Entanglement entropy:
From Field Theory to Condensed Matter

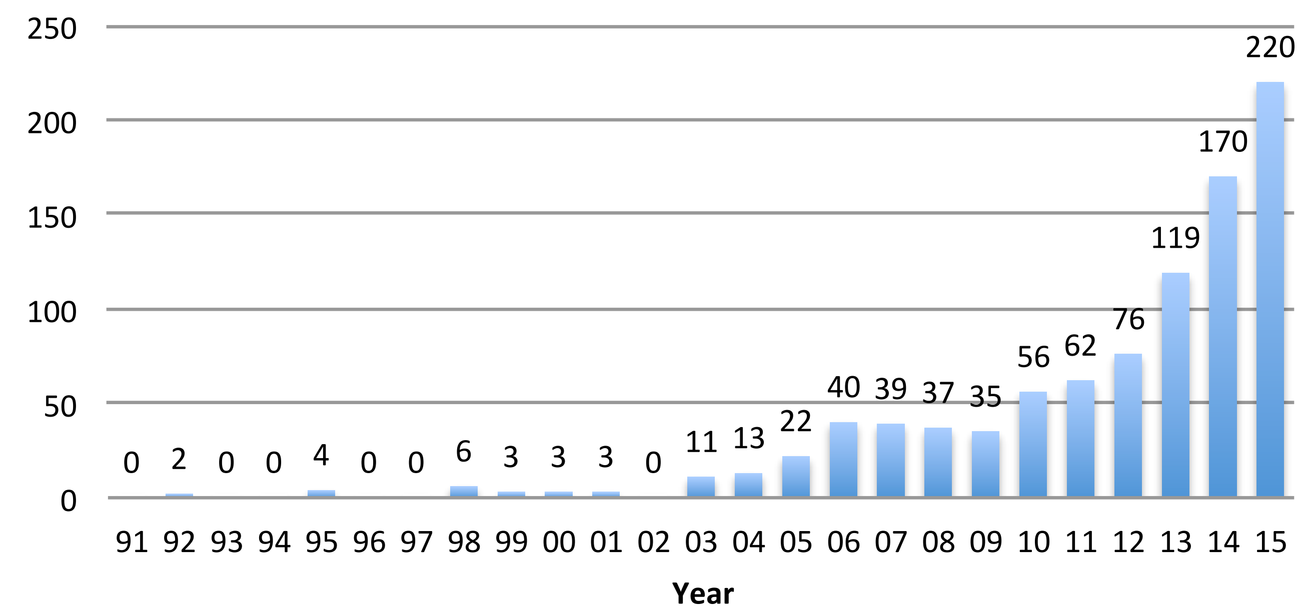
Pasquale Calabrese
SISSA-Trieste

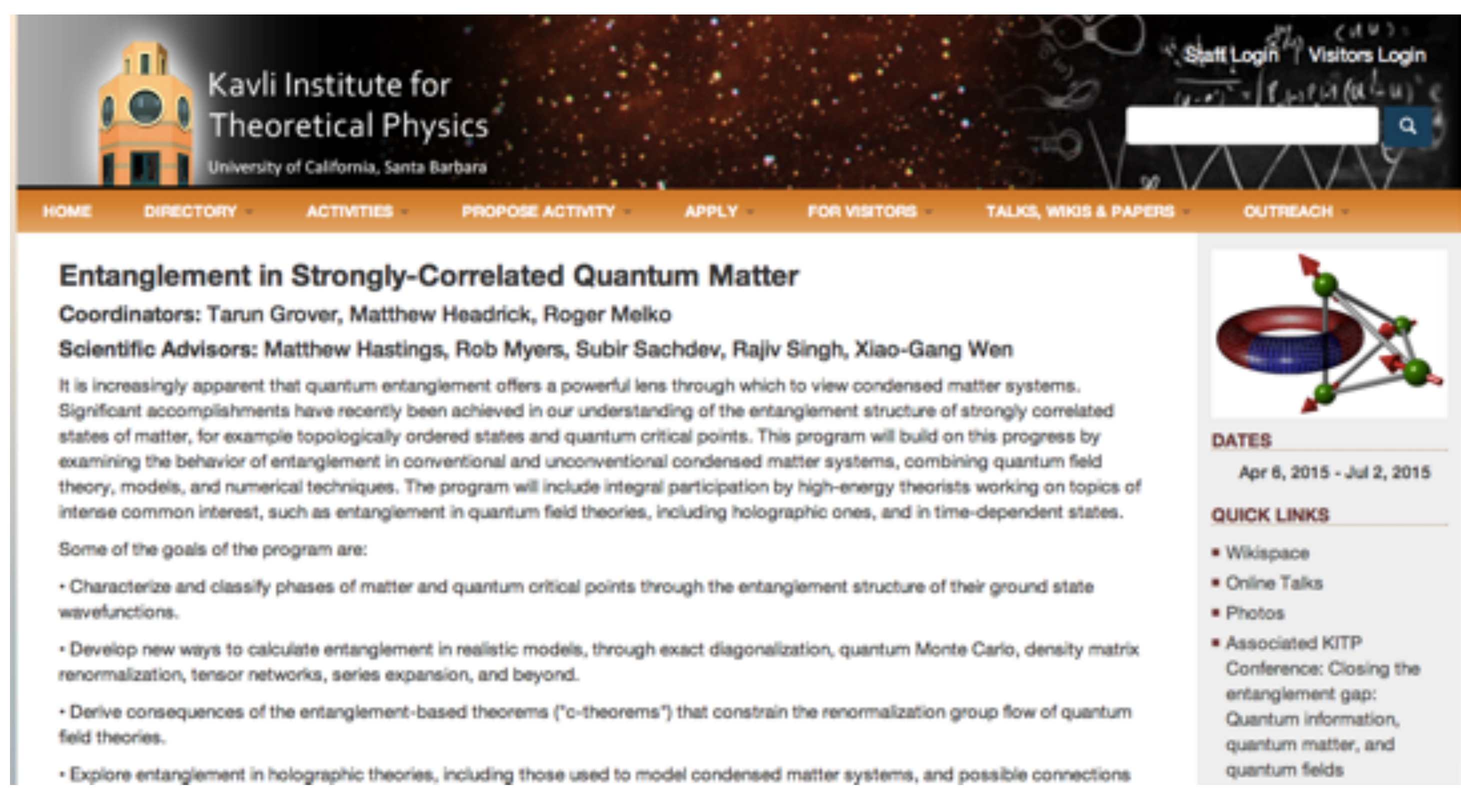
ERG 2016, Trieste, 23/9/2016

Based on collaboration with:
J. Cardy, V. Alba, M. Fagotti, E. Tonni....

Why entanglement entropy?

hep-th arXiv preprints with "entanglement" in the title





Kavli Institute for Theoretical Physics

University of California, Santa Barbara

Staff Login | Visitors Login

HOME DIRECTORY ACTIVITIES PROPOSE ACTIVITY APPLY FOR VISITORS TALKS, WIKIS & PAPERS OUTREACH

Entanglement in Strongly-Correlated Quantum Matter

Coordinators: Tarun Grover, Matthew Headrick, Roger Melko

Scientific Advisors: Matthew Hastings, Rob Myers, Subir Sachdev, Rajiv Singh, Xiao-Gang Wen

It is increasingly apparent that quantum entanglement offers a powerful lens through which to view condensed matter systems. Significant accomplishments have recently been achieved in our understanding of the entanglement structure of strongly correlated states of matter, for example topologically ordered states and quantum critical points. This program will build on this progress by examining the behavior of entanglement in conventional and unconventional condensed matter systems, combining quantum field theory, models, and numerical techniques. The program will include integral participation by high-energy theorists working on topics of intense common interest, such as entanglement in quantum field theories, including holographic ones, and in time-dependent states.

Some of the goals of the program are:

- Characterize and classify phases of matter and quantum critical points through the entanglement structure of their ground state wavefunctions.
- Develop new ways to calculate entanglement in realistic models, through exact diagonalization, quantum Monte Carlo, density matrix renormalization, tensor networks, series expansion, and beyond.
- Derive consequences of the entanglement-based theorems ("c-theorems") that constrain the renormalization group flow of quantum field theories.
- Explore entanglement in holographic theories, including those used to model condensed matter systems, and possible connections

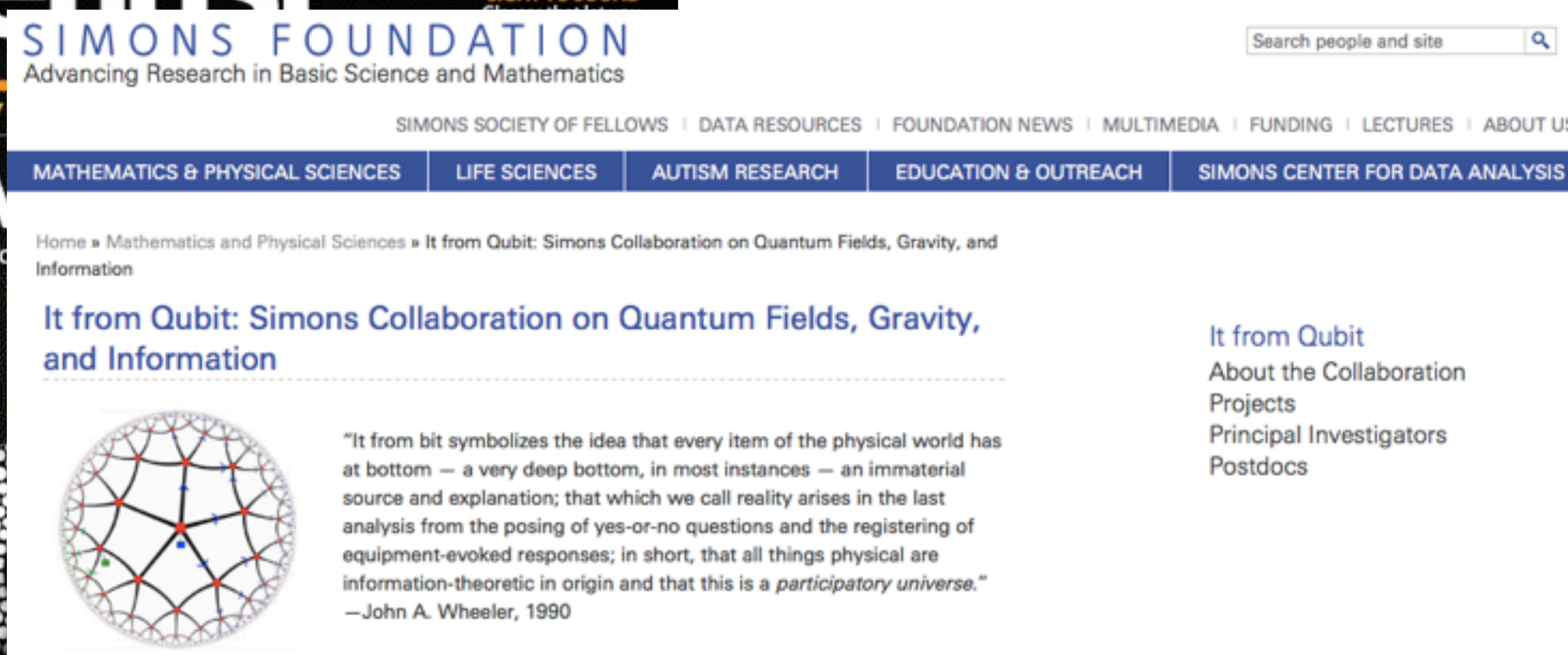
DATES

Apr 6, 2015 - Jul 2, 2015

QUICK LINKS

- Wikispace
- Online Talks
- Photos
- Associated KITP Conference: Closing the entanglement gap: Quantum information, quantum matter, and quantum fields

Why entanglement entropy?



