Gauge invariant flow equation

Gauge invariant effective action

- Variation yields gravitational field equations
- Generates Maxwell equations for QED
  (+ correction terms)

- Involves only one gauge field

Gauge invariant effective action

- Projected second functional derivative is
  inverse propagator for physical fluctuations
- Generates correlation function for physical fluctuations
- Example : power spectrum of cosmic fluctuations

- Involves only one gauge field

Background field formalism

- Background field formalism employs effective action with two gauge fields :
- expectation value of microscopic field : variation yields field equation and inverse propagator
- background field appears in gauge fixing

- effective action is invariant under simultaneous gauge transformation of both gauge fields

Field identification

Identify expectation value and background field :

- Effective action depends only on one gauge field
- Variation does not yield field equation and inverse propagator
- Second functional derivative has zero modes :

  gauge modes

Exact flow equation for effective average action

has been formulated in

background field formalism

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Aim : closed gauge invariant flow equation with single gauge field

problem :

- second functional derivative of gauge invariant effective action cannot generate inverse propagator
- second functional derivative is not invertible
- gauge modes !

Projection on physical fluctuations

$P = P^2$ projects on physical fluctuations

1-P projects on gauge fluctuations

$$\delta_\xi \bar{g} = (1 - P)\delta_\xi \bar{g}$$

Gauge fluctuations correspond to change of gauge field $\bar{g}$ under infinitesimal gauge fluctuation

Fluctuations

$$h = g - \bar{g} = f + a$$

gauge fluctuations $a = \big(1 - P(\bar{g})\big) h$ $P(\bar{g}) a = 0$

gauge transformation at fixed macroscopic gauge field

$$\hat{\delta} h = \bar{P}(\bar{g} + h) \delta_\xi (\bar{g} + h) = \bar{P}(\bar{g}) \delta_\xi (\bar{g}) + \delta_h h$$

$$\bar{P} = 1 - P.$$

$$\delta_{\mathrm{inh}} a = \bar{P}(\bar{g}) \delta_\xi \bar{g}$$

physical fluctuations $f = P(\bar{g}) h$

Yang – Mills theories

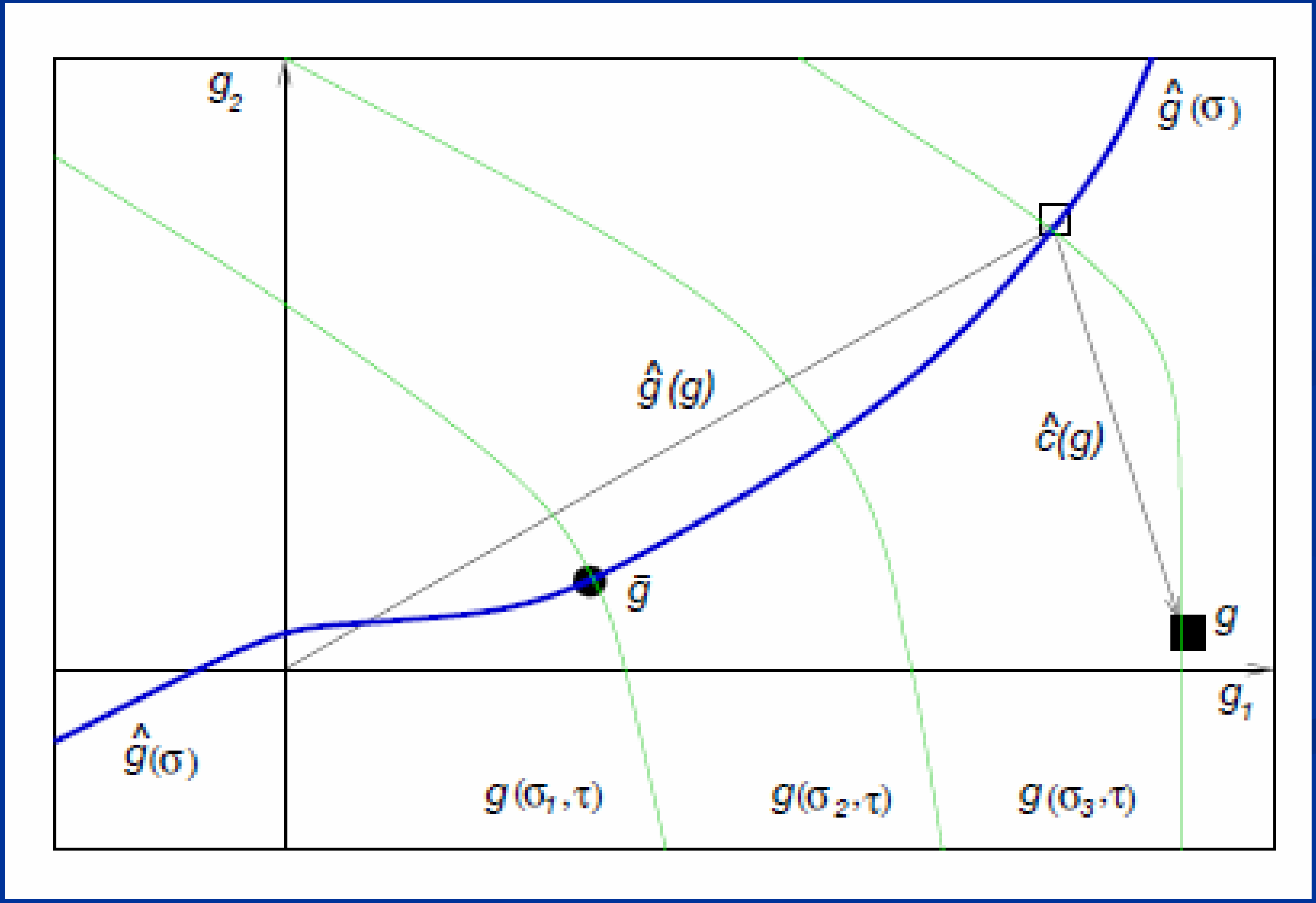
$$P_\mu{}^\nu = \delta_\mu^\nu - \bar{P}_\mu{}^\nu$$

