Momenta fields and the derivative expansion

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Introduction

How to describe an infinite number of momentum-dependent interactions?

Approximation schemes for the functional RG

- Derivative Expansion
- Vertex Expansion
- Scaling Field Expansion (Wegner, 1976)
- Morris (Int. J. Mod. Phys. A **9** (1994))
- Papenbrock and Wetterich (Z. Phys. C **65** (1995))
- Golner (hep-th/9801124)
- Blaizot, Mendez Galain, Wschebor (Phys. Lett. B **632** (2006))
- Hasselmann (Phys. Rev. E **86** (2012))
- ...

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Introduction

Polchinski equation

$$\dot{S}[\phi] = \frac{1}{2}\int_{xy} \frac{\delta S[\phi]}{\delta\phi(x)}\dot{C}(x-y)\frac{\delta S[\phi]}{\delta\phi(y)} - \frac{1}{2}\int_{xy} \frac{\delta}{\delta\phi(x)}\dot{C}(x-y)\frac{\delta S[\phi]}{\delta\phi(y)}$$

Local Lagrangian truncation

$$S_\Lambda[\phi] = \int_x \mathcal{L}(x,\phi(x),\partial\phi(x),\partial^2\phi(x),\partial^3\phi(x),\ldots)$$

it comprehends also some nonlocal terms (by Taylor series about x)

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Local Lagrangian truncation

$$S_\Lambda[\phi] = \int_x \mathcal{L}(x, \phi(x), \partial\phi(x), \partial^2\phi(x), \partial^3\phi(x), \ldots)$$

Derivative Expansion

keep **every** contribution from $\dot{S}[\phi]$ to $O(\partial^m)$, up to $m \leq n$

Alternative idea

keep **some** contribution from $\dot{S}[\phi]$ to $O(\partial^m)$, up to $m \to \infty$

Outline

- Notations
- Exact RG equation: from S to $\mathcal{L}$
- Truncations for $\mathcal{L}$
- Truncated RG equations: from $\mathcal{L}$ to $\mathcal{H}$
- Application: Wilson-Fisher FP
- Summary

Notations

For any function $\phi(x)$

$$\phi_M = \phi_{\mu_1 \ldots \mu_m}(x) = \frac{\mathrm{d}}{\mathrm{d}x^{\mu_m}} \ldots \frac{\mathrm{d}}{\mathrm{d}x^{\mu_1}} \phi(x) = \frac{\mathrm{d}}{\mathrm{d}x^M} \phi(x)$$

$$M \equiv (\mu_1, \mu_2, \ldots, \mu_m), \; m \in \mathbb{N} \qquad (-)^M \equiv (-1)^m$$

$$M = () \text{ corresponds to } \phi_M(x) = \phi(x) \text{ and } m = 0$$

repeated multi-indices are summed over

if $\mathcal{L}$ depends on ϕ_M and also separately on x

$$\frac{\mathrm{d}}{\mathrm{d}x^\lambda} = \frac{\partial}{\partial x^\lambda} + \phi_{M\lambda} \frac{\partial}{\partial \phi_M}$$

From S to $\mathcal{L}$

Functional derivatives: first order

$$\begin{aligned}
\frac{\delta S}{\delta \phi(x)} &= \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}x^{\mu_1}} \frac{\partial \mathcal{L}}{\partial \phi_{\mu_1}} + \frac{\mathrm{d}^2}{\mathrm{d}x^{\mu_1} \mathrm{d}x^{\mu_2}} \frac{\partial \mathcal{L}}{\partial \phi_{\mu_1 \mu_2}} + \dots \\
&= (-)^M \frac{\mathrm{d}}{\mathrm{d}x^M} \frac{\partial \mathcal{L}}{\partial \phi_M}(x) \\
&= \int_y \frac{\partial \mathcal{L}}{\partial \phi_M}(y) \delta_M(y - x)
\end{aligned}$$

and second order

$$\frac{\delta^2 S}{\delta \phi(y) \delta \phi(x)} = (-)^N \frac{\mathrm{d}}{\mathrm{d}y^N} \left[\frac{\partial^2 \mathcal{L}}{\partial \phi_N \partial \phi_M}(y) \delta_M(y - x) \right]$$

