Spatiotemporal correlation functions of fully developed turbulence

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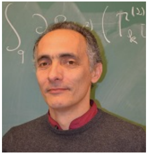
ERG Trieste

19/09/2016

In collaboration with ...



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LC, B. Delamotte, N. Wschebor, Phys. Rev. E **91** (2015)

LC, B. Delamotte, N. Wschebor, Phys. Rev. E **93** (2016)

LC, V. Rossetto, N. Wschebor, G. Balarac, arXiv :1607.03098 (2016)

Presentation outline

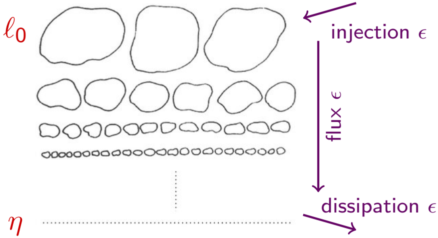
Navier-Stokes turbulence

stationary regime of fully developed isotropic and homogeneous turbulence

- integral scale (energy injection) : ℓ_0
- Kolmogorov scale (energy dissipation) : η

energy cascade

$$\frac{\ell_0}{\eta} \sim R^{3/4}$$



constant energy flux in the inertial range $\eta < r < \ell_0$

Frisch, Turbulence, the legacy of AN Kolmogorov Cambridge Univ. Press (1995)

Scale invariance in the inertial range

velocity structure functions

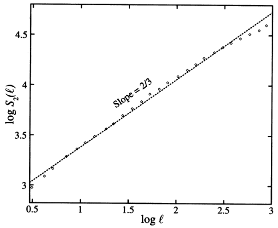
- velocity increments

$$\delta v_{\ell_{\|}} = [\vec{u}(\vec{x} + \vec{\ell}) - \vec{u}(\vec{x})] \cdot \vec{\ell}$$

- structure function

$$S_p(\ell) \equiv \langle (\delta v_{\ell_{\|}})^p \rangle \sim \ell^{\xi_p}$$

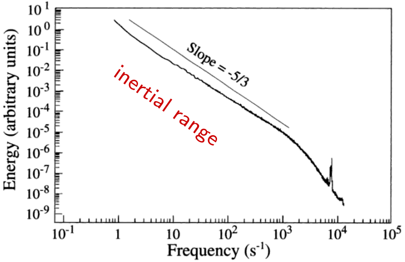
$$\xi_2 = 2/3$$



energy spectrum

$$E(k) = 4\pi k^2 \, \mathrm{TF} \left(\langle \vec{v}(\vec{x}) \cdot \vec{v}(0) \rangle \right)$$

$$\sim k^{-5/3}$$



ONERA wind tunnel Anselmet, Gagne, Hopfinger, Antonia, J. Fluid Mech. **140** (1984).

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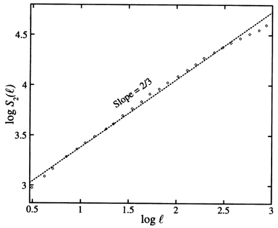

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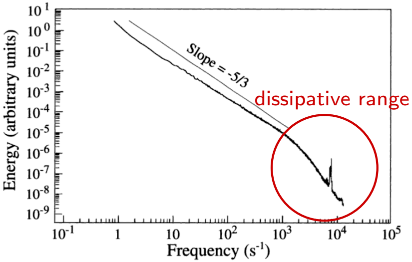

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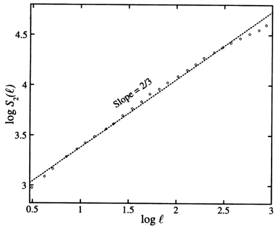

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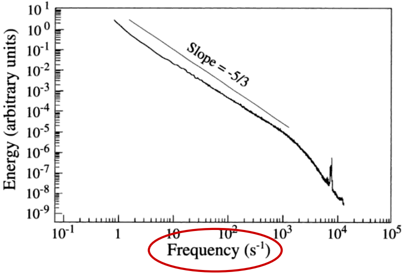


ONERA wind tunnel Anselmet, Gagne, Hopfinger, Antonia, J. Fluid Mech. **140** (1984).