SIMONS FOUNDATION
Advancing Research in Basic Science and Mathematics

Search people and site

SIMONS SOCIETY OF FELLOWS | DATA RESOURCES | FOUNDATION NEWS | MULTIMEDIA | FUNDING | LECTURES | ABOUT US

MATHEMATICS & PHYSICAL SCIENCES | LIFE SCIENCES | AUTISM RESEARCH | EDUCATION & OUTREACH | SIMONS CENTER FOR DATA ANALYSIS

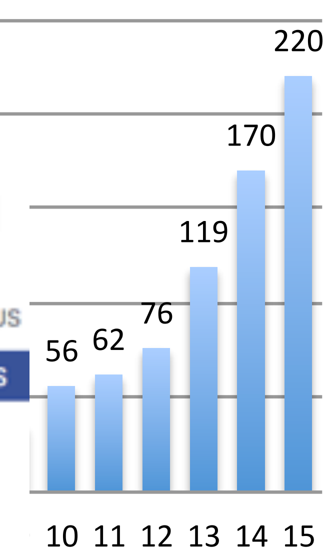
Home » Mathematics and Physical Sciences » It from Qubit: Simons Collaboration on Quantum Fields, Gravity, and Information

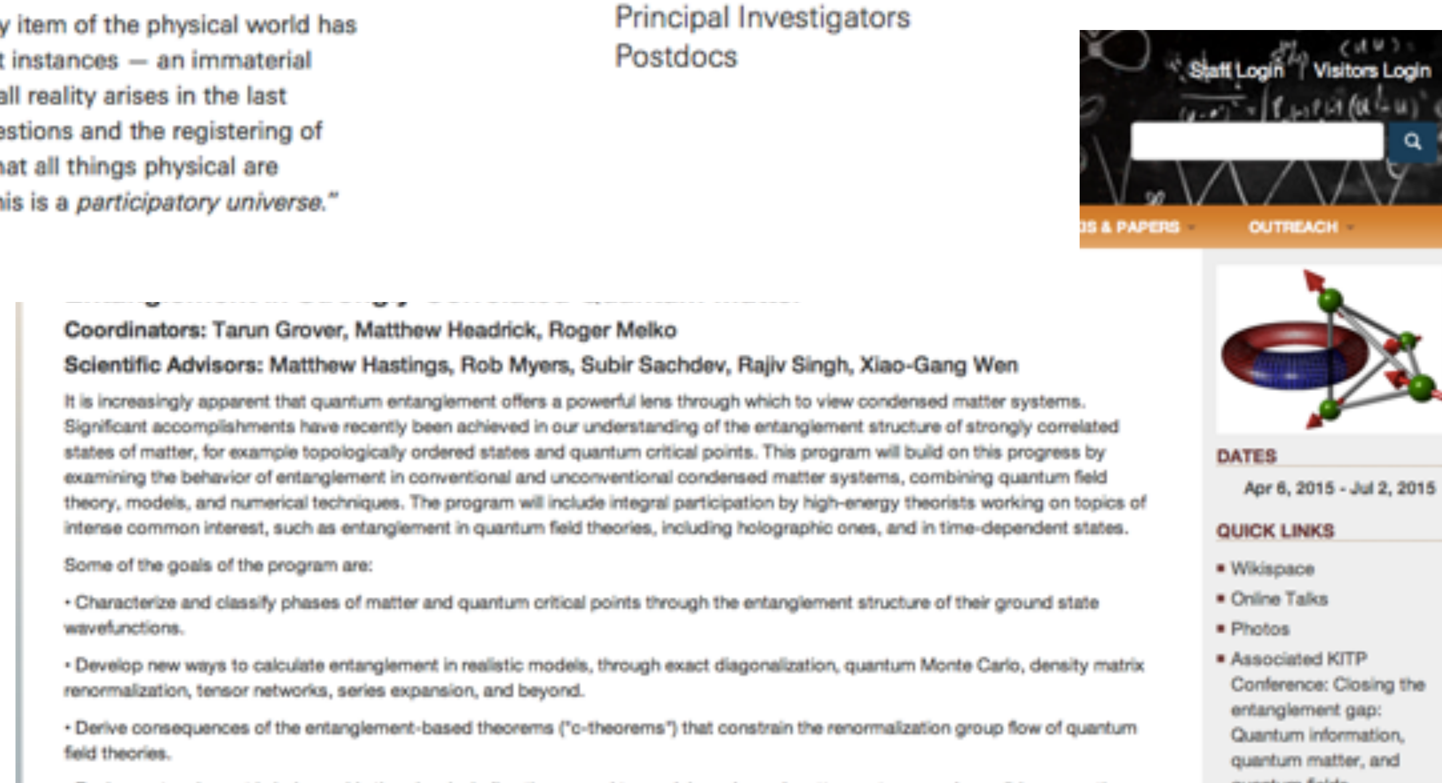
It from Qubit: Simons Collaboration on Quantum Fields, Gravity, and Information

"It from bit symbolizes the idea that every item of the physical world has at bottom — a very deep bottom, in most instances — an immaterial source and explanation; that which we call reality arises in the last analysis from the posing of yes-or-no questions and the registering of equipment-evoked responses; in short, that all things physical are information-theoretic in origin and that this is a *participatory universe.*"
—John A. Wheeler, 1990

It from Qubit
About the Collaboration
Projects
Principal Investigators
Postdocs

hep-th arXiv preprints with "entanglement" in the title





Staff Login | Visitors Login

S & PAPERS | OUTREACH

Coordinators: Tarun Grover, Matthew Headrick, Roger Melko

Scientific Advisors: Matthew Hastings, Rob Myers, Subir Sachdev, Rajiv Singh, Xiao-Gang Wen

It is increasingly apparent that quantum entanglement offers a powerful lens through which to view condensed matter systems. Significant accomplishments have recently been achieved in our understanding of the entanglement structure of strongly correlated states of matter, for example topologically ordered states and quantum critical points. This program will build on this progress by examining the behavior of entanglement in conventional and unconventional condensed matter systems, combining quantum field theory, models, and numerical techniques. The program will include integral participation by high-energy theorists working on topics of intense common interest, such as entanglement in quantum field theories, including holographic ones, and in time-dependent states.

Some of the goals of the program are:

• Characterize and classify phases of matter and quantum critical points through the entanglement structure of their ground state wavefunctions.

• Develop new ways to calculate entanglement in realistic models, through exact diagonalization, quantum Monte Carlo, density matrix renormalization, tensor networks, series expansion, and beyond.

• Derive consequences of the entanglement-based theorems ("c-theorems") that constrain the renormalization group flow of quantum field theories.

• Explore entanglement in holographic theories, including those used to model condensed matter systems, and possible connections

DATES
Apr 6, 2015 - Jul 2, 2015

QUICK LINKS
• Wikispace
• Online Talks
• Photos
• Associated KITP Conference: Closing the entanglement gap: Quantum information, quantum matter, and quantum fields

Many-body quantum systems

When many particles do not interact, their properties follow straightforwardly from those of few

Free Fermions
= metals

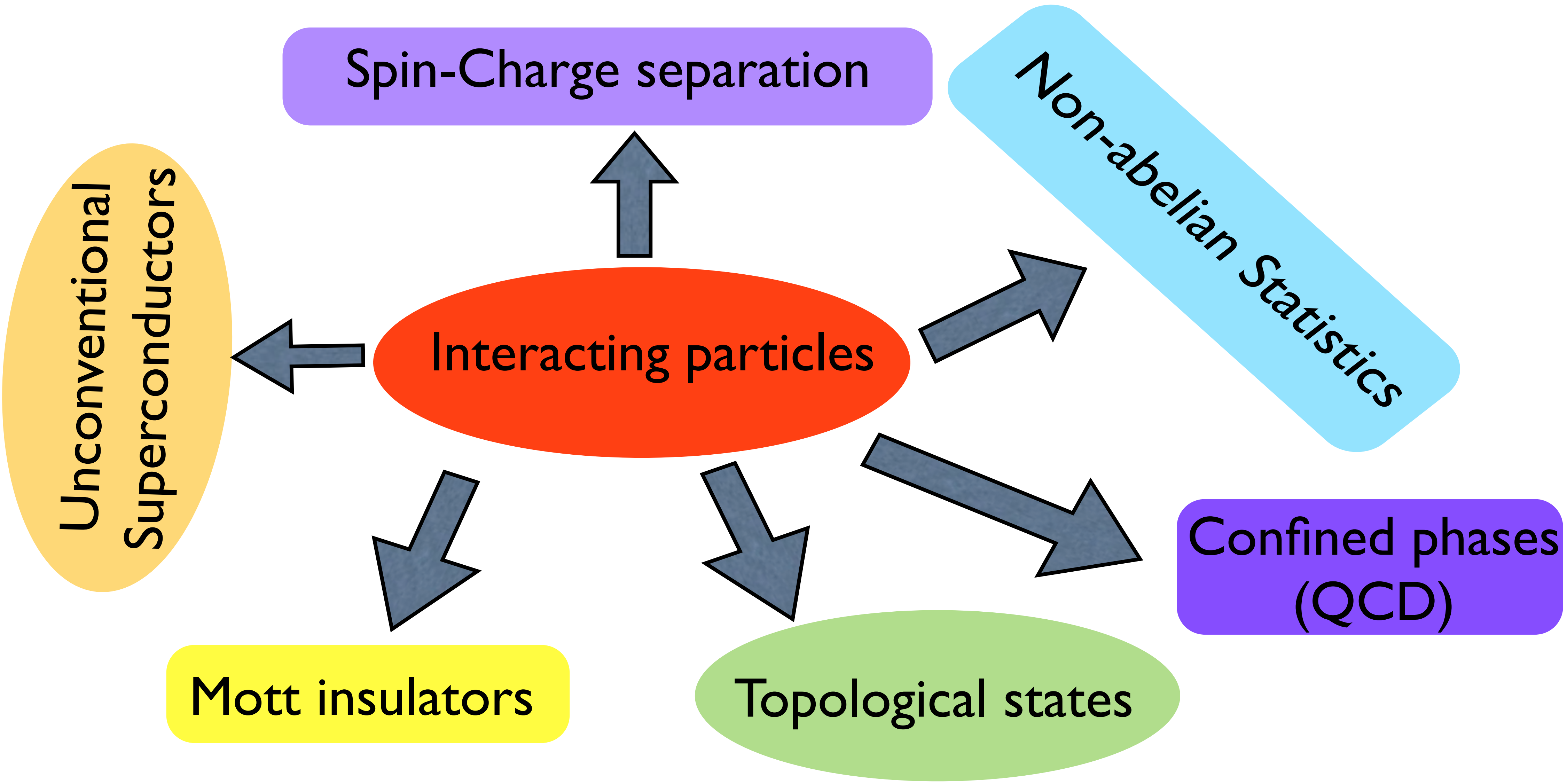
Free Bosons
= superfluids

Interactions dramatically change this paradigm especially in low dimensions

"More is different"

PW Anderson 1972

Interactions give rise to new phases of matter



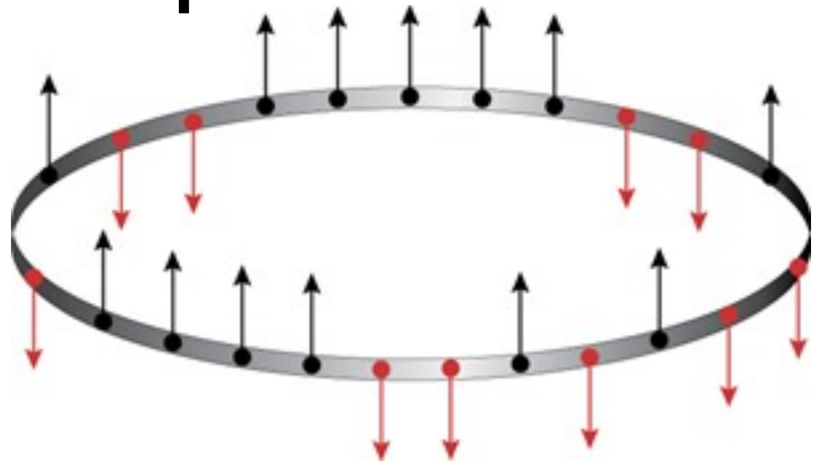
The properties of many do not follow simply from those of few: "more is truly different!"

"The complexity frontier"

How to describe these many-body systems?

