^m_l$

Flow eq. $\dot{\Gamma}_k = \tfrac{1}{2} G^{mn} \dot{R}_{mn}$

Wetterich

Compatibility: $\dot{\mathcal{N}}_i = -\tfrac{1}{2} (G \dot{R} G)^{qp} (\mathcal{N}_i)_{;pq}$

Standard Effective Action:

Search for a special solution of $\boxed{\Gamma_{,i} + \Gamma_{;j}\langle \xi^j_{,i}\rangle = 0}$

The last factor is in general very non local.
Starting from a local algebraic splitting at the bare action level…

$$\xi_{,\,i} = \left(a_{0i} + a_{1i}\xi + \sum_n a_{ni}(\xi \cdots \xi) \right) \longrightarrow \quad \langle \xi_{,\,i} \rangle = \left(a_{0i} + a_{1i}\bar{\xi} + F(\Gamma) \right)$$

one gets a complicated non local and non linear higher derivative PDE for Γ.

Then the definition of $\phi(\varphi, \xi)$ would be implicit, through $\Gamma[\phi(\varphi, \xi)]$

To avoid this and look for a dynamical independent split we look at the simple case

$$\boxed{\xi^k_{,i} = \alpha^k_i(\varphi) - \beta^k_{ij}(\varphi)\,\xi^j} \longrightarrow \quad \Gamma_{,i} + \Gamma_{;k}\,\bar{\xi}^k_{,i} = 0$$

This admits solutions if (integrability Froebenius conditions)

$$\boxed{\left[\frac{\partial}{\partial \varphi_i} + \bar{\xi}^k_{,i}\frac{\partial}{\partial \bar{\xi}_k} \,,\, \frac{\partial}{\partial \varphi_j} + \bar{\xi}^l_{,j}\frac{\partial}{\partial \bar{\xi}_l} \right] = 0}$$

spWI: a particular solution

The Froebenius condition are written as zero torsion and curvature Cartan equation

$$d\alpha^k + \beta^k_j \wedge \alpha^j = 0, \qquad d\beta^k_j + \beta^k_l \wedge \beta^l_j = 0.$$

tensor values one-forms

solved by $\beta = U^{-1} dU$, $\alpha = -U^{-1} df$

$$\boxed{\beta^k_{ij} = (U^{-1})^k_a \partial_i U^a_j, \qquad \alpha^k_i = -(U^{-1})^k_a \partial_i f^a}$$

This gives also $\quad U d\xi = -dU\xi - df$

$$\xi^k(\varphi,\phi) = -(U^{-1}(\varphi))^k_a \left(f^a(\varphi) - (g^{-1})^a(\phi)\right) \qquad \text{or} \qquad \phi^k(\varphi,\xi) = g^k\left[f(\varphi) + U(\varphi)\xi\right]$$

If f is not singular one can always write

$$\boxed{\phi^i(\varphi,\xi) = \left[f^{-1}\left(f(\varphi) + U\xi\right)\right]^i}$$

This splitting is an <u>exponential map</u> if one redefines ξ so that $U^a_i(\varphi) = f^a_{,i}(\varphi)$

$$[Exp_\varphi\, \xi]^i_\Gamma \equiv \varphi^i + \xi^i - \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma^i_{i_1 i_2 \ldots i_n}(\varphi)\, \xi^{i_1} \cdots \xi^{i_n} = \left[f^{-1}\left(f(\varphi) + \partial f \xi\right)\right]^i$$

based on the flat <u>connection</u> $\quad \Gamma^k_{ij} = \beta^k_{ij} = (f^{-1})^k_{,b}\, f^b_{,ij} = -(f^{-1})^{\,k}_{,ab}\; f^a_{,i}\; f^b_{,j}$

Covariance

We have therefore obtained an effective action which is

- covariant: field obtained by a geodesic construction from
  a base point (background) and
  a quantum field (vector)

- dependent on a total field (background independent) thanks to the flat connection.
  It is possible to write all in terms of the total field.

$$e^{-\Gamma[\bar{\phi}]} = \int D\phi\; \mu(\phi)\, e^{-S[\phi] + \frac{\delta\Gamma}{\delta\bar{\phi}^i}\left[(\partial f)^{-1}(f(\bar{\phi}))\right]^i_a \left[f(\phi) - f(\bar{\phi})\right]^a}$$

$\Gamma[\bar{\phi}]$ is clearly a scalar under field reparameterization and depends on f, since the parameterization is chosen independently on the dynamics.

$$\varphi \to h(\varphi), \quad \xi \to \partial h\, \xi, \quad f \to f \circ h^{-1}, \quad U \to U(\partial h)^{-1} \longrightarrow \quad \phi \to h(\phi)$$

This family of splitting can be obtained by a linear splitting and a change of variable!