Kolmogorov K41 theory for isotropic 3D turbulence

Kolmogorov original work A.N. Kolmogorov, Dokl. Akad. Nauk. SSSR **30, 31, 32** (1941)

Assumptions

- symmetries restored in a statistical sense : homogeneity, isotropy
- finite dissipation rate per unit mass ϵ in the limit $\nu \to 0$

$\Longrightarrow$ derivation of energy flux constancy relation

exact result "four-fifth law" $S_3(\ell) = -\frac{4}{5}\,\epsilon\,\ell$

Assuming universality in the inertial range

- self-similarity $\delta\vec{v}_{||}(\vec{r},\lambda\vec{\ell}) = \lambda^h \delta\vec{v}_{||}(\vec{r},\vec{\ell})$
- dimensional analysis

$\Longrightarrow$ scaling predictions

$$S_p(\ell) = C_p\,\epsilon^{p/3}\,\ell^{p/3} \qquad E(k) = C_K\,\epsilon^{2/3}\,k^{-5/3}$$

Intermittency, multi-scaling

deviations from K41

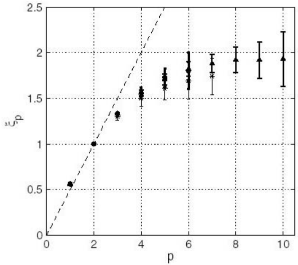
in experiments and numerical simulations

$$S_p(\ell) \equiv \langle (\delta v_{\ell_{\|}})^p \rangle \sim \ell^{\xi_p}$$

$$\xi_p \neq p/3$$

- violation of simple scale-invariance
  $\Longrightarrow$ multi-scaling
- non-Gaussian statistics of velocity differences
  $\Longrightarrow$ intermittency

illustration :
von Kárman swirling flow



●	exp.
▲ , *	num.
- - -	K41

Mordant, Lévêque, Pinton,

New J. Phys. **6** (2004)

Intermittency, multi-scaling

theoretical challenge : understand K41 and intermittency
from first principles (microscropic description)

Various perturbative RG approaches

formal expansion parameter through the forcing profile $N_{\alpha\beta}(\vec{p}) \propto p^{4-d-2\epsilon}$

- *early works* de Dominicis, Martin, PRA **19** (1979) , Fournier, Frisch, PRA **28** (1983) Yakhot, Orszag, PRL **57** (1986)
- *reviews* Zhou, Phys. Rep. **488** (2010)

  Adzhemyan *et al.*, *The Field Theoretic RG in Fully Developed Turbulence*, Gordon Breach, 1999

Non-Perturbative (functional) RG approaches

Tomassini, Phys. Lett. B **411** (1997), Mejía-Monasterio, Muratore-Ginnaneschi, PRE **86** (2012)

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NPRG without truncations : exact closure based on symmetries !

LC, Delamotte, Wschebor, PRE **93** (2016), LC, Rossetto, Wschebor, Balarac, arXiv :1607.03098

Microscopic theory

Navier Stokes equation with forcing for incompressible fluids

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} = -\frac{1}{\rho} \vec{\nabla} p + \nu \vec{\nabla}^2 \vec{v} + \vec{f}$$

$$\vec{\nabla} \cdot \vec{v}(t, \vec{x}) = 0$$

- $\vec{v}(\vec{x}, t)$ velocity field and $p(\vec{x}, t)$ pressure field
- ρ density and ν kinematic viscosity
- $\vec{f}(\vec{x}, t)$ gaussian stochastic stirring force with variance

$$\left\langle f_\alpha(t, \vec{x}) f_\beta(t', \vec{x}\,') \right\rangle = 2\delta_{\alpha\beta}\delta(t - t') N_{\ell_0}(|\vec{x} - \vec{x}\,'|).$$

with N_{ℓ_0} peaked at the integral scale (energy injection)