From S to $\mathcal{L}$

Polchinski equation for $\mathcal{L}$

$$\int_x \dot{\mathcal{L}}(x) = \frac{(-)^N}{2}\left\{\int_{xy} \frac{\partial \mathcal{L}}{\partial \phi_M}(x)\dot{C}_{MN}(x-y)\frac{\partial \mathcal{L}}{\partial \phi_N}(y) - \dot{C}_{MN}(0)\int_x \frac{\partial^2 \mathcal{L}}{\partial \phi_M \partial \phi_N}(x)\right\}$$

no x-derivatives act on $\mathcal{L}$, but the flow equation generates nonlocalities through the first (classical) term

Expand non-localities to higher derivatives

$$\frac{\partial \mathcal{L}}{\partial \phi_N}(y) = \frac{\partial \mathcal{L}}{\partial \phi_N}(x) + \frac{(y-x)^L}{L!}\frac{\mathrm{d}}{\mathrm{d}x^L}\frac{\partial \mathcal{L}}{\partial \phi_N}(x)$$

where $L \neq 0$.

From S to $\mathcal{L}$

$$\frac{\partial \mathcal{L}}{\partial \phi_N}(y) = \frac{\partial \mathcal{L}}{\partial \phi_N}(x) + \frac{(y-x)^L}{L!}\frac{\mathrm{d}}{\mathrm{d}x^L}\frac{\partial \mathcal{L}}{\partial \phi_N}(x)$$

where $L \neq 0$.

$$\frac{\mathrm{d}}{\mathrm{d}x^\lambda} = \frac{\partial}{\partial x^\lambda} + \phi_{M\lambda}\frac{\partial}{\partial \phi_M}$$

From S to $\mathcal{L}$

$$\frac{\partial \mathcal{L}}{\partial \phi_N}(y) = \frac{\partial \mathcal{L}}{\partial \phi_N}(x) + \frac{(y-x)^L}{L!}\frac{\mathrm{d}}{\mathrm{d}x^L}\frac{\partial \mathcal{L}}{\partial \phi_N}(x)$$

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$$\frac{\mathrm{d}}{\mathrm{d}x^\lambda} = \frac{\partial}{\partial x^\lambda} + \phi_{M\lambda}\frac{\partial}{\partial \phi_M}$$

We restrict to pointlike interactions: no explicit x-dependence of $\mathcal{L}$

From S to $\mathcal{L}$

$$\frac{\partial \mathcal{L}}{\partial \phi_N}(y) = \frac{\partial \mathcal{L}}{\partial \phi_N}(x) + \frac{(y-x)^L}{L!}\frac{\mathrm{d}}{\mathrm{d}x^L}\frac{\partial \mathcal{L}}{\partial \phi_N}(x)$$

where $L \neq 0$.

$$\frac{\mathrm{d}}{\mathrm{d}x^\lambda} = \frac{\partial}{\partial x^\lambda} + \phi_{M\lambda}\frac{\partial}{\partial \phi_M}$$

We restrict to pointlike interactions: no explicit x-dependence of $\mathcal{L}$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^{\lambda_1}\mathrm{d}x^{\lambda_2}} = \phi_{M_1\lambda_1\lambda_2}\frac{\partial}{\partial \phi_{M_1}} + \phi_{M_1\lambda_1}\phi_{M_2\lambda_2}\frac{\partial^2}{\partial \phi_{M_1}\partial \phi_{M_2}}$$

In general

$$\frac{\mathrm{d}}{\mathrm{d}x^L} = \sum_{j=1}^{L}\phi_{(M_1}\ldots\phi_{M_j)_L}\frac{\partial^j}{\partial \phi_{M_1}\ldots\partial \phi_{M_j}}$$

Exact RG equation for $\mathcal{L}$

$$\int_x \dot{\mathcal{L}}(x) = \frac{(-)^N}{2} \left\{ \int_{xy} \frac{\partial \mathcal{L}}{\partial \phi_M}(x) \dot{C}_{MN}(x-y) \frac{\partial \mathcal{L}}{\partial \phi_N}(y) - \dot{C}_{MN}(0) \int_x \frac{\partial^2 \mathcal{L}}{\partial \phi_M \partial \phi_N}(x) \right\}$$