Numerically? Too difficult, e.g. for a spin-chain

$$|\Psi\rangle = \sum_{s_i=\pm} A_{s_1 s_2 \ldots s_N} |s_1, s_2, \ldots s_N\rangle$$



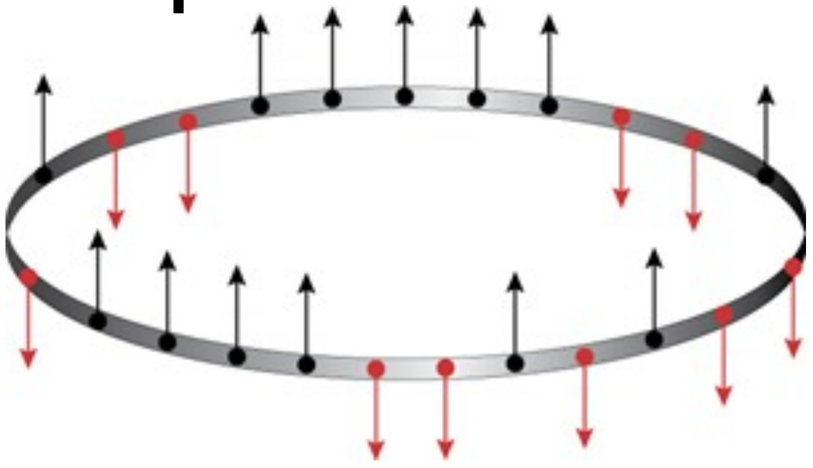
2^N coefficients: too many for a classical PC

"The complexity frontier"

How to describe these many-body systems?

Numerically? Too difficult, e.g. for a spin-chain

$$|\Psi\rangle = \sum_{s_i=\pm} A_{s_1 s_2 \ldots s_N} |s_1, s_2, \ldots s_N\rangle$$



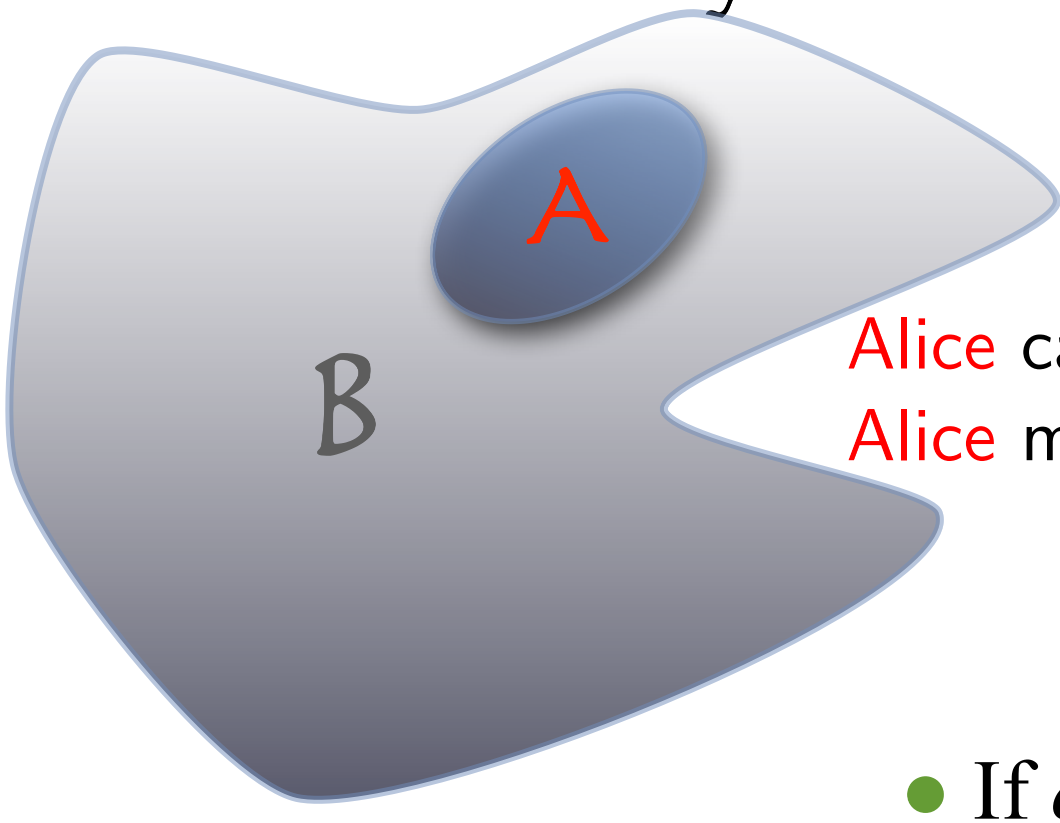
2^N coefficients: too many for a classical PC

We need a criterion that sets physical states apart from the others

Entanglement is this criterion

Entanglement entropy

Consider a system in a quantum state $|\psi\rangle$ ($\rho=|\psi\rangle\langle\psi|$)



$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Alice can measure only in A, while Bob in the remainder B

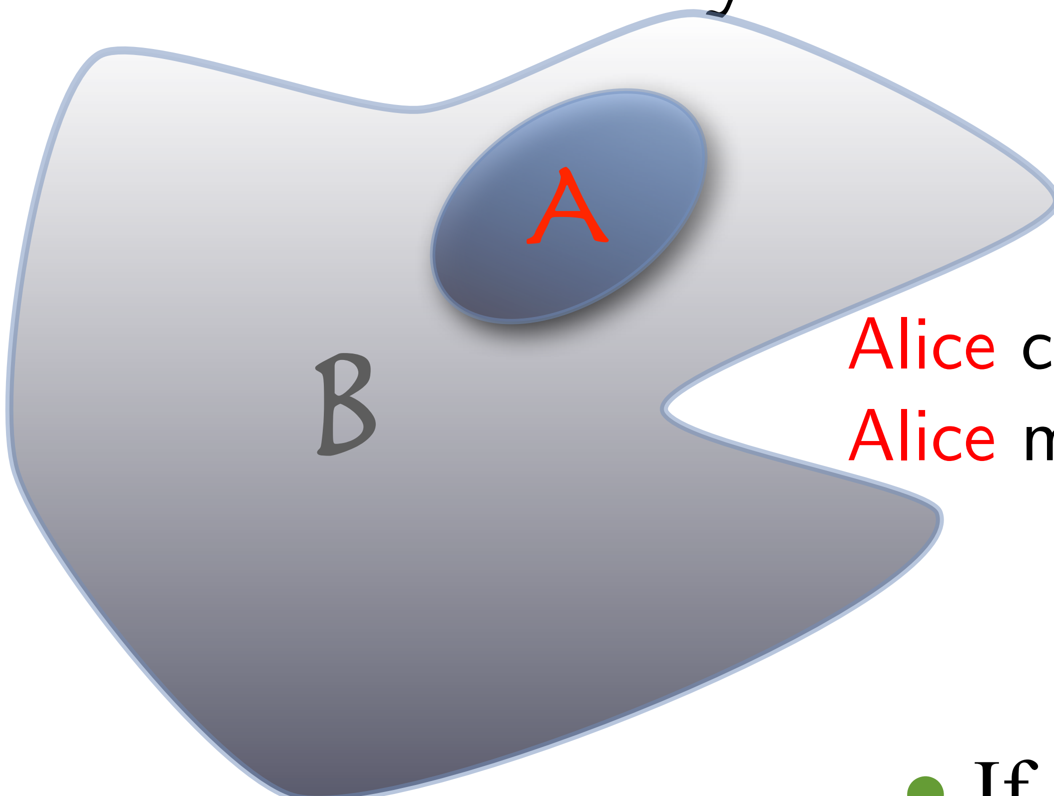
Alice measures are entangled with Bob's ones: Schmidt deco

$$|\Psi\rangle = \sum_n c_n |\Psi_n\rangle_A |\Psi_n\rangle_B \qquad c_n \geq 0, \sum_n c_n^2 = 1$$

- If $c_1=1 \Rightarrow |\psi\rangle$ unentagled
- If c_i all equal $\Rightarrow |\psi\rangle$ maximally entangled

Entanglement entropy

Consider a system in a quantum state $|\psi\rangle$ ($\rho=|\psi\rangle\langle\psi|$)



$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Alice can measure only in A, while Bob in the remainder B

Alice measures are entangled with Bob's ones: Schmidt deco

$$|\Psi\rangle = \sum_n c_n |\Psi_n\rangle_A |\Psi_n\rangle_B \qquad c_n \geq 0, \sum_n c_n^2 = 1$$

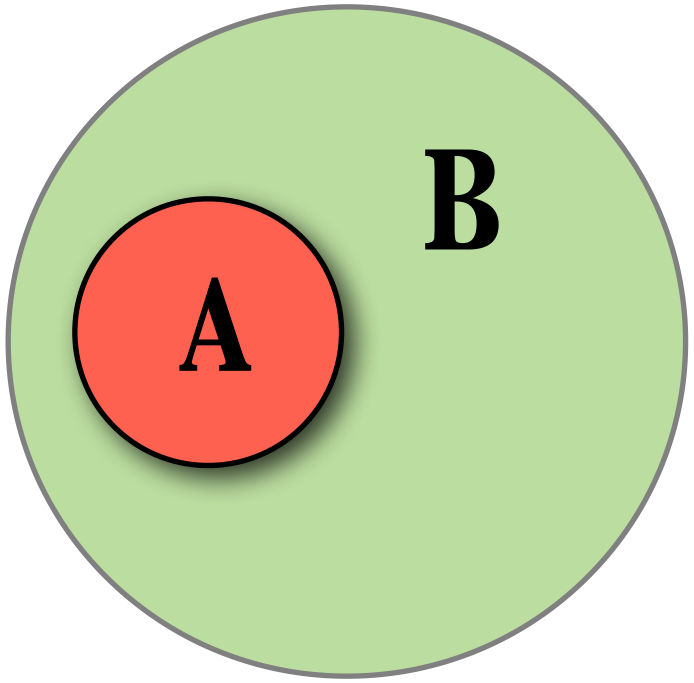
- If $c_1=1 \Rightarrow |\psi\rangle$ unentagled
- If c_i all equal $\Rightarrow |\psi\rangle$ maximally entangled

A natural measure is the entanglement entropy ($\rho_A = \mathrm{Tr}_B\, \rho$)

$$S_A \equiv -\mathrm{Tr}\, \rho_A \ln \rho_A = S_B$$
$$= -\sum c_n^2 \ln c_n^2$$

basis independent

Entanglement in extended systems



If $|\psi\rangle$ is the ground state of a local Hamiltonian

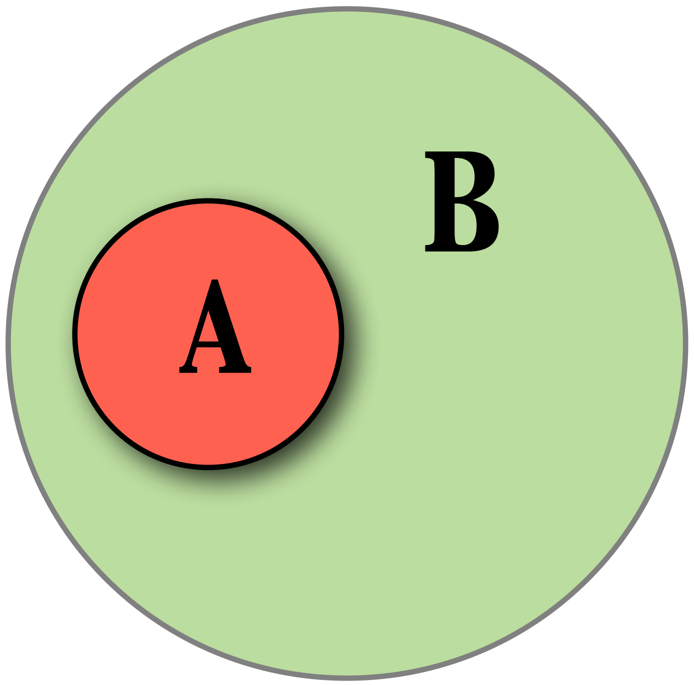
Area Law

$S_A \propto$ Area separating A and B

[Srednicki '93 +many more]