Non-Perturbative Renormalisation Group for NS

MSR Janssen de Dominicis formalism : NS field theory

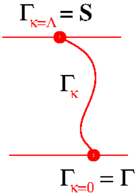
Martin, Siggia, Rose, PRA **8** (1973), Janssen, Z. Phys. B **23** (1976), de Dominicis, J. Phys. Paris **37** (1976)

$$\mathcal{S}_0 = \int_{t,\vec{x}} \bar{v}_\alpha \left[\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha + \frac{1}{\rho}\partial_\alpha p - \nu \nabla^2 v_\alpha\right] + \bar{p}\left[\partial_\alpha v_\alpha\right]$$
$$- \int_{t,\vec{x},\vec{x}'} \bar{v}_\alpha \left[N_{\ell_0}(|\vec{x}-\vec{x}'|)\right] \bar{v}_\alpha$$

Non-Perturbative Renormalization Group approach

Wetterich's equation for scale-dependent effective actions Γ_κ

$$\partial_\kappa \Gamma_\kappa = \frac{1}{2}\mathrm{Tr}\int_{\vec{q}} \partial_\kappa \mathcal{R}_\kappa \left[\Gamma_\kappa^{(2)} + \mathcal{R}_\kappa\right]^{-1} = \frac{1}{2}\mathrm{Tr}\int_{\vec{q}} \partial_\kappa \mathcal{R}_\kappa \cdot G_\kappa$$



C. Wetterich, Phys. Lett. B **301** (1993)

Non-Perturbative Renormalisation Group for NS

Aim : compute correlation function and response function

$$\langle v_\alpha(t,\vec{x}) v_\beta(0,0) \rangle \quad \text{and} \quad \langle v_\alpha(t,\vec{x}) f_\beta(0,0) \rangle$$

Wetterich's equation for the 2-point functions

$$\partial_\kappa \Gamma^{(2)}_{\kappa,ij}(\mathbf{p}) = \mathrm{Tr}\int_{\mathbf{q}} \partial_\kappa \mathcal{R}_\kappa(\mathbf{q}) \cdot G_\kappa(\mathbf{q}) \cdot \Big(-\frac{1}{2}\Gamma^{(4)}_{\kappa,ij}(\mathbf{p},-\mathbf{p},\mathbf{q}) + \Gamma^{(3)}_{\kappa,i}(\mathbf{p},\mathbf{q}) \cdot G_\kappa(\mathbf{p}+\mathbf{q}) \cdot \Gamma^{(3)}_{\kappa,j}(-\mathbf{p},\mathbf{p}+\mathbf{q}) \Big) \cdot G_\kappa(\mathbf{q})$$

infinite hierarchy of flow equations

- approximation scheme : truncation of higher-order vertices

  Tomassini, Phys. Lett. B **411** (1997), Mejía-Monasterio, Muratore-Ginnaneschi, PRE **86** (2012)

  LC, Delamotte, Wschebor, PRE **93** (2016)

- exact closure exploiting (time-gauged) symmetries of NS action

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NPRG at Leading Order (LO) approximation

general form of the effective action from the symmetries

$$\Gamma[\vec{u},\vec{\bar{u}},p,\bar{p}] = \int_{t,\vec{x}} \left\{ \bar{u}_\alpha \left(\partial_t u_\alpha + u_\beta \partial_\beta u_\alpha + \frac{\partial_\alpha p}{\rho} \right) + \bar{p}\, \partial_\alpha u_\alpha \right\} + \hat{\Gamma}[\vec{u},\vec{\bar{u}}]$$

Ansatz for $\hat{\Gamma}_\kappa$ at LO approximation

$$\hat{\Gamma}_\kappa[\vec{u},\vec{\bar{u}}] = \int_{t,\vec{x},\vec{x}'} \left\{ \bar{u}_\alpha\, f^{\nu}_{\kappa,\alpha\beta}(\vec{x}-\vec{x}')\, u_\beta - \bar{u}_\alpha\, f^{D}_{\kappa,\alpha\beta}(\vec{x}-\vec{x}')\, \bar{u}_\beta \right\}$$

truncation at quadratic order in the fields

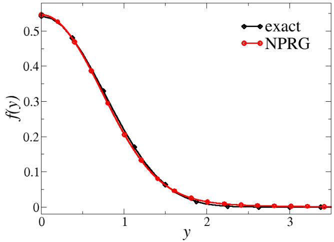
two flowing functions $f^{\nu}_{\kappa}(\vec{k})$ and $f^{D}_{\kappa}(\vec{k})$