$$\bar{P}_\mu{}^\nu = D_\mu D^{-2} D^\nu$$

projector depends on gauge field

gravity more complicated

Physical fluctuations generate gauge invariant fields no global projector



Gauge invariance of effective action

$$\frac{\partial \bar{\Gamma}}{\partial \bar{g}} = \frac{\partial \bar{\Gamma}}{\partial \bar{g}} P$$

$$\delta_\xi \bar{\Gamma} = \frac{\partial \bar{\Gamma}}{\partial \bar{g}} P \delta_\xi \bar{g} = 0$$

$$\delta_\xi \bar{g} = (1 - P) \delta_\xi \bar{g}$$

Gauge invariant flow equation involves projection on physical fluctuations

$$k\partial_k\bar{\Gamma} = \zeta_k = \pi_k + \delta_k - \epsilon_k$$

$$\pi_k = \frac{1}{2} Str(k\partial_k \bar{R}_P G_P)$$

G_P : propagator for physical fluctuations

$$PG_P = G_P P^T = G_P$$

$$\delta_\xi \bar{g} = (1-P)\delta_\xi \bar{g}$$

measure contributions $\delta_k - \epsilon_k$ do not depend on effective action

Closed flow equation

projection on physical fluctuations makes second functional derivative invertible

$$\bar{\Gamma}_P^{(2)} = P^T \bar{\Gamma}^{(2)} P$$

$$\bar{\Gamma}^{(2)ij} = \frac{\partial^2 \bar{\Gamma}}{\partial \bar{g}_i \partial \bar{g}_j}$$

$$\left(\bar{\Gamma}_P^{(2)} + \bar{R}_P \right) G_P = P^T$$

Projector depends on macroscopic gauge field

- Projector , gauge fixing and infrared cutoff are all formulated with dynamical macroscopic gauge field $\bar{g}$
- Macroscopic gauge field replaces fixed background field
- Implicit definition of functional integral and effective action

Expectation value

- Expectation value of microscopic gauge field g can be expressed in terms of macroscopic gauge field $\bar{g}$
- Macroscopic gauge field is not identical with expectation value of microscopic gauge field
- This is not important

Functional integral

$$Z(L,\bar{g}) \;=\; \int \mathcal{D}g' M_k(g',\bar{g}) \exp\Big\{ - S(g') - S_{gf}(g',\bar{g}) \\ -\Delta S_k(g',\bar{g}) + L^T g' \Big\},$$

$$M_k = M(g',\bar{g}) E_k(\bar{g})$$

$$\epsilon_k(\bar{g}) = \mathrm{tr}\Big\{ \ln k\partial_k E_k(\bar{g}) \Big\}$$

$$\Delta S_k(g',\bar{g}) = \frac{1}{2}(g'-\bar{g})^T R_k(\bar{g})(g'-\bar{g})$$

$$W(L,\bar{g}) \;=\; \ln Z(L,\bar{g})$$

Effective action and exact flow equation

$$\tilde{\Gamma}(g,\bar{g}) = -W(L,\bar{g}) + L^T g$$

$$g = \frac{\partial W(L,\bar{g})}{\partial L} = \langle g' \rangle$$

$$L = \frac{\partial \tilde{\Gamma}(g,\bar{g})}{\partial g}$$

$$\Gamma_k(g,\bar{g}) = \tilde{\Gamma}_k(g,\bar{g}) - \frac{1}{2} h^T R_k(\bar{g}) h$$