If the Hamiltonian has a gap

Entanglement in extended systems



If $|\psi\rangle$ is the ground state of a local Hamiltonian

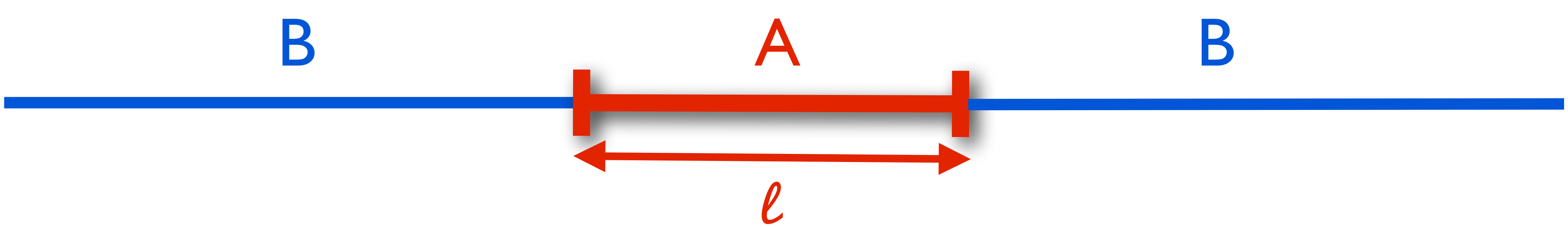
Area Law

$S_A \propto$ Area separating A and B

[Srednicki '93
+many more]

If the Hamiltonian has a gap

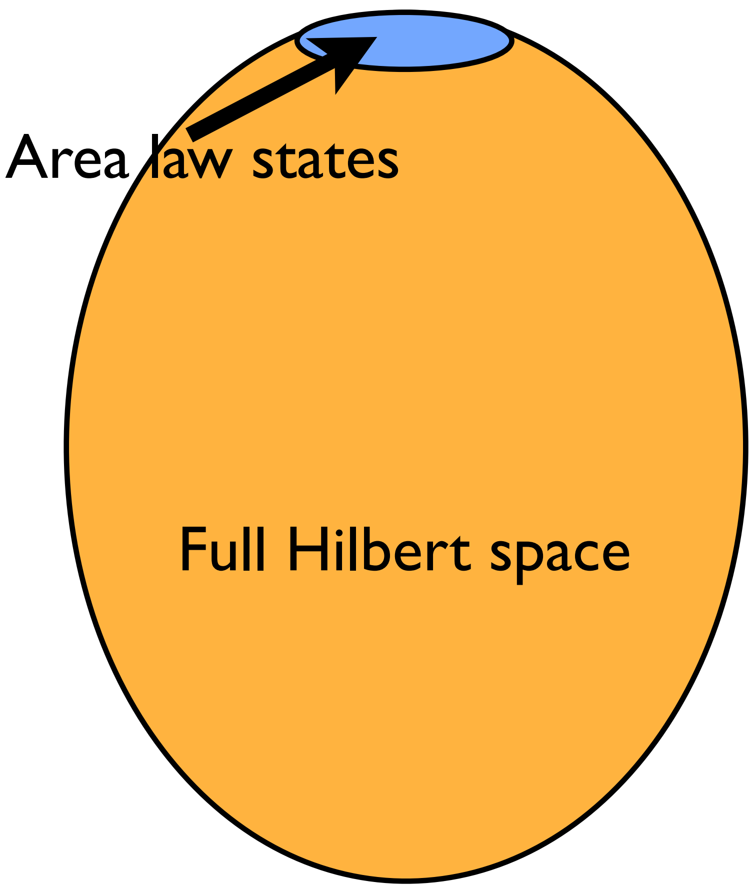
In a 1+1 D CFT Holzhey, Larsen, Wilczek '94



$$S_A = \frac{c}{3} \ln \ell$$

This is the most effective way to determine the central charge

Importance



Only a tiny fraction of states satisfy the area law (or small violations)

→ If we can limit the search for the ground state to this small subset, the complexity of the problem is exponentially reduced

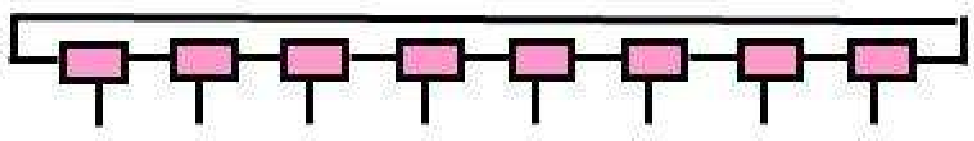
→ One meaning of S_A:

S_A gives the amount of classical information required to specify $|\Psi\rangle$

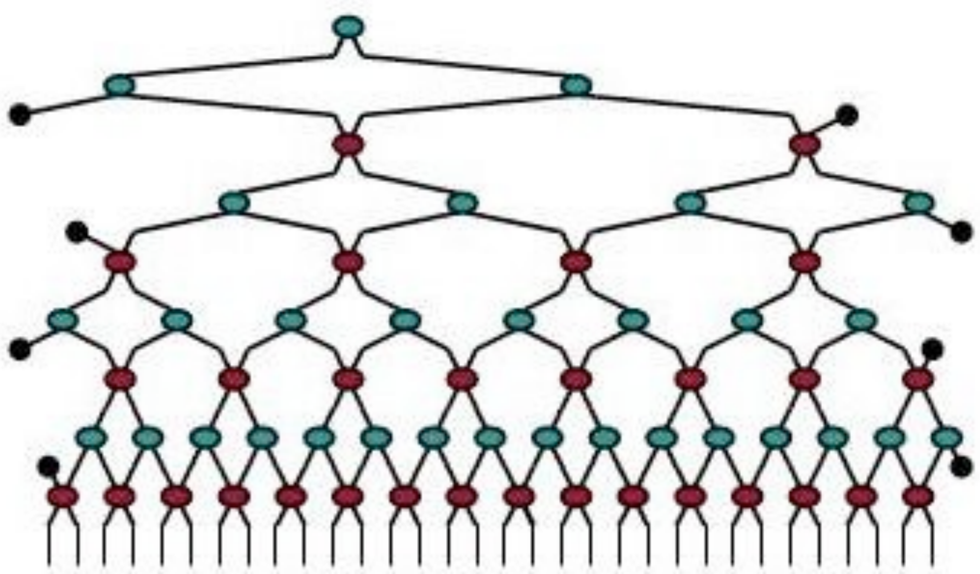
Tensor network states

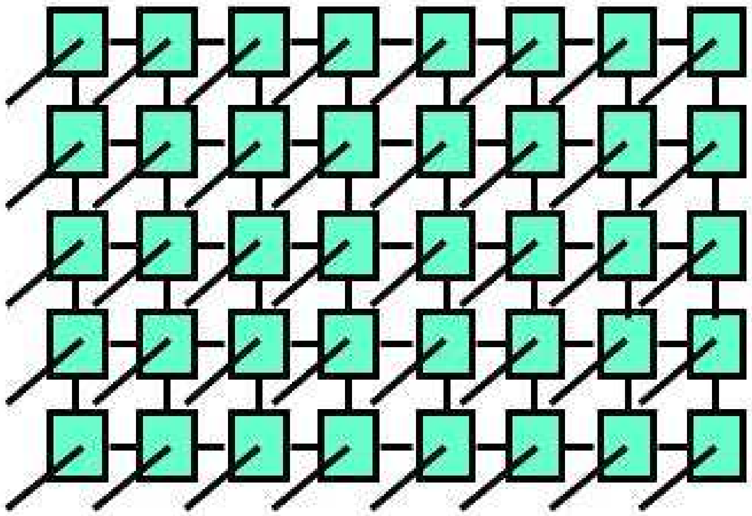
A new and powerful set of numerical methods based on entanglement content of quantum states

MPS



MERA





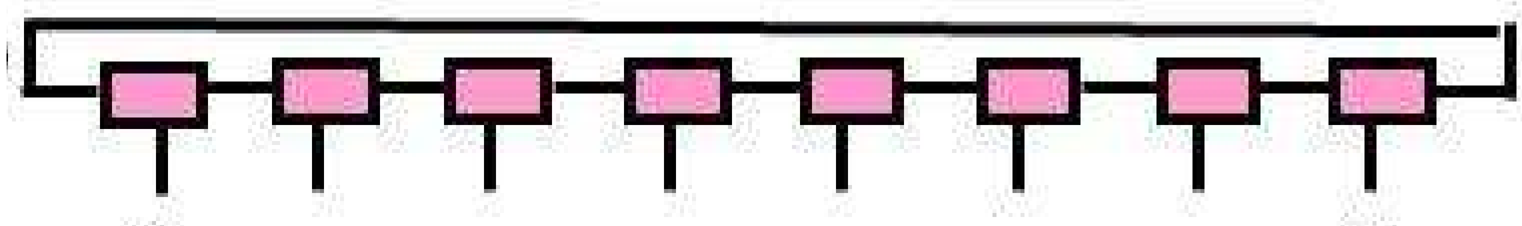
PEPS

"Alphabet soup of proposals"

Subir Sachdev

Matrix Product States (MPS)

$$|\psi\rangle = \sum_{s_1,\ldots,s_N=\pm} \mathrm{Tr}\left[A^{[1]}_{s_1} \ldots A^{[N]}_{s_N}\right] |s_1 \ldots s_N\rangle.$$



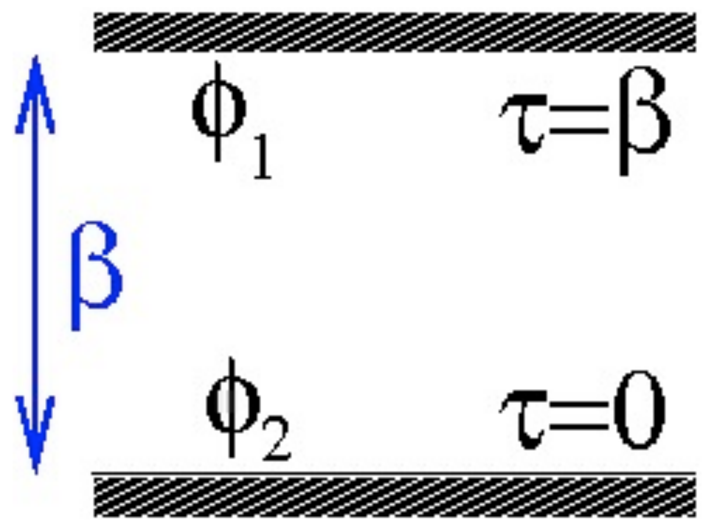
- For each site there are two matrices $A^{[i]}_{\pm}$ of finite dimension $\chi\times\chi$. More entanglement can be stored as χ increases.
- The famous DMRG is a practical way to find a variational MPS
- At fixed χ, the maximum entanglement entropy of an MPS is $\ln\chi$
- 1D area is a number $\Rightarrow$ entanglement entropy constant $\Rightarrow$ an MPS with finite χ can describe it
- In d dimensions, area law $N^{d-1} \Rightarrow \chi$ needs to be $\chi \sim \exp(N^{d-1})$

Entanglement entropy and path integral

PC, J Cardy 2004

The density matrix at temperature β^{-1}

$$\langle \Phi_1 | \rho | \Phi_2 \rangle =$$

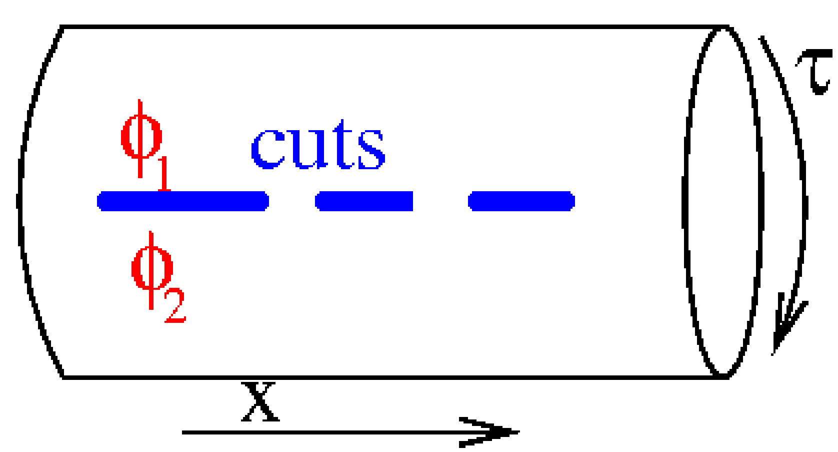


$$= \int \frac{[d\phi(x,\tau)]}{Z} \prod_x \delta(\phi(x,0) - \phi_2(x)) \prod_x \delta(\phi(x,\beta) - \phi_1(x))\, e^{-S_E}$$

The trace sews together the edges along $\tau = 0$ and $\tau = \beta$ to form a cylinder of circumference β.