Taylor-expand:

$$\frac{\partial \mathcal{L}}{\partial \phi_N}(y) = \frac{\partial \mathcal{L}}{\partial \phi_N}(x) + \frac{(y-x)^L}{L!} \frac{\mathrm{d}}{\mathrm{d}x^L} \frac{\partial \mathcal{L}}{\partial \phi_N}(x)$$

$$J_{L,MN} = \int_y y^L \dot{C}_{MN}(y)$$

and obtain

$$\begin{aligned} \dot{\mathcal{L}} &= \frac{1}{2} \dot{\hat{C}}(0) \left(\frac{\partial \mathcal{L}}{\partial \phi} \right)^2 - \frac{(-)^N}{2} \dot{C}_{MN}(0) \frac{\partial^2 \mathcal{L}}{\partial \phi_M \partial \phi_N} \\ &+ \frac{(-)^M J_{L,MN}}{2L!} \frac{\partial \mathcal{L}}{\partial \phi_M} \phi_{(M_1} \dots \phi_{M_j)_L} \frac{\partial^{j+1} \mathcal{L}}{\partial \phi_{M_1} \dots \partial \phi_{M_j} \partial \phi_N} \end{aligned}$$

still integrated over x

Exact RG equation for $\mathcal{L}$

$$
\begin{aligned}
\dot{\mathcal{L}} &= \frac{1}{2}\dot{\hat{C}}(0)\left(\frac{\partial\mathcal{L}}{\partial\phi}\right)^2 - \frac{(-)^N}{2}\dot{C}_{MN}(0)\frac{\partial^2\mathcal{L}}{\partial\phi_M\partial\phi_N} \\
&+ \frac{(-)^M J_{L,MN}}{2L!}\frac{\partial\mathcal{L}}{\partial\phi_M}\phi_{(M_1}\cdots\phi_{M_j)_L}\frac{\partial^{j+1}\mathcal{L}}{\partial\phi_{M_1}\ldots\partial\phi_{M_j}\partial\phi_N}
\end{aligned}
$$

Sums over:

$$
\begin{aligned}
&j = 1, \ldots, L \\
&L \neq 0\ , \ \ L = \{\lambda_1, (\lambda_1\lambda_2), (\lambda_1\lambda_2\lambda_3), \ldots\} \\
&M_1, \ldots, M_j, M, N \geq 0
\end{aligned}
$$

Do truncations make the sums finite?

Only if they remove the generation of infinitely many higher derivative interactions during an infinitesimal RG step

Truncations for $\mathcal{L}$

Truncation criteria:

1. neglect the dependence of $\mathcal{L}$ on $\phi_{\mu_1 \ldots \mu_m}$ for $m > n$

2. truncate the explicit dependence of $\dot{\mathcal{L}}$ on $\phi_{\mu_1 \ldots \mu_m}$ up to some power

Truncations for $\mathcal{L}$

Truncation 1: project on $\mathcal{L}(\phi, \phi_\mu)$

we need $M_1 = \cdots = M_j = 0$ but still the infinite sum over j remains

$$\begin{aligned}
\dot{\mathcal{L}} \;=\; & \frac{1}{2}\dot{\hat{C}}(0)\left(\frac{\partial \mathcal{L}}{\partial \phi}\right)^2 - \frac{1}{2}\dot{C}(0)\frac{\partial^2 \mathcal{L}}{\partial \phi^2} + \frac{1}{2}\dot{C}_{\mu\nu}(0)\frac{\partial^2 \mathcal{L}}{\partial \phi_\mu \partial \phi_\nu} \\
& + \frac{J_{\lambda_1 \cdots \lambda_j, \mu}}{2j!}\phi_{\lambda_1}\ldots\phi_{\lambda_j}\left(\frac{\partial \mathcal{L}}{\partial \phi}\frac{\partial^{j+1}\mathcal{L}}{\partial \phi^j \partial \phi_\mu} - \frac{\partial \mathcal{L}}{\partial \phi_\mu}\frac{\partial^{j+1}\mathcal{L}}{\partial \phi^{j+1}}\right) \\
& - \frac{J_{\lambda_1 \cdots \lambda_j, \mu\nu}}{2j!}\phi_{\lambda_1}\ldots\phi_{\lambda_j}\frac{\partial \mathcal{L}}{\partial \phi_\mu}\frac{\partial^{j+1}\mathcal{L}}{\partial \phi^j \partial \phi_\nu} \\
& + \frac{J_{\lambda_1 \cdots \lambda_j, 0}}{2j!}\phi_{\lambda_1}\ldots\phi_{\lambda_j}\frac{\partial \mathcal{L}}{\partial \phi}\frac{\partial^{j+1}\mathcal{L}}{\partial \phi^{j+1}}
\end{aligned}$$