$$\partial_k \Gamma = \frac{1}{2} \mathrm{Str}(\partial_k R G) - \epsilon_k$$

$$G = (\Gamma^{(2)} + R)^{-1}$$

Reuter, CW

Projection on physical fluctuations and gauge fixing

$$\Gamma(g,\bar{g}) = \hat{\Gamma}(g,\bar{g}) + \Gamma_{gf}(g,\bar{g})$$

gauge fixing term is
quadratic in gauge fluctuations
diverges for α⟶0,
vanishes for solution of
field equations

$$\Gamma_{gf}(g,\bar{g}) = \frac{1}{2\alpha} a^T \bar{Q}(\bar{g}) a$$

on microscopic level :

$$\Gamma_{gf}(g',g) = \frac{1}{2\alpha} a'^T \bar{Q}(\bar{g}) a'$$

$$a' = \left(1 - P(\bar{g})\right)(g' - \bar{g})$$

Yang - Mills theory

$$\Gamma_{gf} = \frac{1}{2\alpha}\int_x G^z G_z^* \ , \ G^z = \left(D^\mu (A'_\mu - \bar{A}_\mu)\right)^z$$

Gauge invariant effective action involving only one gauge field

$$\Gamma(g,\bar{g}) = \hat{\Gamma}(g,\bar{g}) + \Gamma_{gf}(g,\bar{g})$$

Choose $\bar{g}$ (g) consistent with gauge transformation

$$\bar{g}(g) = g - f\left(\bar{g}(g)\right)$$

$$\begin{aligned}\bar{\Gamma}(\bar{g}) &= \Gamma\left(g = \bar{g} + f(\bar{g}),\ \bar{g}\right) - C(\bar{g}) \\ &= \hat{\Gamma}\left(g = \bar{g} + f(\bar{g}), \bar{g}\right) - C(\bar{g}),\end{aligned}$$

Choice of macroscopic gauge field

Use freedom in precise definition of macroscopic gauge field in order to obtain simple expression of physical propagator in terms of gauge invariant effective action

$$\bar{\Gamma}_P^{(2)} = P^T \bar{\Gamma}^{(2)} P \qquad \bar{\Gamma}^{(2)ij} = \frac{\partial^2 \bar{\Gamma}}{\partial \bar{g}_i \partial \bar{g}_j}$$

$$\left(\bar{\Gamma}_P^{(2)} + \bar{R}_P \right) G_P = P^T$$

Projected propagator in flow equation

$$R_k = \bar{R}_k(\bar{g}) + \frac{1}{\alpha}\big(1 - P^T(\bar{g})\big) R_{k,gf}(\bar{g}) \big(1 - P(\bar{g})\big)$$

$$\begin{aligned}\tilde{\Gamma}^{(2)} &= \hat{\Gamma}^{(2)} + \bar{R}_k \\ &\quad + \frac{1}{\alpha}(1 - P^T)(\bar{Q} + R_{gf})(1 - P)\end{aligned}$$

$$G = G_P + \alpha \Delta G \ , \ G_P = P G P^T$$

$$\tilde{\Gamma}_P^{(2)} G_P = P^T \ , \ \tilde{\Gamma}_P^{(2)} = P^T(\hat{\Gamma}^{(2)} + \bar{R})P$$

For α→0 flow equation decomposes into projected parts

Measure terms in flow equation

$$k\partial_k \Gamma(g,\bar{g}) = \frac{1}{2} S\mathrm{tr}(k\partial_k \bar{R} G_P) + \delta_k - \epsilon_k$$

$$\delta_k = \frac{1}{2}\mathrm{tr}\{k\partial_k R_{gf}(1-P)(\bar{Q}+R_{gf})^{-1}(1-P^T)\}$$

does not involve effective action

$$\epsilon_k = 2\delta_k$$

Particular gauge fixing for quantum gravity

$$S_{gf} = \frac{1}{2\alpha} \int_x \sqrt{\bar{g}} \left(D^{\mu} h'_{\mu\nu} \right)^2 \ , \ h'_{\mu\nu} = g'_{\mu\nu} - \bar{g}_{\mu\nu}$$

Conclusion

- Closed gauge invariant flow equation involving only one gauge field exists.
- Is effective action local enough to admit simple truncations ?