A = (u, v): ρ_A sews together only those points x which are not in A, leaving an open cut along the $\tau = 0$.

$$\langle \Phi_1(x) | \rho_A | \Phi_2(x) \rangle =$$



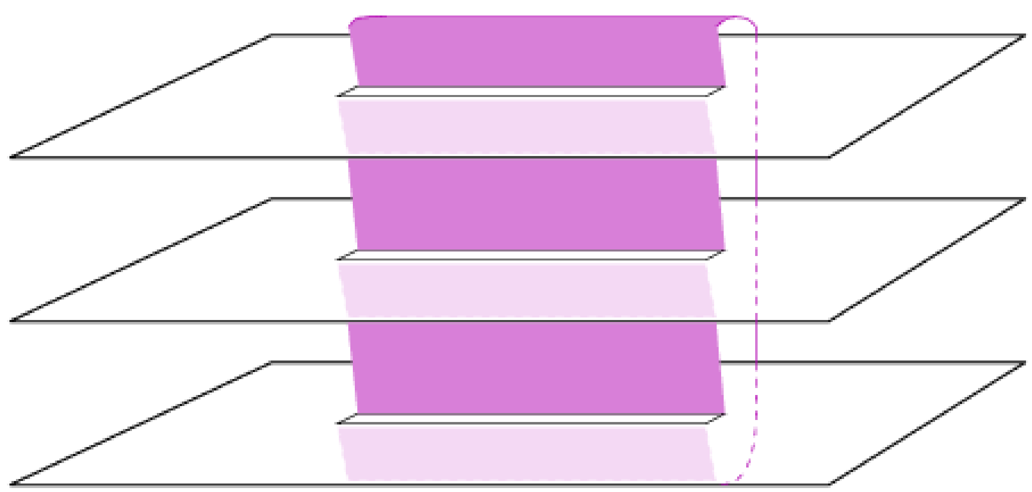
Replicas and Riemann surfaces

PC, J Cardy 2004

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A = -\lim_{n\to 1}\frac{\partial}{\partial n}\mathrm{Tr}\rho_A^n$$

For n integer, $\mathrm{Tr}\,\rho_A^n$ is obtained by sewing cyclically n cylinders above.
This is the partition function on a n-sheeted Riemann surface

$$\mathrm{Tr}\rho_A^n =$$

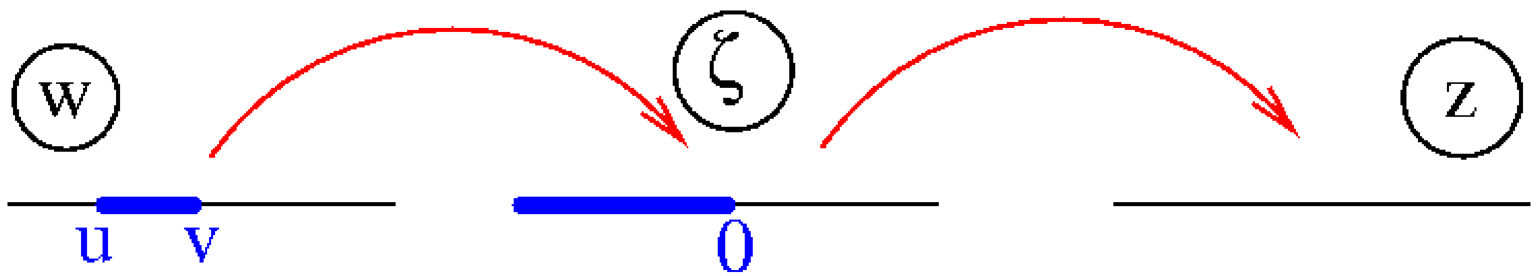


Renyi EE: $S_A \equiv 1/(1-n) \ln \mathrm{Tr}\,\rho_A^n$

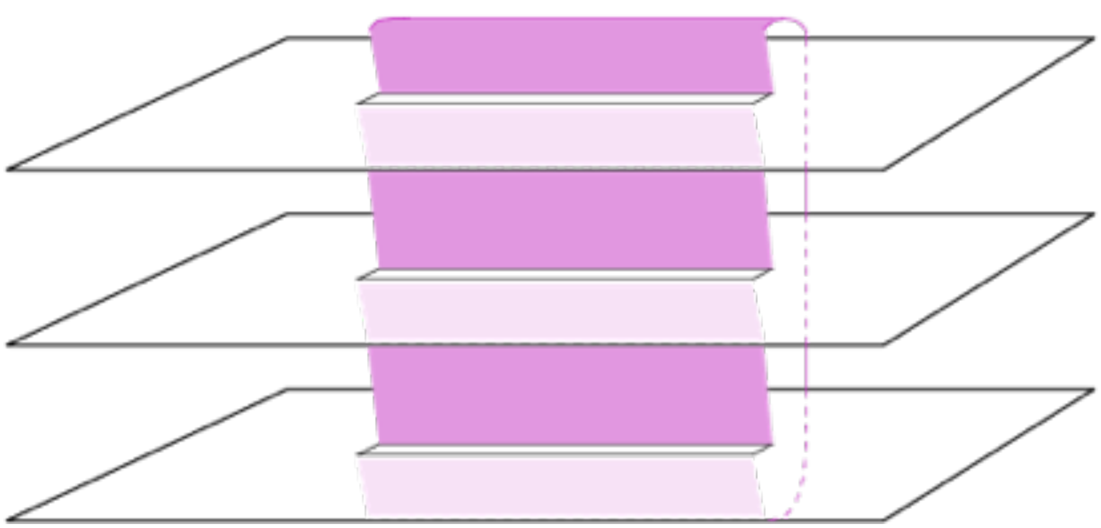
Riemann surfaces and CFT

This Riemann surface is mapped to the plane by **PC, J Cardy 2004**

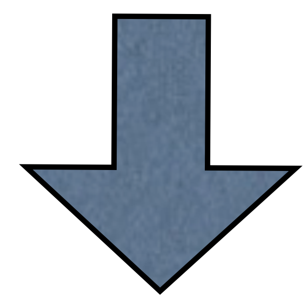
$$w \to \zeta = \frac{w-u}{w-v};\ \zeta \to z = \zeta^{1/n} \Rightarrow\ w \to z = \left(\frac{w-u}{w-v}\right)^{1/n}$$



$\mathrm{Tr}\,\rho_A^n =$



$$= c_n |u-v|^{-\frac{c}{6}(n-1/n)}$$



$|u-v| = \ell$

$$S_A = -\lim_{n\to 1}\frac{\partial}{\partial n}\mathrm{Tr}\rho_A^n = \frac{c}{3}\log \ell$$

$\mathrm{Tr}\,\rho_A^n$ is equivalent to the 2-point function of **twist fields**

$\mathrm{Tr}\rho_A^n = \langle \mathcal{T}_n(u)\,\bar{\mathcal{T}}_n(v)\rangle$ with scaling dimension

$$\Delta_{\mathcal{T}_n} = \frac{c}{12}\left(n - \frac{1}{n}\right)$$

From CFT to cold atoms

ARTICLE

doi:10.1038/nature15750

Nature 528, 77 (2015)

Measuring entanglement entropy in a quantum many-body system

Rajibul Islam[1], Ruichao Ma[1], Philipp M. Preiss[1], M. Eric Tai[1], Alexander Lukin[1], Matthew Rispoli[1] & Markus Greiner[1]

Entanglement is one of the most intriguing features of quantum mechanics. It describes non-local correlations between quantum objects, and is at the heart of quantum information sciences. Entanglement is now being studied in diverse fields ranging from condensed matter to quantum gravity. However, measuring entanglement remains a challenge. This is especially so in systems of interacting delocalized particles, for which a direct experimental measurement of spatial entanglement has been elusive. Here, we measure entanglement in such a system of itinerant particles using quantum interference of many-body twins. Making use of our single-site-resolved control of ultracold bosonic atoms in optical lattices, we prepare two identical copies of a many-body state and interfere them. This enables us to directly measure quantum purity, Rényi entanglement entropy, and mutual information. These experiments pave the way for using entanglement to characterize quantum phases and dynamics of strongly correlated many-body systems.

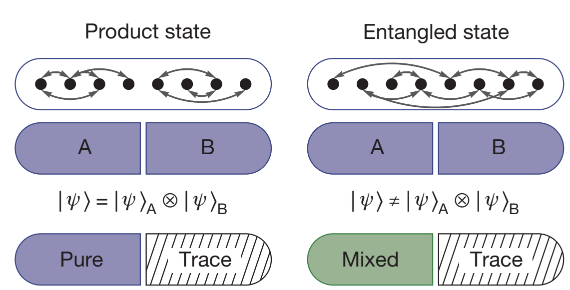


Figure 1 | Bipartite entanglement and partial measurements. A generic pure quantum many-body state has quantum correlations (shown as arrows) between different parts. If the system is divided into two subsystems A and B, the subsystems will be bipartite entangled with each other when there are quantum correlations between them (right column). Only when there is no bipartite entanglement present, the partitioned system $|\psi_{AB}\rangle$ can be described as a product of subsystem states $|\psi_A\rangle$ and $|\psi_B\rangle$ (left column). A path for measuring the bipartite entanglement emerges from the concept of partial measurements: ignoring all information about subsystem B (indicated as 'Trace') will put subsystem A into a statistical mixture, to a degree given by the amount of bipartite entanglement present. Finding ways of measuring the many-body quantum state purity of the system and comparing that of its subsystems would then enable measurements of entanglement. For an entangled state, the subsystems will have less purity than the full system.