$\Longrightarrow$ works very accurately for

Kardar-Parisi-Zhang equation

LC, Chaté, Delamotte, Wschebor, PRL **104** (2010), PRE **84** (2011)

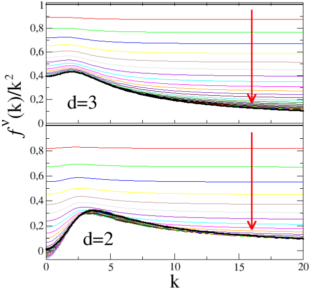
Kloss, LC, Wschebor, PRE **86** (2012) , PRE **89** (2014)



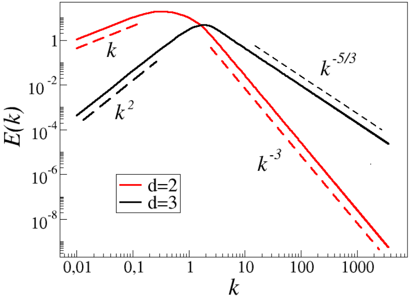
NPRG at Leading Order (LO) approximation

Numerical integration at LO LC, Delamotte, Wschebor, PRE **93** (2016)

fixed-point in $d = 2$ and $d = 3$



kinetic energy spectrum



- Kolmogorov scaling $k^{-5/3}$ in $d = 3$
- Kraichnan-Batchelor scaling k^{-3} in $d = 2$

Presentation outline

1. NPRG approach to Navier-Stokes equation
   - Fully developed turbulence
   - Navier-Stokes equation
   - NPRG formalism for NS
   - Leading Order approximation

2. Exact correlation function in the limit of large wave-numbers
   - Exact flow equations in the limit of large wave-numbers
   - Solution in the inertial range
   - Solution in the dissipative range

3. Perspectives

Ingredient 1 : Symmetries of the NS field theory

- infinitesimal time-gauged galilean transformations

$$\mathcal{G}(\vec{\epsilon}(t)) = \begin{cases} \vec{x} \to \vec{x} + \vec{\epsilon}(t) \\ \vec{v} \to \vec{v} - \dot{\vec{\epsilon}}(t) \end{cases} \qquad \begin{cases} \delta v_\alpha(t,\vec{x}) & = -\dot{\epsilon}_\alpha(t) + \epsilon_\beta(t)\partial_\beta v_\alpha(t,\vec{x}) \\ \delta \bar{v}_\alpha(t,\vec{x}) & = \epsilon_\beta(t)\partial_\beta \bar{v}_\alpha(t,\vec{x}) \\ \delta p(t,\vec{x}) & = \epsilon_\beta(t)\partial_\beta p(t,\vec{x}) \\ \delta \bar{p}(t,\vec{x}) & = \epsilon_\beta(t)\partial_\beta \bar{p}(t,\vec{x}) \end{cases}$$

- infinitesimal time-gauged response field shift *not identified yet !*

$$\mathcal{R}(\vec{\bar{\epsilon}}(t)) = \begin{cases} \delta \bar{v}_\alpha(t,\vec{x}) & = \bar{\epsilon}_\alpha(t) \\ \delta \bar{p}(t,\vec{x}) & = v_\beta(t,\vec{x})\bar{\epsilon}_\beta(t) \end{cases}$$

LC, Delamotte, Wschebor, Phys. Rev. E **91** (2015)

infinite set of *local in time* exact Ward identities
for all vertices with one zero momentum

$$\Gamma^{(2,1)}_{\alpha\beta\gamma}(\omega, \vec{q}=\vec{0}; \nu, \vec{p}) = -\frac{p^\alpha}{\omega}\Big(\Gamma^{(1,1)}_{\beta\gamma}(\omega+\nu,\vec{p}) - \Gamma^{(1,1)}_{\beta\gamma}(\nu,\vec{p})\Big)$$

$$\Gamma^{(2,2)}_{\alpha\beta\gamma\delta}(\omega,\vec{0},-\omega,\vec{0},\nu,\vec{p}) = \frac{p^\alpha p^\beta}{\omega^2}\Big[\Gamma^{(0,2)}_{\gamma\delta}(\nu+\omega,\vec{p}) - 2\Gamma^{(0,2)}_{\gamma\delta}(\nu,\vec{p}) + \Gamma^{(0,2)}_{\gamma\delta}(\nu-\omega,\vec{p})\Big]$$