One loop effective action

How covariance and background independence are maintained at one loop?

$$\phi^i(\varphi,\xi) = \left[f^{-1}\left(f(\varphi)+U\xi\right)\right]^i \qquad \phi^p_{;i} = (f^{-1})^p_{,a}(f(\phi))U^a_i(\varphi), \qquad \phi^p_{;ij} = (f^{-1})^p_{,ab}(f(\phi))U^a_i(\varphi)U^b_j(\varphi)$$

One loop correction: one need the second fluctuation derivative

$$\begin{aligned} S_{;ij}[\phi(\varphi,\xi)] &= S_{,p}\,(f^{-1})^p_{,mn}\,U^m_i U^n_j + S_{,pq}\,(f^{-1})^p_{,m}\,U^m_i (f^{-1})^q_{,n}\,U^n_j \\ &= \left[S_{,pq} - \Gamma^k_{pq} S_{,k}\right](f^{-1})^p_{,m}\,U^m_i (f^{-1})^q_{,n}\,U^n_j \\ &= \nabla_p \nabla_q S\;\; (f^{-1})^p_{,m}\,U^m_i (f^{-1})^q_{,n}\,U^n_j, \end{aligned}$$

Some factors are cancelled by the Jacobian $\dfrac{\delta\phi^i}{\delta\xi^p} = (f^{-1})^i_{;a}\,U^a_p$

$$\Gamma_f^{1-\text{loop}}[\phi] = S[\phi] + \frac{\hbar}{2}\mathrm{Tr}\log\left[S_{,ij} - \Gamma^k_{ij} S_{,k}\right]$$

Even doing a computation at background level one can reconstruct
the full field dependence (and the dependence at all order in the fluctuations)

Symmetries

When are preserved?

Denote with G all the tensors characterizing the effective action.
Consider a transformation

$$\phi \to \phi'$$

Because of covariance: $\Gamma_{G',f'}[\phi'] = \Gamma_{G,f}[\phi]$

There is an extra dependence in f because of the flat connection

It is a symmetry if: $G' = G \longrightarrow \Gamma_{G,f'}[\phi'] = \Gamma_{G,f}[\phi]$

but also: $\Gamma'^{k}_{ij} = \Gamma^{k}_{ij}$

In the reference frame of zero connection the transformation must be of first order in the fields

$$0 = \delta\Gamma^{k}_{ij}(\phi') = \frac{\partial \phi'^{a}}{\partial \phi^{i}} \frac{\partial \phi'^{b}}{\partial \phi^{j}} \frac{\partial \phi^{k}}{\partial \phi'^{a} \partial \phi'^{b}}.$$

Therefore in this covariant background independent construction we can state

A symmetry is preserved by quantization in the effective action iff it is linearizable

But when a symmetry is linearizable?

A symmetry group acting on a field manifold

Linearizable Symmetries

Consider a field manifold M and a symmetry group H acting on M

The CWZ lemma states:

Coleman,Wess, Zumino

if the group H has a fixed point in M, then at least in a neighbourhood of this point there is a choice of coordinates on which the group H acts linearly

In this frame the fixed point has zero field coordinates

Are there interesting non trivial cases?

The fixed point may lie in the natural frame of interest outside the manifold!
For example at infinity. So we consider

- the possibility to extend the manifold M in order to have a fixed point of H
- the possibility to remove from the manifold M the fixed point under the action of H

In any case the action of the symmetry group is linearizable!

A QFT is instead linearizable requiring more!
The action must be well defined in the linear coordinate system.

Example:
O(N) non linear sigma model has a target manifold S^{N-1} and a non linearizable symmetry.
Adding an extra field to have a cylindrical target manifold $\mathbb{R}\times S^{N-1}$ the manifold can be extendend to have the fixed point.