Truncations for $\mathcal{L}$

Truncation 1: project on $\mathcal{L}(\phi, \phi_\mu)$

input: $\mathcal{L}_t = \frac{1}{2}\phi_\mu\phi^\mu + v(\phi)$

$$
\begin{aligned}
\dot{\mathcal{L}} \;=\; & \frac{1}{2}\dot{\hat{C}}(0)\left(\frac{\partial\mathcal{L}}{\partial\phi}\right)^2 - \frac{1}{2}\dot{C}(0)\frac{\partial^2\mathcal{L}}{\partial\phi^2} + \frac{1}{2}\dot{C}_{\mu\nu}(0)\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\phi_\nu} \\
& + \frac{J_{\lambda_1\cdots\lambda_j,\mu}}{2j!}\phi_{\lambda_1}\ldots\phi_{\lambda_j}\left(\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial^{j+1}\mathcal{L}}{\partial\phi^j\partial\phi_\mu} - \frac{\partial\mathcal{L}}{\partial\phi_\mu}\frac{\partial^{j+1}\mathcal{L}}{\partial\phi^{j+1}}\right) \\
& - \frac{J_{\lambda_1\cdots\lambda_j,\mu\nu}}{2j!}\phi_{\lambda_1}\ldots\phi_{\lambda_j}\frac{\partial\mathcal{L}}{\partial\phi_\mu}\frac{\partial^{j+1}\mathcal{L}}{\partial\phi^j\partial\phi_\nu} \\
& + \frac{J_{\lambda_1\cdots\lambda_j,0}}{2j!}\phi_{\lambda_1}\ldots\phi_{\lambda_j}\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial^{j+1}\mathcal{L}}{\partial\phi^{j+1}}
\end{aligned}
$$

output: $\mathcal{L}_{t+\epsilon} = \mathcal{L}_t + \delta\mathcal{L}(\phi, \phi_\mu)$

Truncations for $\mathcal{L}$

Truncation 2: the explicit ϕ_μ-dependence of $\dot{\mathcal{L}}$ be quadratic

$$\begin{aligned}
\dot{\mathcal{L}} &= \frac{1}{2}\dot{\hat{C}}(0)\left(\frac{\partial\mathcal{L}}{\partial\phi}\right)^2 - \frac{1}{2}\dot{C}(0)\frac{\partial^2\mathcal{L}}{\partial\phi^2} + \frac{1}{2}\dot{C}_{\mu\nu}(0)\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\phi_\nu} \\
&+ \frac{J_{\lambda,\mu}}{2}\phi_\lambda\left(\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial^2\mathcal{L}}{\partial\phi\partial\phi_\mu} - \frac{\partial\mathcal{L}}{\partial\phi_\mu}\frac{\partial^2\mathcal{L}}{\partial\phi^2}\right) \\
&+ \frac{J_{\lambda\mu,0}}{4}\phi_\mu\phi_\lambda\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial^3\mathcal{L}}{\partial\phi^3}
\end{aligned}$$