Ingredient 2 : limit of large wave-numbers

Wetterich's equation for the 2-point functions

$$\partial_\kappa \Gamma^{(2)}_{\kappa,ij}(\nu,\vec{p}) = \mathrm{Tr}\int_{\nu,\vec{q}} \partial_\kappa \mathcal{R}_\kappa(\vec{q})\cdot G_\kappa(\omega,\vec{q})\cdot\Big(-\frac{1}{2}\,\Gamma^{(4)}_{\kappa,ij}(\nu,\vec{p};-\nu,-\vec{p};\omega,0)$$
$$+\Gamma^{(3)}_{\kappa,i}(\nu,\vec{p};\omega,0)\cdot G_\kappa(\nu+\omega,\vec{p}+\vec{q})\cdot\Gamma^{(3)}_{\kappa,j}(-\omega,-0;\nu+\omega,\vec{p}+0)\Big)\cdot G_\kappa(\omega,\vec{q})$$

regime of large wave-vector $|\vec{p}|\gg\kappa$ or $\kappa\to 0 \Longrightarrow |\vec{q}|\ll|\vec{p}|$
set $\vec{q}=0$ in all vertices and close with Ward identities

$$\partial_s\Gamma^{(1,1)}_\perp(\nu,\vec{p}) = p^2\int_\omega\Big\{-\left[\frac{\Gamma^{(1,1)}_\perp(\omega+\nu,\vec{p})-\Gamma^{(1,1)}_\perp(\nu,\vec{p})}{\omega}\right]^2 G^{u\bar{u}}_\perp(-\omega-\nu,\vec{p})$$
$$+\frac{1}{2\omega^2}\left[\Gamma^{(1,1)}_\perp(\omega+\nu,\vec{p})-2\Gamma^{(1,1)}_\perp(\nu,\vec{p})+\Gamma^{(1,1)}_\perp(-\omega+\nu,\vec{p})\right]\Big\}$$
$$\times\frac{(d-1)}{d}\tilde{\partial}_s\int_{\vec{q}} G^{uu}_\perp(\omega,\vec{q})$$

$$\partial_s\Gamma^{(0,2)}_\perp(\nu,\vec{p}) = \dots$$

LC, Delamotte, Wschebor, PRE **93** (2016)

Exact flow equations in the large wave-number limit

exact equation for $C_\kappa(\omega, \vec{k})$ when $|\vec{k}| \gg \kappa$ and $\omega \gg \kappa^z$

$$\kappa \partial_\kappa C_\kappa(\omega, \vec{k}) = -\frac{1}{3} k^2 I_\kappa \partial_\omega^2 C_\kappa(\omega, \vec{k})$$

$$I_\kappa = -\int_{\nu, \vec{q}} \Big\{ 2\partial_s N_s(\vec{q}) \, |G_\kappa(\nu, \vec{q})|^2 - 2\partial_s R_s(\vec{q}) \, C_\kappa(\nu, \vec{q}) \Re G_\kappa(\nu, \vec{q}) \Big\}$$

$\Longrightarrow$ also exact equation for response function

LC, Delamotte, Wschebor, PRE **93** (2016)

exact analytical solutions of the fixed-point equations

two regimes :

- $I_* > 0$: solution in the inertial range
- $I_* < 0$: solution in the dissipative range

LC, Rossetto, Wschebor, Balarac, arXiv :1607.03098 (2016)

Analytical solutions I : inertial range

analytical solution in the inertial range

$$C(\omega, k) = \frac{c_C}{k^{13/3}} \frac{1}{\sqrt{4\pi\alpha k^{2/3}}} \exp\left[-\frac{(\omega/k)^2}{4\alpha}\right] \qquad \alpha = 3I_*/2$$

- kinetic energy spectrum (in wave-vector)

$$E(k) = 4\pi k^2 \int_0^\infty \frac{d\omega}{\pi} C(\omega, k) \propto k^{-5/3}$$