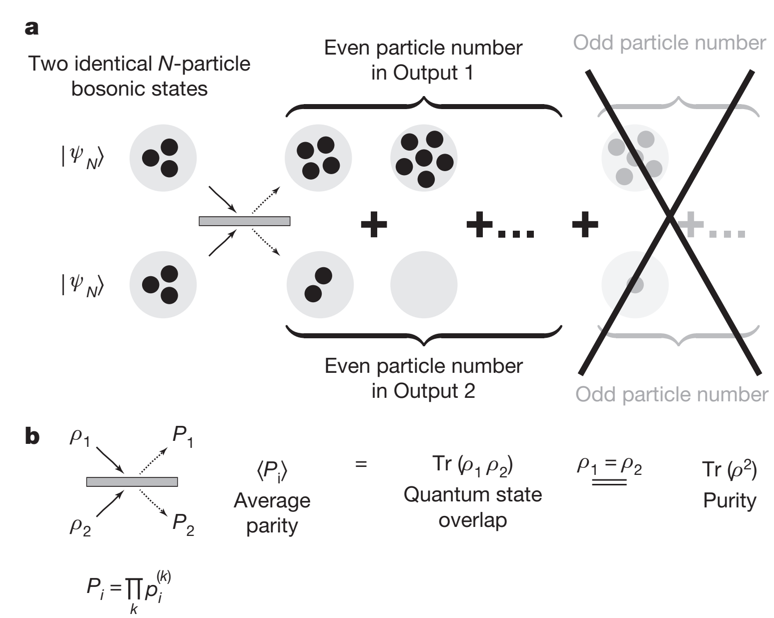
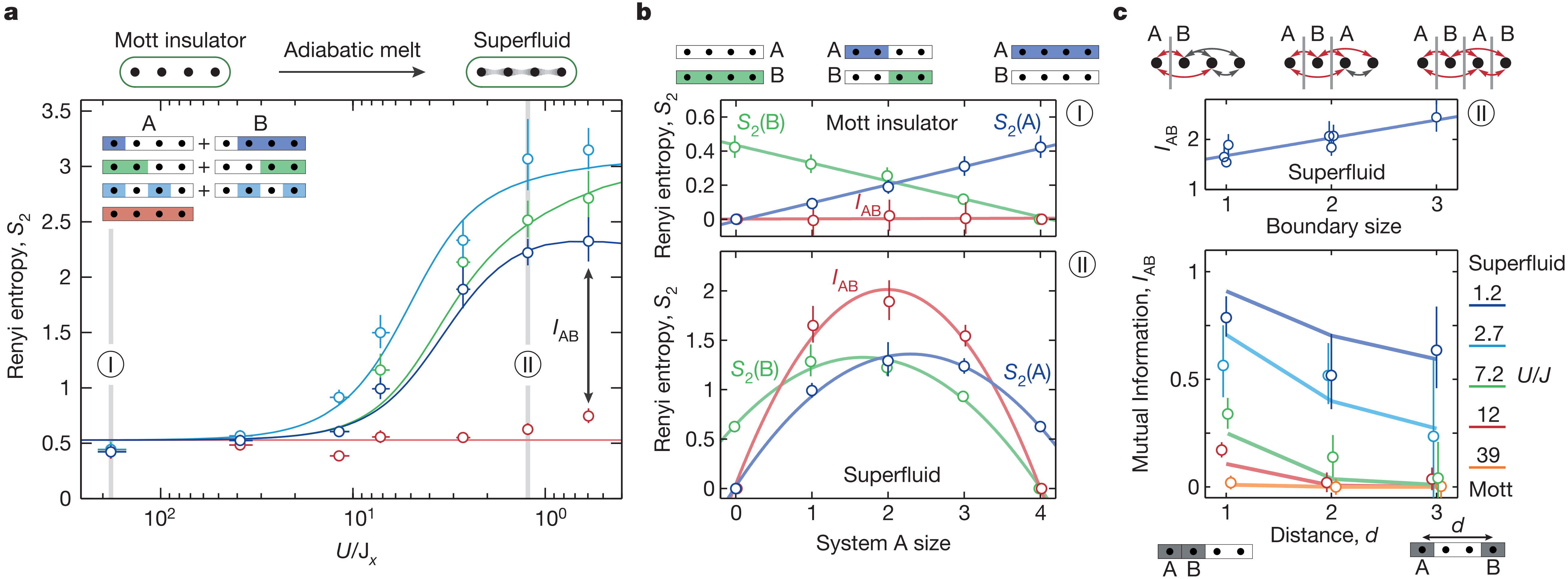


Figure 2 | Measurement of quantum purity with many-body bosonic interference of quantum twins. a, When two N-particle bosonic systems that are in identical pure quantum states are interfered on a 50%–50% beam splitter, they always produce output states with an even number

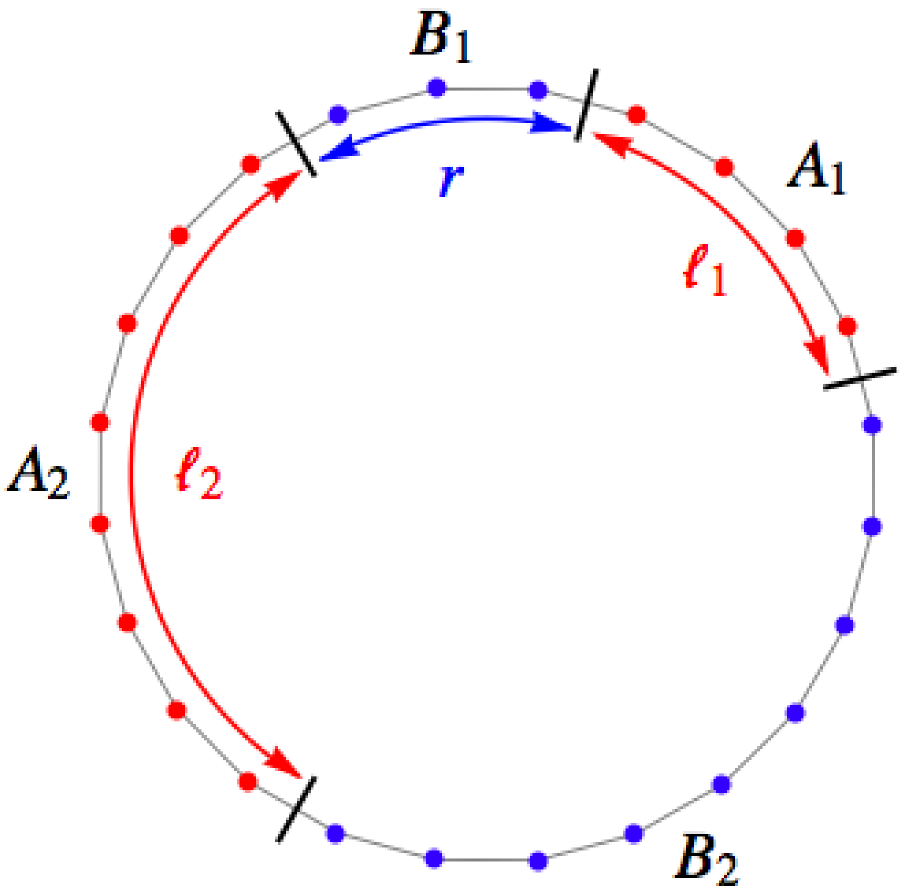
From CFT to cold atoms



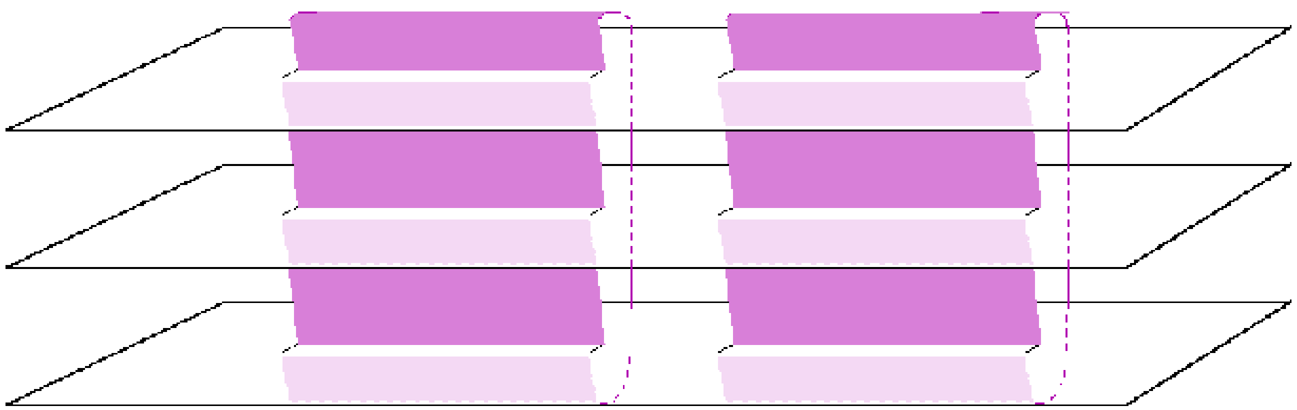
A look to a more difficult problem

PC, J Cardy, E Tonni 2009/10

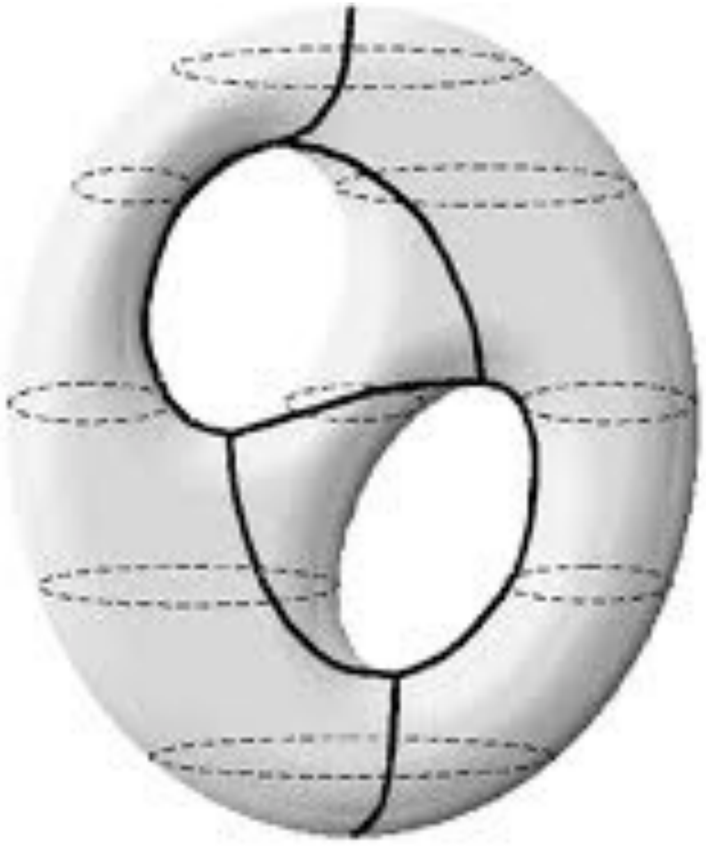
A = Disconnected regions:



More complex Riemann surface:



$\mathcal{R}_{n,2}$ of genus $(n-1)$



$\mathrm{Tr}\,\rho_A^n,\ S_A$??

Disjoint intervals

PC, J Cardy, E Tonni 2009/10

$$\mathrm{Tr}\rho_A^n = c_n^2 \left(\frac{|u_1 - u_2||v_1 - v_2|}{|u_1 - v_1||u_2 - v_2||u_1 - v_2||u_2 - v_1|} \right)^{\frac{c}{6}(n-1/n)} F_n(x)$$

$$x = \frac{(u_1-v_1)(u_2-v_2)}{(u_1-u_2)(v_1-v_2)} = 4 - \text{point ratio}$$

$F_n(x)$ is a calculable function depending on the full operator content

$$\mathrm{Tr}\,\rho_A^n = c_n^2 (\ell_1 \ell_2)^{-\frac{c}{6}(n-\frac{1}{n})} \sum_{\{k_j\}} \left(\frac{\ell_1 \ell_2}{n^2 r^2} \right)^{\sum_j (\Delta_j + \overline{\Delta}_j)} \langle \prod_{j=1}^{n} \phi_{k_j}\left(e^{2\pi i j/n}\right) \rangle_{\mathsf{C}}^2$$

Can we get $F_n(x)$ for some explicit models??

The compactified boson

PC, J Cardy, E Tonni 2009/2010

Using old results of CFT on orbifolds Dixon et al 86

$$F_n(x) = \frac{\Theta(0|\eta\Gamma)\,\Theta(0|\Gamma/\eta)}{[\Theta(0|\Gamma)]^2}$$

Γ is an $(n-1)\times(n-1)$ matrix

$$\Gamma_{rs} = \frac{2i}{n}\sum_{k=1}^{n-1}\sin\left(\pi\frac{k}{n}\right)\beta_{\frac{k}{n}}\cos\left[2\pi\frac{k}{n}(r-s)\right]$$

$$\text{with} \qquad \beta_y = \frac{H_y(1-x)}{H_y(x)}, \qquad H_y(x) = {}_2F_1(y, 1-y; 1; x)$$