Truncations for $\mathcal{L}$

Truncation 2: the explicit ϕ_μ-dependence of $\dot{\mathcal{L}}$ be quadratic

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\begin{aligned}
\dot{\mathcal{L}} \;&=\; \frac{1}{2}\dot{\hat{C}}(0)\left(\frac{\partial \mathcal{L}}{\partial \phi}\right)^2 - \frac{1}{2}\dot{C}(0)\frac{\partial^2 \mathcal{L}}{\partial \phi^2} + \frac{1}{2}\dot{C}_{\mu\nu}(0)\frac{\partial^2 \mathcal{L}}{\partial \phi_\mu \partial \phi_\nu} \\
&- \;\frac{\dot{\hat{C}}(0)}{2}\phi_\mu\left(\frac{\partial \mathcal{L}}{\partial \phi}\frac{\partial^2 \mathcal{L}}{\partial \phi \partial \phi_\mu} - \frac{\partial \mathcal{L}}{\partial \phi_\mu}\frac{\partial^2 \mathcal{L}}{\partial \phi^2}\right) \\
&+ \;\frac{J_{\lambda\mu,0}}{4}\phi_\mu \phi_\lambda \frac{\partial \mathcal{L}}{\partial \phi}\frac{\partial^3 \mathcal{L}}{\partial \phi^3}
\end{aligned}
$$

Truncations for $\mathcal{L}$

Truncation 2: the explicit ϕ_μ-dependence of $\dot{\mathcal{L}}$ be quadratic

$$\begin{aligned}
\dot{\mathcal{L}} &= -\Lambda^{\eta-2} K_0 \left(\frac{\partial \mathcal{L}}{\partial \phi}\right)^2 + \Lambda^{d-2+\eta} I_0 \frac{\partial^2 \mathcal{L}}{\partial \phi^2} + \Lambda^{d+\eta} \frac{I_1}{d} \frac{\partial^2 \mathcal{L}}{\partial \phi_\mu \partial \phi^\mu} \\
&+ \Lambda^{\eta-2} K_0 \, \phi_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi} \frac{\partial^2 \mathcal{L}}{\partial \phi \partial \phi_\mu} - \frac{\partial \mathcal{L}}{\partial \phi_\mu} \frac{\partial^2 \mathcal{L}}{\partial \phi^2} \right) \\
&- \Lambda^{\eta-4} K_1 \, \phi_\mu \phi^\mu \frac{\partial \mathcal{L}}{\partial \phi} \frac{\partial^3 \mathcal{L}}{\partial \phi^3}
\end{aligned}$$

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&+ \Lambda^{\eta-2} K_0 \, \phi_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi} \frac{\partial^2 \mathcal{L}}{\partial \phi \partial \phi_\mu} - \frac{\partial \mathcal{L}}{\partial \phi_\mu} \frac{\partial^2 \mathcal{L}}{\partial \phi^2} \right) \\
&- \Lambda^{\eta-4} K_1 \, \phi_\mu \phi^\mu \frac{\partial \mathcal{L}}{\partial \phi} \frac{\partial^3 \mathcal{L}}{\partial \phi^3}
\end{aligned}$$

upon integration by parts, the last term becomes

$$+ \Lambda^{\eta-4} K_1 \, \phi_\mu \phi^\mu \left(\frac{\partial^2 \mathcal{L}}{\partial \phi^2} \right)^2$$

Truncations for $\mathcal{L}$

input: $\mathcal{L}_t = \frac{1}{2}\phi_\mu\phi^\mu + v(\phi)$

$$\begin{aligned}\dot{\mathcal{L}} &= -\Lambda^{\eta-2}K_0\left(\frac{\partial\mathcal{L}}{\partial\phi}\right)^2 + \Lambda^{d-2+\eta}I_0\ \frac{\partial^2\mathcal{L}}{\partial\phi^2} + \Lambda^{d+\eta}\frac{I_1}{d}\ \frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\phi^\mu} \\ &+ \Lambda^{\eta-2}K_0\ \phi_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial^2\mathcal{L}}{\partial\phi\partial\phi_\mu} - \frac{\partial\mathcal{L}}{\partial\phi_\mu}\frac{\partial^2\mathcal{L}}{\partial\phi^2}\right) \\ &+ \Lambda^{\eta-4}K_1\ \phi_\mu\phi^\mu\left(\frac{\partial^2\mathcal{L}}{\partial\phi^2}\right)^2\end{aligned}$$

output: $\mathcal{L}_{t+\epsilon} = \frac{1}{2}z(\phi)\phi_\mu\phi^\mu + v(\phi)$