Possible applications

Note also that one can consider a theory with a non flat target space and use a flat connection for the splitting. In such a case while the covariance and single field dependence are maintained, the non linear symmetry is broken.

O(N) effective theories (Beyond SM extensions with custodial symmetry)

$$\mathcal{L} = \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2}F^2(h)\, g_{\alpha\beta}(\chi)\partial_\mu \chi^\alpha \partial^\mu \chi^\beta - V(h).$$

Setting $F(h)$ to be constant (cylynder topology $\mathbb{R} \times S^{N-1}$)
one has a model which has the same universality class as the non linear model and the action of the symmetry group is linearizable.

Example:
O(N) non linear sigma model has a target manifold S^{N-1} and a non linearizable symmetry. Adding an extra field to have a cylindrical target manifold $\mathbb{R} \times S^{N-1}$ the manifold can be extendend to have the fixed point.

Effective average action

Is it possible to keep the properties of the previous solution in the case of the IR regulated object?

- Consider a <u>k independent splitting</u>

$$\boxed{\Gamma_{,i} + \Gamma_{;j}\langle \xi^j_{,i}\rangle = 0}$$

- Consider an IR regulator compatible

$$\boxed{0 = \frac{1}{2}G^{mn}(R_{nm})_{,i} - G^{np}R_{pm}\langle \xi^m_{;i}\rangle_{;n}}$$

$$\begin{tikzcd} \Gamma_k[\phi] \arrow[r, "\text{reparam.}"] \arrow[d, "\text{flow}"'] & \Gamma'_k[\phi'] \arrow[d] \\ \Gamma_{\bar{k}}[\phi] \arrow[r] & \Gamma'_{\bar{k}}[\phi'] \end{tikzcd}$$

We consider the case where flow and reparametrization do commute

IR regulator

$$\boxed{0 = \frac{1}{2} G^{mn} (R_{nm})_{,i} - G^{np} R_{pm} \langle \xi^m_{,i} \rangle_{;n}} \xrightarrow[\nabla_i = \partial_i + \beta_i]{} \boxed{0 = \frac{1}{2} G^{mn} \nabla_i R_{nm} + \cancel{G^{mp} R_{pn} \langle \nabla_i \xi^n \rangle_{;m}}}$$

- From the already chosen form $\boxed{\xi^k_{,i} = \alpha^k_i(\varphi) - \beta^k_{ij}(\varphi)\, \xi^j} \longrightarrow \nabla_i \xi^n = \alpha^n_i$

- <u>Define</u> the background dependent IR regulator as

$$R_{mn} = U^a_m\, \tilde{R}(-\Box)\, U^a_n = U^a_m U^a_n\, \tilde{R}(-\nabla^2) = \bar{g}_{mn}\, \tilde{R}(-\nabla^2), \qquad \bar{g}_{mn} \equiv U^a_m U^a_n, \quad \nabla_k \bar{g}_{mn} = 0.$$

Then we get $\boxed{\nabla_i R_{mn} = (R_{mn})_{,i} - \beta^k_{im}\, R_{kn} - R_{mk}\, \beta^k_{in} = 0}$ $\qquad \beta^k_{im} = (U^{-1})^k_b (U^b_m)_{,i}$

Therefore the mspWI are satisfied along the flow.

RG flow

The one can define the flow for a covariant and single field dependent Effective Average Action

$$\boxed{\dot{\Gamma}_{G,f}[\bar{\phi}] = \frac{1}{2}\,\mathrm{Tr}\left[\left(\frac{\delta^2\Gamma_{G,f}}{\delta\bar{\xi}\delta\bar{\xi}}[\bar{\phi}] + R(\varphi)\right)^{-1}\dot{R}(\varphi)\right]}$$

$$\frac{\delta^2\Gamma_{G,f}}{\delta\bar{\xi}^i\delta\bar{\xi}^j}[\bar{\phi}] = U_i^m (f^{-1})^p_{;m}\,\nabla_p\nabla_q\Gamma_{G,f}[\bar{\phi}]\,(f^{-1})^q_{;n}\,U_j^n$$

Redistributing the factors one can write a manifest covariant single field eq.