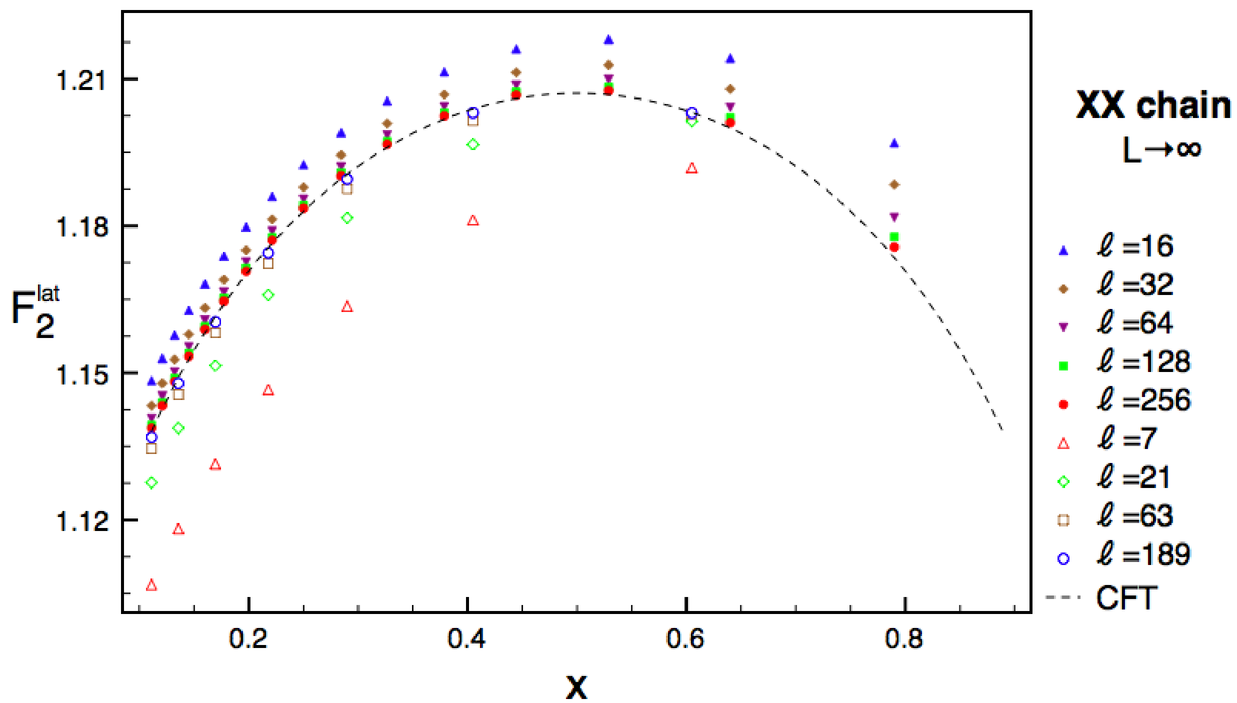
Riemann theta function $\Theta(z|\Gamma) \equiv \sum_{m\in \mathbf{Z}^{n-1}} \exp\left[\, i\pi\, m\cdot\Gamma\cdot m + 2\pi i m\cdot z\,\right]$

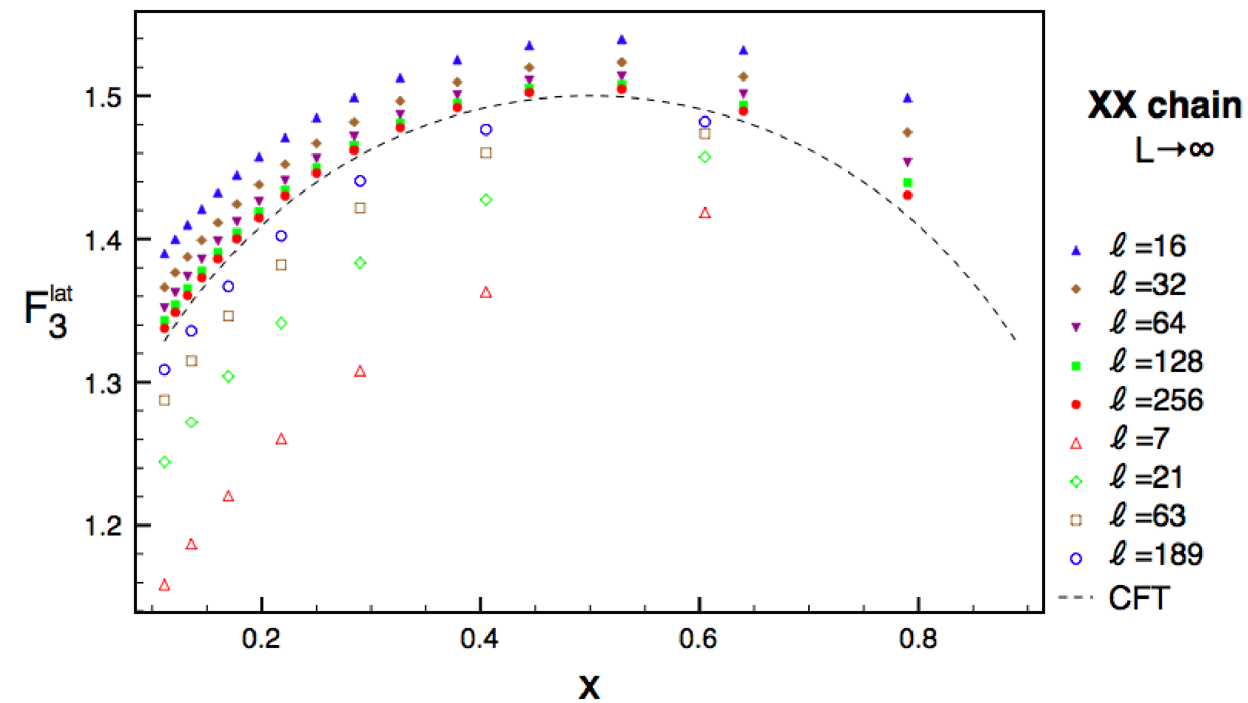
Nowadays generalized to many other cases: Ising (PC, Cardy, Tonni), Askhin-Teller (Alba, Tagliacozzo, PC), Fusion-twist (Rajabpour, Gliozzi), merged models (Fagotti).....

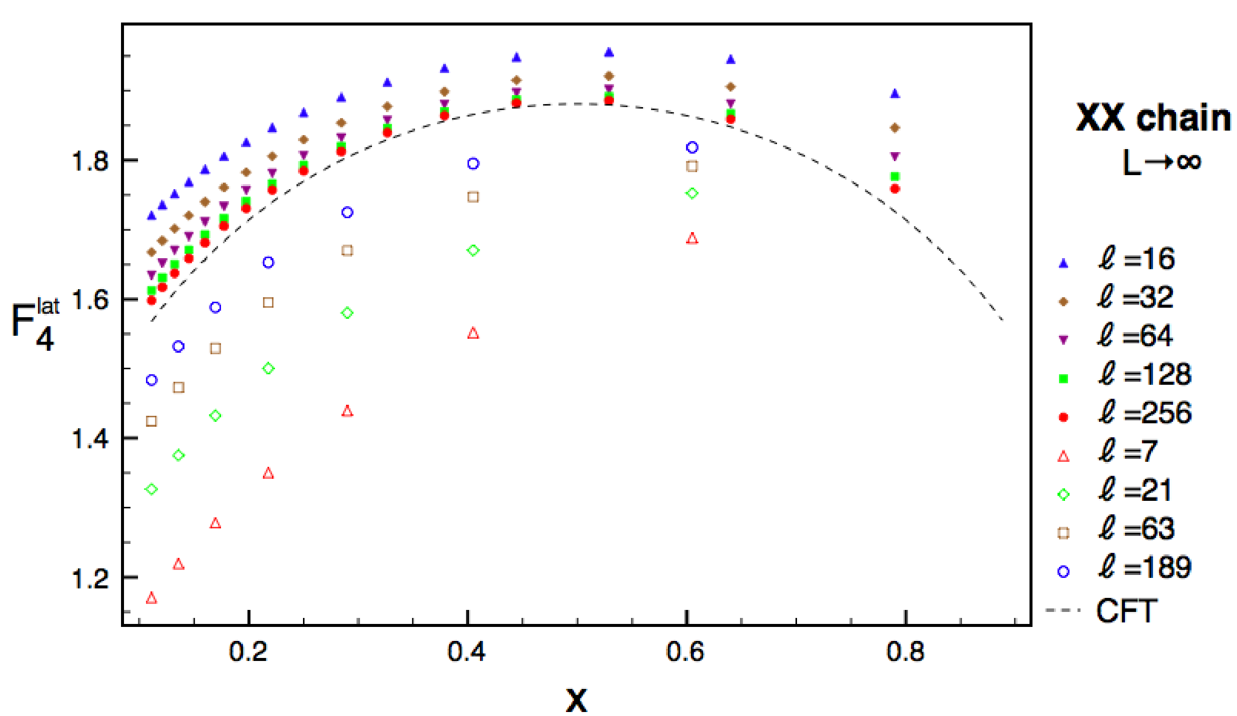
Does it work?

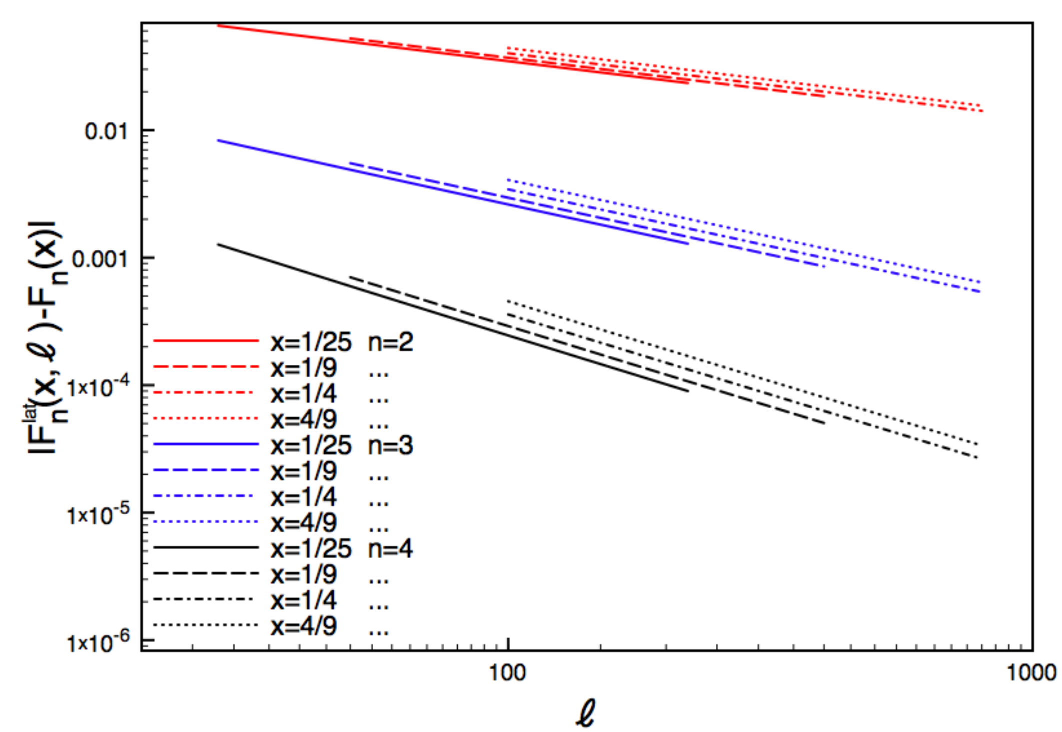
M Fagotti, PC, 2010

The RDM of two intervals is not trivial because of JW string Igloi-Peschel









Reviews (up to 2009)

IOP | electronic journals

Journal of Physics A: Mathematical and Theoretical

Volume 42. Number 50. 18 December 2009

SPECIAL ISSUE: ENTANGLEMENT ENTROPY IN EXTENDED QUANTUM SYSTEMS

INTRODUCTION

ISSN 1751-8113

Journal of Physics A

Mathematical and Theoretical

Volume 42 Number 50 18 December 2009

Special issue

Entanglement entropy in extended quantum systems

Guest Editors: Pasquale Calabrese, John Cardy and Benjamin Doyon

Further developments

- Detect and characterize quantum criticality

  In random quantum spin chains $S_A \propto \ln \ell$

  Refael and Moore, Laflorencie, Santachiara, Jacobsen, Saleur ...

  Universal corrections to the scaling

  PC, Essler, Cardy, Ravanini, Franchini,Ercolessi, Alcaraz....