- kinetic energy spectrum (in frequency)

$$E(\omega) = 4\pi \int_0^\infty k^2 C(\omega, k) dk \propto \omega^{-5/3}$$

$\Longrightarrow$ sweeping effect ! (random Taylor hypothesis Tennekes, J. Fluid Mech. **67** (1975))

standard scaling theory with $z = 2/3 \Longrightarrow E(\omega) \propto \omega^{-2}$
observed for Lagrangian velocities, but not Eulerian ones

Chevillard, Roux, Lévêque, Mordant, Pinton, Arnéodo, PRL **95** (2005)

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$$E(k) = 4\pi k^2 \int_0^\infty \frac{d\omega}{\pi} C(\omega, k) \propto k^{-5/3}$$

- kinetic energy spectrum (in frequency)

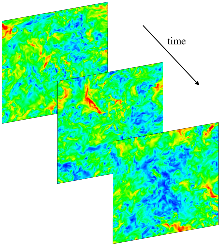
$$E(\omega) = 4\pi \int_0^\infty k^2 C(\omega, k) dk \propto \omega^{-5/3}$$

- time-dependence

$$C(t, k) = \Re\left\{\int_0^\infty \frac{d\omega}{\pi} C(\omega, k) e^{-i\omega t}\right\} \propto \frac{1}{k^{11/3}} e^{-\alpha k^2 t^2}$$

Analytical solutions I : inertial range

numerical data



- our simulations

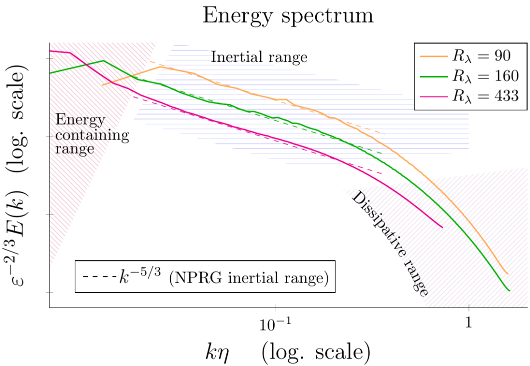
based on pseudo-spectral code

Lagaert, Balarac, Cottet,
J. Comp. Phys. **260** (2014)

- JHTBD

Johns Hopkins TurBulence Database

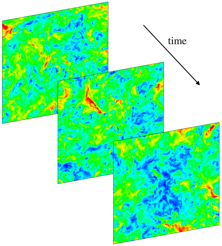
http ://turbulence.pha.jhu.edu/

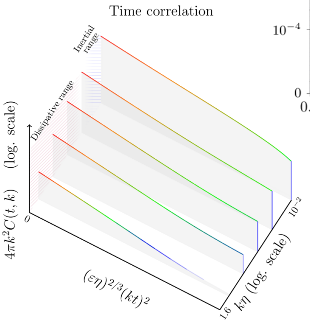


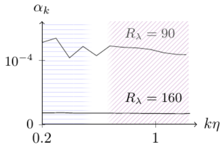
LC, Rossetto, Wschebor, Balarac, arXiv :1607.03098

Analytical solutions I : inertial range

numerical data

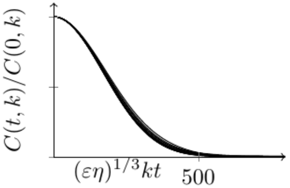






analytical prediction

$$C(t,k) \propto \exp(-\alpha k^2 t^2)$$



Analytical solutions II : dissipative range

analytical solution in the dissipative range

$$C(\omega, k) = \frac{c_C}{k^{13/3}} \exp\left[-\left(\mu k^{2/3} + A\frac{\omega}{k^{2/3}}\right)\right] \qquad A = \sqrt{-\frac{2\mu}{3I_*}}$$

- kinetic energy spectrum

$$E(k) = 4\pi k^2 \int_0^\infty \frac{d\omega}{\pi} C(\omega, k) \propto \frac{1}{k^{5/3}} \exp\left[-\mu k^{2/3}\right]$$

several empirical propositions $\exp[-ck^\gamma]$ with $\gamma = 1/2,\ 3/2,\ 4/3,\ 2, \ldots$