Truncations for $\mathcal{L}$

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$$+\ \Lambda^{\eta-2}K_0\ \phi_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial^2\mathcal{L}}{\partial\phi\partial\phi_\mu} - \frac{\partial\mathcal{L}}{\partial\phi_\mu}\frac{\partial^2\mathcal{L}}{\partial\phi^2}\right)$$
$$+\ \Lambda^{\eta-4}K_1\ \phi_\mu\phi^\mu\left(\frac{\partial^2\mathcal{L}}{\partial\phi^2}\right)^2$$

output: $\mathcal{L}_{t+\epsilon} = \frac{1}{2}z(\phi)\phi_\mu\phi^\mu + v(\phi)$

$\mathcal{L}_{t+2\epsilon} = \frac{1}{2}z(\phi)\phi_\mu\phi^\mu + v(\phi) + \frac{1}{4}w(\phi)(\phi_\mu\phi^\mu)^2$

Truncations for $\mathcal{L}$

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output: $\mathcal{L}_{t+\epsilon} = \frac{1}{2}z(\phi)\phi_\mu\phi^\mu + v(\phi)$

$$\mathcal{L}_{t+2\epsilon} = \frac{1}{2}z(\phi)\phi_\mu\phi^\mu + v(\phi) + \frac{1}{4}w(\phi)(\phi_\mu\phi^\mu)^2$$

$$\mathcal{L}_{t+\Delta t} = \mathcal{L}_t + \Delta\mathcal{L}(\phi,\phi_\mu)$$

Truncations for $\mathcal{L}$

Given the approximate RG equation

$$\begin{aligned}\dot{\mathcal{L}} &= -\Lambda^{\eta-2}K_0\left(\frac{\partial\mathcal{L}}{\partial\phi}\right)^2 + \Lambda^{d-2+\eta}I_0\,\frac{\partial^2\mathcal{L}}{\partial\phi^2} + \Lambda^{d+\eta}\frac{I_1}{d}\,\frac{\partial^2\mathcal{L}}{\partial\phi_\mu\partial\phi^\mu}\\ &+ \Lambda^{\eta-2}K_0\,\phi_\mu\left(\frac{\partial\mathcal{L}}{\partial\phi}\frac{\partial^2\mathcal{L}}{\partial\phi\partial\phi_\mu} - \frac{\partial\mathcal{L}}{\partial\phi_\mu}\frac{\partial^2\mathcal{L}}{\partial\phi^2}\right)\\ &+ \Lambda^{\eta-4}K_1\,\phi_\mu\phi^\mu\left(\frac{\partial^2\mathcal{L}}{\partial\phi^2}\right)^2\end{aligned}$$

how can we account for the flow of the infinitely many couplings inside $\mathcal{L}$, both in the ϕ and in the ϕ_μ sector?

if $\phi_\mu = const$ then $\phi(x) = \phi_\mu x^\mu + const'$ and the x-integral is hard to compute

From $\mathcal{L}$ to $\mathcal{H}$

Natural solution: a covariant Hamiltonian formalism

ϕ_μ and ϕ cannot be simultaneously constant, but π^μ and ϕ can!

so let us describe the RG flow in phase space $\mathcal{H}(\phi, \pi^\mu)$

(G.P.Vacca, L.Z., Phys. Rev. D **86** (2012))

From $\mathcal{L}$ to $\mathcal{H}$

Replace derivatives with fields at any scale

introduce tensor fields π^M $\qquad (M \neq 0)$

$$\mathcal{H}(x, \phi, \pi^M) = \mathrm{ext}_{\phi_M}\left\{\mathrm{i}\,\pi^M \phi_M + \mathcal{L}(x, \phi, \phi_M)\right\}$$

$$\pi^M(x) = \mathrm{i}\frac{\partial \mathcal{L}}{\partial \phi_M}(x) \quad , \quad \phi_M(x) = -\mathrm{i}\frac{\partial \mathcal{H}}{\partial \pi^M}(x)$$

equations of motion

$$\mathrm{i}(-)^M \partial_M \pi^M(x) = \frac{\partial \mathcal{H}}{\partial \phi}(x)$$

$$\phi_M(x) = -\mathrm{i}\frac{\partial \mathcal{H}}{\partial \pi^M}(x)$$

if $\pi^M \neq 0$ is a solution $\Rightarrow$ rotations/Lorentz SSB

RG equation for $\mathcal{H}$

Setting π^M and ϕ to constants the x-integral factorizes

the ERGE becomes a PDE for a function of infinitely many variables

any truncated version reduces to a standard PDE

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Setting π^M and ϕ to constants the x-integral factorizes