$$\boxed{\dot{\Gamma}_{G,f}[\bar{\phi}] = \frac{1}{2}\,\mathrm{Tr}\left[\left(\nabla\nabla\,\Gamma_{G,f}[\bar{\phi}] + \hat{R}(\bar{\phi})\right)^{-1}\dot{\hat{R}}(\bar{\phi})\right]}$$

having defined

$$(\hat{R})_{ij}(\bar{\phi}) = f^a_{;i}(\bar{\phi})\,\tilde{R}(-\Box)\,f^a_{;j}(\bar{\phi}) \qquad (= f^a_{;i}(\bar{\phi}) f^a_{;j}(\bar{\phi})\,\tilde{R}(-\nabla^2), \quad \nabla_\mu \equiv \partial_\mu + \Gamma_\mu)$$

Regarding the <u>symmetries</u> the same said for $\Gamma[\bar{\phi}]$ can be said for $\Gamma_k[\bar{\phi}]$

requirig invariance for the connection $\phi^p_{;m'}\,\phi^q_{;n'}\,(\Gamma^l_{pq}(\bar{\phi}) - \phi'^r_{;pq}\,\phi^l_{;r'})\phi'^k_{;l} = \Gamma^k_{mn}(\bar{\phi}')$

Simple example

- Single scalar field $\Gamma_k[\phi] = \int \left[\frac{1}{2} J_k(\phi)\, \partial_\mu\phi\, \partial^\mu\phi + V_k(\phi)\right]$

any splitting

$\phi(\varphi, \xi)$

The flow is covariantly transformed: $J'(h(\phi)) = [(h^{-1})'(h(\phi))]^2 J(\phi), \qquad V'(h(\phi)) = V(\phi)$

$$\begin{array}{l} \dot V_k(\phi) = \beta_V[V_k, J_k, \phi] \\ \dot J_k(\phi) = \beta_J[V_k, J_k, \phi] \end{array} \xrightarrow{\phi' = h(\phi)} \begin{array}{l} \dot V'_k(\phi') = \beta_V[V'_k, J'_k, \phi'] \\ \dot J'_k(\phi') = \beta'_J[V'_k, J'_k, \phi'] \end{array}$$

Example: $f(x) = M \log(x/M) \longrightarrow \phi = \varphi e^{\xi/\varphi} = \varphi + \xi + \frac{1}{2\varphi}\xi^2 + \cdots$

A background computation gives the full dependence and a covariant result

Gaussian and non tirivial fixed point in LPA in the linear parameterization are related to a more complicated truncation in another parameterization. Computation in any frame. Clearly the critical exponents are the same.

- O(2) model in polar coordinates: $\mathcal{L} = \frac{1}{2}\, \partial_\mu\rho\partial^\mu\rho + \frac{1}{2}\rho^2\, \partial_\mu\theta\partial^\mu\theta + V(\rho)$

Natural split: $\rho = \sqrt{(\rho_0 + \xi_\rho)^2 + \rho_0^2\, \xi_\theta^2} \quad , \quad \theta = \arctan \frac{\sin\theta_0\, (\rho_0 + \xi_\rho) + \rho_0 \cos\theta_0\, \xi_\theta}{\cos\theta_0\, (\rho_0 + \xi_\rho) - \rho_0 \sin\theta_0\, \xi_\theta}$

$$\rho = \rho_0 + \xi_\rho + \frac{1}{2}\rho_0\, \xi_\theta^2 - \frac{1}{2}\xi_\rho\xi_\theta^2 + \cdots \quad , \quad \theta = \theta_0 + \xi_\theta - \frac{1}{\rho_0}\xi_\rho\xi_\theta + \frac{1}{\rho_0^2}\xi_\rho^2\xi_\theta - \frac{1}{3}\xi_\theta^3 + \cdots$$

A background computation gives the full dependence and a covariant result

Conclusions: not yet….

- In a restricted context: one can construct a covariant and background independent off-shell effective action and functional RG flow for the effective average action

- It can be useful for non gauge theories, like non linear sigma models.

- In more general terms the problem looks very hard
  - gauge theories
  - and in particular for the Asymptotic Safety program for gravity

- On going work also with alternative proposals

Thank you!