Monin and Yaglom, *Statistical Fluid Mechanics : Mechanics of Turbulence* (1973)

common wisdom : approximately exponential decay

Analytical solutions II : dissipative range

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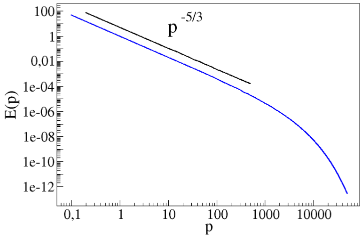
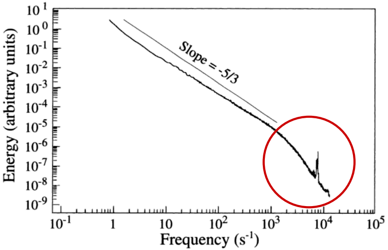
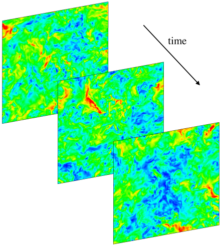




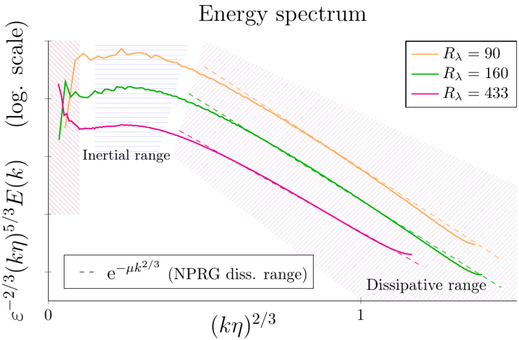
Analytical solutions II : dissipative range

numerical data



analytical prediction

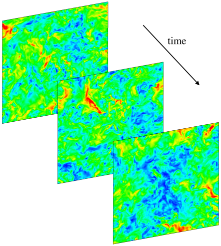
$E(k) \propto \exp(-\mu k^{2/3})$



LC, Rossetto, Wschebor, Balarac, arXiv :1607.03098

Analytical solutions II : dissipative range

numerical data



analytical prediction

$$E(k) \propto \exp(-\mu k^{2/3})$$

and experiments !

in preparation ...

Dubue, Kuzzay, Saw, Daviaud, Dubrulle,

Wschebor, LC, Rossetto (2016)

Conclusion and perspectives

Conclusion

- analytical solution for $C(\omega, \vec{k})$ in $d = 3$
- confirmed by numerical simulations

bidimensional turbulence

both energy and enstrophy are conserved

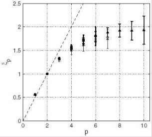
- direct cascade
- inverse cascade
- energy spectra

Numerical solution for $C(\omega, \vec{k})$

- interplay of the two regimes
- intermittency effects

structure functions

- derive flow equations for $S_p(\ell)$, $p = 3, 4, \ldots$
- intermittency exponents ξ_p




Thank you for attention !

NPRG formalism for NS

Navier-Stokes action

$$\mathcal{S}_0 = \int_{t,\vec{x}} \bar{v}_\alpha \left(\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha + \frac{1}{\rho} \partial_\alpha p - \nu \nabla^2 v_\alpha \right) + \bar{p}\, \partial_\alpha v_\alpha$$
$$- \int_{t,\vec{x},\vec{x}'} \bar{v}_\alpha\, N_{L^{-1},\alpha\beta}(|\vec{x} - \vec{x}'|)\, \bar{v}_\beta$$

early NPRG setting proposed in R. Collina and P. Tomassini, Phys. Lett. B **411** (1997)

using the inverse integral scale L^{-1} as the RG scale

NPRG formalism for NS I

improved regulator term L. Canet, B. Delamotte, N. Wschebor, PRE (2016)

$$\Delta\mathcal{S}_\kappa[\vec{v},\bar{\vec{v}}] = -\int_{t,\vec{x},\vec{x}'} \bar{v}_\alpha(t,\vec{x}) N_{k,\alpha\beta}(|\vec{x}-\vec{x}'|)\bar{v}_\beta(t,\vec{x}') + \int_{t,\vec{x},\vec{x}'} \bar{v}_\alpha(t,\vec{x}) R_{\kappa,\alpha\beta}(|\vec{x}-\vec{x}'|) v_\beta(t,\vec{x}')$$