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any truncated version reduces to a standard PDE, e.g.

$$\begin{aligned}
\dot{\mathcal{L}} &= -\Lambda^{\eta-2} K_0 \left(\frac{\partial \mathcal{L}}{\partial \phi}\right)^2 + \Lambda^{d-2+\eta} I_0 \, \frac{\partial^2 \mathcal{L}}{\partial \phi^2} + \Lambda^{d+\eta} \frac{I_1}{d} \, \frac{\partial^2 \mathcal{L}}{\partial \phi_\mu \partial \phi^\mu} \\
&+ \Lambda^{\eta-2} K_0 \, \phi_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi} \frac{\partial^2 \mathcal{L}}{\partial \phi \partial \phi_\mu} - \frac{\partial \mathcal{L}}{\partial \phi_\mu} \frac{\partial^2 \mathcal{L}}{\partial \phi^2}\right) \\
&+ \Lambda^{\eta-4} K_1 \, \phi_\mu \phi^\mu \left(\frac{\partial^2 \mathcal{L}}{\partial \phi^2}\right)^2
\end{aligned}$$

in terms of $\mathcal{H}\,(\varpi \equiv \pi^\mu \pi_\mu/2, \ \phi)$

RG equation for $\mathcal{H}$

Setting π^M and ϕ to constants the x-integral factorizes

the ERGE becomes a PDE for a function of infinitely many variables

any truncated version reduces to a standard PDE, e.g.

$$\begin{aligned}
\dot{\mathcal{H}} \;=\; & \Lambda^{\eta-2} K_0 \Bigg[-\mathcal{H}^{(01)\,2} + 2\varpi \mathcal{H}^{(10)} \left(\frac{\mathcal{H}^{(01)}\mathcal{H}^{(11)}}{\mathcal{H}^{(10)} + 2\varpi\mathcal{H}^{(20)}} + \mathcal{H}^{(02)} \right) \\
& - \left(\frac{2\varpi\mathcal{H}^{(10)}\mathcal{H}^{(11)}}{\mathcal{H}^{(10)} + 2\varpi\mathcal{H}^{(20)}} \right)^2 \Bigg] - 2\Lambda^{\eta-4} K_1\, \varpi \left(\mathcal{H}^{(10)}\mathcal{H}^{(02)} \right)^2 \\
& + \Lambda^{d-2+\eta} I_0\, \mathcal{H}^{(02)} + \frac{\Lambda^{d+\eta}}{d} I_1 \left(\frac{d-1}{\mathcal{H}^{(10)}} + \frac{1}{\mathcal{H}^{(10)} + 2\varpi\mathcal{H}^{(20)}} \right)
\end{aligned}$$

in terms of $\mathcal{H}\,(\varpi \equiv \pi^\mu \pi_\mu/2,\ \phi)$

Rescaling $\mathcal{H}$, ϖ and ϕ: $\quad K_0, I_0, I_1 \to 1$ and $K_1 \to B$

RG equations for Z and V

Dimensionless renormalized fields: $d_\phi = (d - 2 + \eta)/2\,, \quad d_\pi = (d - \eta)/2$

$$\mathcal{H} \to \Lambda^d \mathcal{H}\ , \qquad \varpi \to \Lambda^{2d_\pi} \varpi\ , \qquad \phi \to \Lambda^{d_\phi} \phi$$

Project on: $\mathcal{H}(\varpi, \phi) = \varpi / Z(\phi) + V(\phi)$

$$\begin{aligned} \dot{V} &= dV - \frac{d-2+\eta}{2}\phi V' - (V')^2 + V'' + Z \\ \dot{Z} &= -\eta Z - \frac{d-2+\eta}{2}\phi Z' - 2ZV'' + Z'' - 2\frac{(Z')^2}{Z} + 2B\,(V'')^2 \end{aligned}$$