with $N_{k,\alpha\beta}(\vec{q}) = \delta_{\alpha\beta}\, D_\kappa\, \hat{n}\,(|\vec{q}|/\kappa)$
and $R_{\kappa,\alpha\beta}(\vec{q}) = \delta_{\alpha\beta}\, \nu_\kappa\, \vec{q}^{\,2}\hat{r}\,(|\vec{q}|/\kappa)$

- physically : RG (volume) scale κ and integral scale k^{-1} can be kept independent
- technically : flow equations regularized down to $d=2$

Symmetries of the NS field theory

- infinitesimal gauged shifts in the pressure sector

$$p(t,\vec{x}) \to p(t,\vec{x}) + \epsilon(t,\vec{x})$$
$$\bar{p}(t,\vec{x}) \to \bar{p}(t,\vec{x}) + \bar{\epsilon}(t,\vec{x})$$

- infinitesimal time-gauged galilean transformations

$$\mathcal{G}(\vec{\epsilon}(t)) = \begin{cases} \vec{x} \to \vec{x} + \vec{\epsilon}(t) \\ \vec{v} \to \vec{v} - \dot{\vec{\epsilon}}(t) \end{cases} \qquad \begin{cases} \delta v_\alpha(t,\vec{x}) & = -\dot{\epsilon}_\alpha(t) + \epsilon_\beta(t)\partial_\beta v_\alpha(t,\vec{x}) \\ \delta \bar{v}_\alpha(t,\vec{x}) & = \epsilon_\beta(t)\partial_\beta \bar{v}_\alpha(t,\vec{x}) \\ \delta p(t,\vec{x}) & = \epsilon_\beta(t)\partial_\beta p(t,\vec{x}) \\ \delta \bar{p}(t,\vec{x}) & = \epsilon_\beta(t)\partial_\beta \bar{p}(t,\vec{x}) \end{cases}$$

- infinitesimal time-gauged response field shift

$$\mathcal{R}(\vec{\bar{\epsilon}}(t)) = \begin{cases} \delta \bar{v}_\alpha(t,\vec{x}) & = \bar{\epsilon}_\alpha(t) \\ \delta \bar{p}(t,\vec{x}) & = v_\beta(t,\vec{x})\bar{\epsilon}_\beta(t) \end{cases}$$

LC, Delamotte, Wschebor, Phys. Rev. E **91** (2015)

not identified yet !

Symmetries of the NS field theory

fully gauged shift symmetry $\bar{\epsilon}_\alpha(t) \to \bar{\epsilon}_\alpha(\vec{x}, t)$
in the presence of a local source term $v_\alpha L_{\alpha\beta} v_\beta$

local functional Ward identity for $\mathcal{W} = \ln \mathcal{Z}$

$$\Big[-\partial_t + \nu\nabla^2 + \bar{K}\Big] \frac{\delta\mathcal{W}}{\delta J_\alpha} - \frac{1}{\rho}\partial_\alpha \frac{\delta\mathcal{W}}{\delta K} + \bar{J}_\alpha - \partial_\beta \frac{\delta\mathcal{W}}{\delta L_{\alpha\beta}} + \int_{\vec{x}'} \Big\{ 2\frac{\delta\mathcal{W}}{\delta \bar{J}_\beta} N_{\alpha\beta} \Big\} = 0$$

LC, Delamotte, Wschebor, Phys. Rev. E **91** (2015)

from which can be derived :

- Kármán-Howarth-Monin relation
  $\Longrightarrow$ "four-fifth" Kolmogorov law : $S_3(\ell) = -\frac{4}{5}\,\epsilon\,\ell$
- exact relation for a pressure-velocity correlation function
  $\langle \vec{v}(\vec{r}) p(\vec{r}) \vec{v}^{\,2}(0) \rangle \propto \vec{r}$ Falkovich, Fouxon, Oz, J. Fluid Mech. **644**, (2010).
- infinite set of generalized exact local relations