RG equations for Z and V

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\end{aligned}$$

the $O(\partial^2)$ derivative expansion gives (Ball et al., Phys. Lett. B **347** (1995))

$$\begin{aligned}
\dot{V} &= dV - \frac{d-2+\eta}{2}\phi V' - (V')^2 + V'' + 2AZ \\
\dot{Z} &= -\eta Z - \frac{d-2+\eta}{2}\phi Z' - 4ZV'' + Z'' - 2Z'V' - \frac{\eta}{2} + \tilde{B}\,(V'')^2
\end{aligned}$$

RG equations for Z and V

Dimensionless renormalized fields: $d_\phi = (d-2+\eta)/2\,, \quad d_\pi = (d-\eta)/2$

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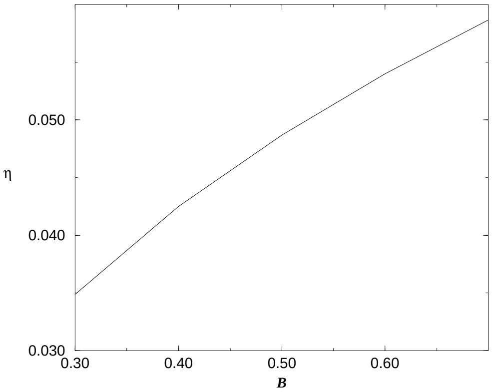
Wilson-Fisher FP

(Ball et al., Phys. Lett. B **347** (1995), Comellas, Nucl. Phys. B **509** (1998))

In the $O(\partial^2)$ derivative expansion: no preferred value of η

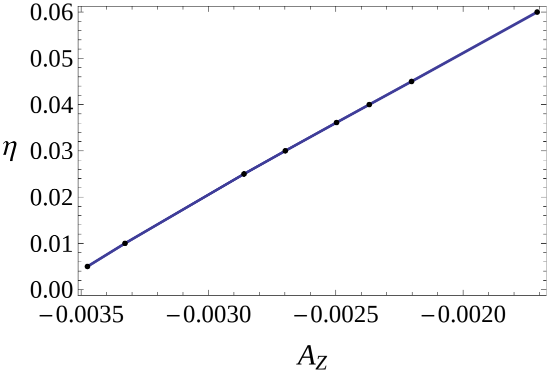


requiring $V''(0)$ to be close to the LPA value: $\eta(A, \tilde{B}, Z(0)) \sim 0.042$

Wilson-Fisher FP

(A.Ugolotti and G.P.Vacca, Master Thesis, Bologna U.)

In a scheme with $B = 0$: no preferred value of η

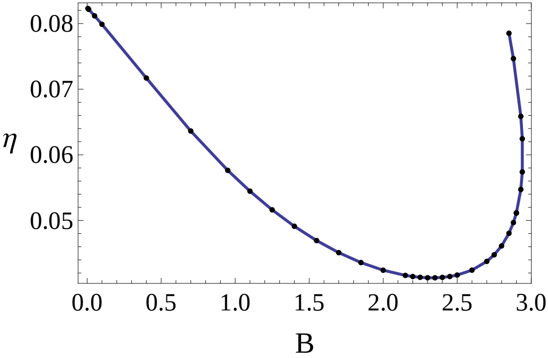


where $Z(\phi)^{-1} \sim A_Z \, \phi^{\frac{4}{d-2+\eta}}$

Wilson-Fisher FP

(A.Ugolotti and G.P.Vacca, Master Thesis, Bologna U.)

In a scheme with $B \neq 0$



minimum at: $\eta = 0.0408$

Summary

- We enjoyed the freedom to devise truncations inspired by the derivative expansion, retaining less information for each $O(\partial^m)$, but including some information for any order
- This poses the challenge to simultaneously describe infinitely many derivative and non-derivative interactions. We addressed this issue by means of a covariant Hamiltonian formalism
- We derived a first example of a PDE that accomplishes this goal
- Wen truncated to $O(\pi^2)$, this PDE does not contain all the information of an $O(\partial^2)$ derivative expansion. We compared the two truncations by analyzing the 3D Wilson-Fisher FP