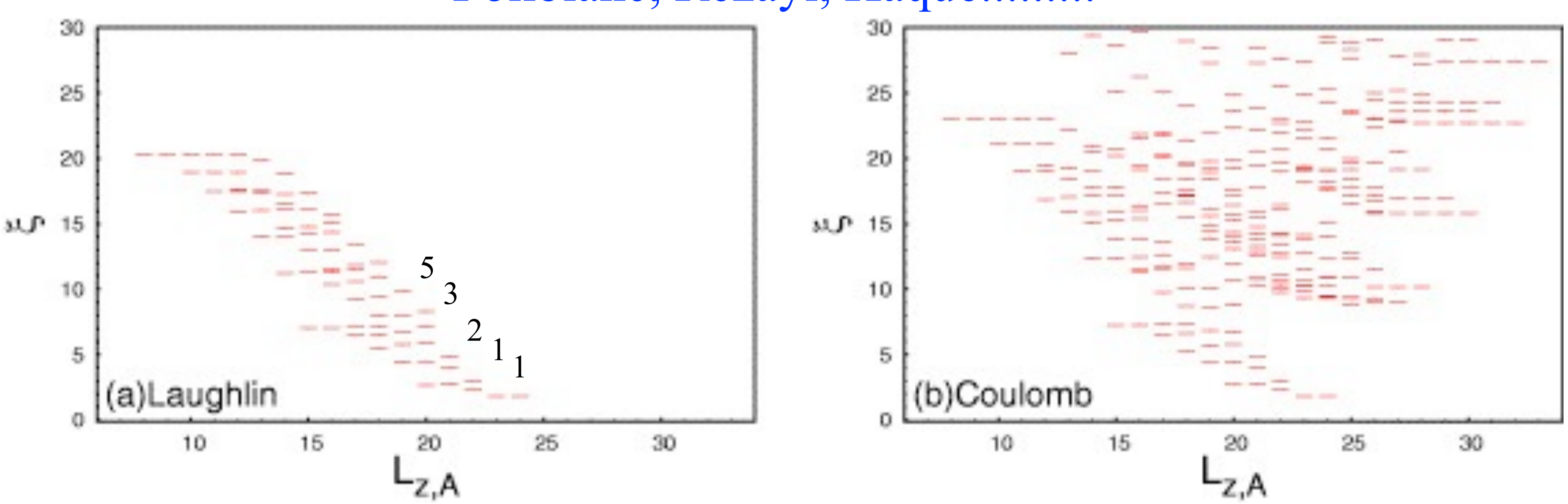
- Topological entanglement entropy

  $S_A = \alpha L - \gamma$ γ is the topological charge

  Kitaev and Preskill, Levin and Wen, Fradkin and Moore, Schoutens et al....

- Entanglement spectrum Haldane, Regnault, Read, Ludwig, Bernevig, Poilblanc, Rezayi, Haque.........

Eigenvalues of ρ_A



Further developments (II)

- Holography: S_A=length of the geodesic in the AdS bulk

  Ryu and Takayanagi. Headrick, Maldacena, Myers

- c-theorem analogues with S_A Casini and Huerta, Myers,

- Entanglement out of equilibrium (quenches)

  PC and Cardy, Vidal, Schollwoeck, Kollath, Eisert, Cirac...

- Other measures of entanglement (eg mixed states)

  Fazio, Amico, Vidal....

  - Entanglement negativity **PC, Cardy, Tonni 2012/13**
  - Shannon information Stephan, Pasquier, Oshikawa Alcaraz

- Too many more to be mentioned here

Further developments (II)

- Holography: S_A=length of the geodesic in the AdS bulk

  Ryu and Takayanagi. Headrick, Maldacena, Myers

- c-theorem analogues with S_A Casini and Huerta, Myers,

- Entanglement out of equilibrium (quenches)

  PC and Cardy, Vidal, Schollwoeck, Kollath, Eisert, Cirac...

- Other measures of entanglement (eg mixed states)

  Fazio, Amico, Vidal....

  - Entanglement negativity **PC, Cardy, Tonni 2012/13**
  - Shannon information Stephan, Pasquier, Oshikawa Alcaraz

- Too many more to be mentioned here

Many particles out of equilibrium

How do the Gibbs distribution emerge in QM?

von Neumann in 1929 posed the question [1003.2133]

Quantum Quench idea:

1) prepare a many-body system in a pure state that is not an eigenstate of the Hamiltonian

2) let it evolve according to QM laws (no coupling to environment)

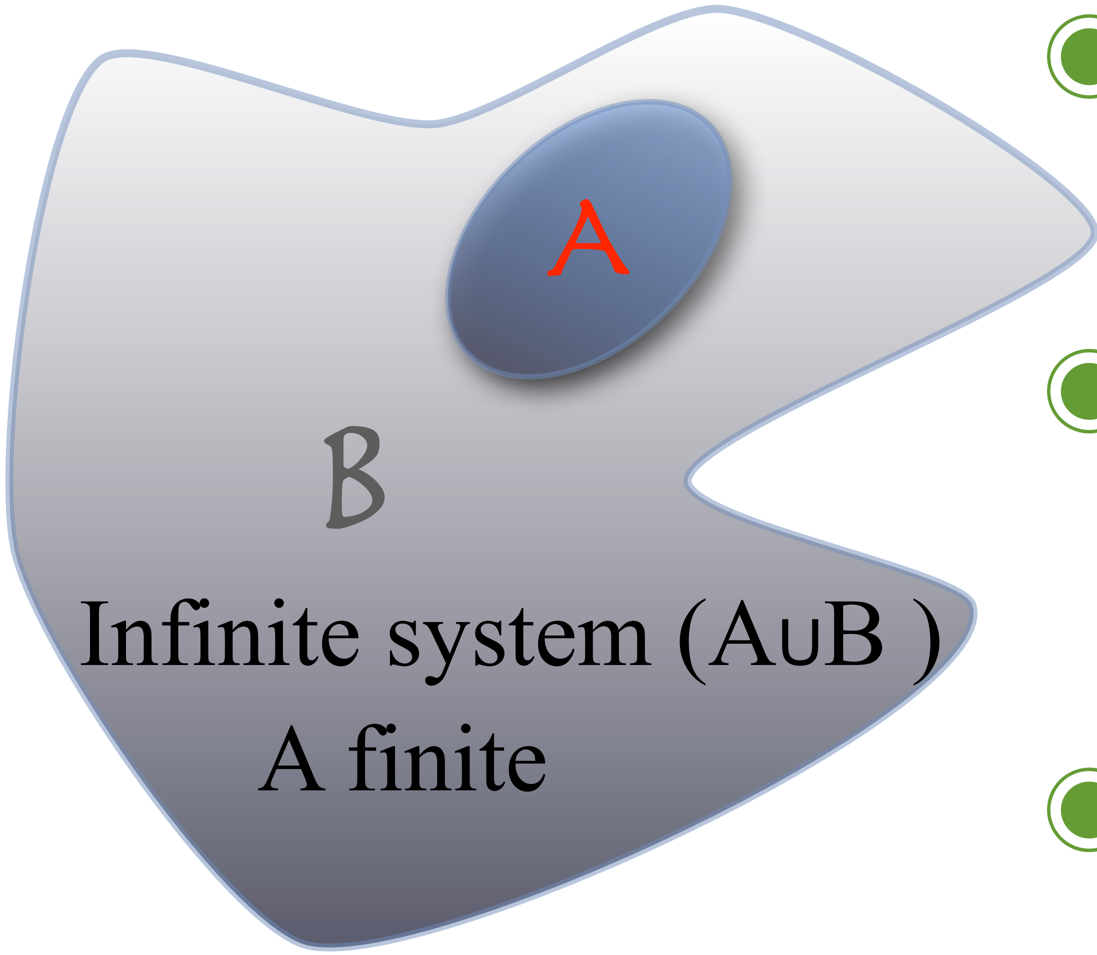
$$|\psi(t)\rangle = e^{-iHt}|\psi_0\rangle$$

Questions:

- How can we describe the dynamics?
- Does it exist a stationary state? In which sense?

$|\Psi(t)\rangle$ remains pure for any t

The Reduced density matrix helps!



- $|\Psi(t)\rangle$ time dependent pure state

  Reduced density matrix: $\rho_A(t) = \mathrm{Tr}_B\, \rho(t)$

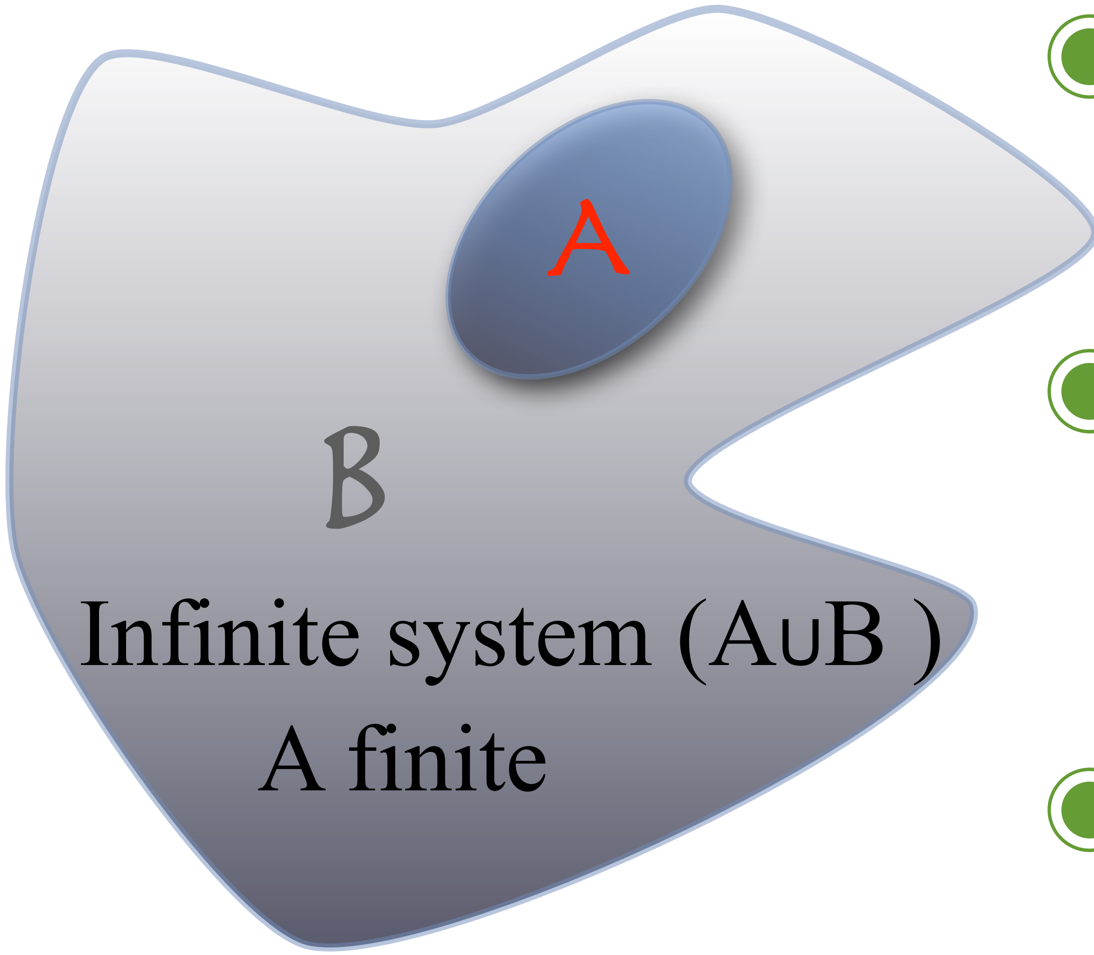
- The expectation values of all local observables within A are

  $$\langle \Psi(t)| O_A(x) |\Psi(t)\rangle = \mathrm{Tr}[\rho_A(t)\, O_A(x)]$$

- Stationary state: if exists the limit

  $$\lim_{t\to\infty} \rho_A(t) = \rho_A(\infty)$$

The Reduced density matrix helps!



- $|\Psi(t)\rangle$ time dependent pure state

  Reduced density matrix: $\rho_A(t) = \mathrm{Tr}_B\, \rho(t)$

- The expectation values of all local observables within A are

  $$\langle \Psi(t)| O_A(x) |\Psi(t)\rangle = \mathrm{Tr}[\rho_A(t)\, O_A(x)]$$

- Stationary state: if exists the limit

  $$\lim_{t\to\infty} \rho_A(t) = \rho_A(\infty)$$

Thermalization *vs* Generalized Gibbs

$\rho_T = e^{-\beta_{\mathrm{eff}} H}/Z$ *vs* $\rho_{GGE} = e^{-\sum \lambda_m I_m}/Z$

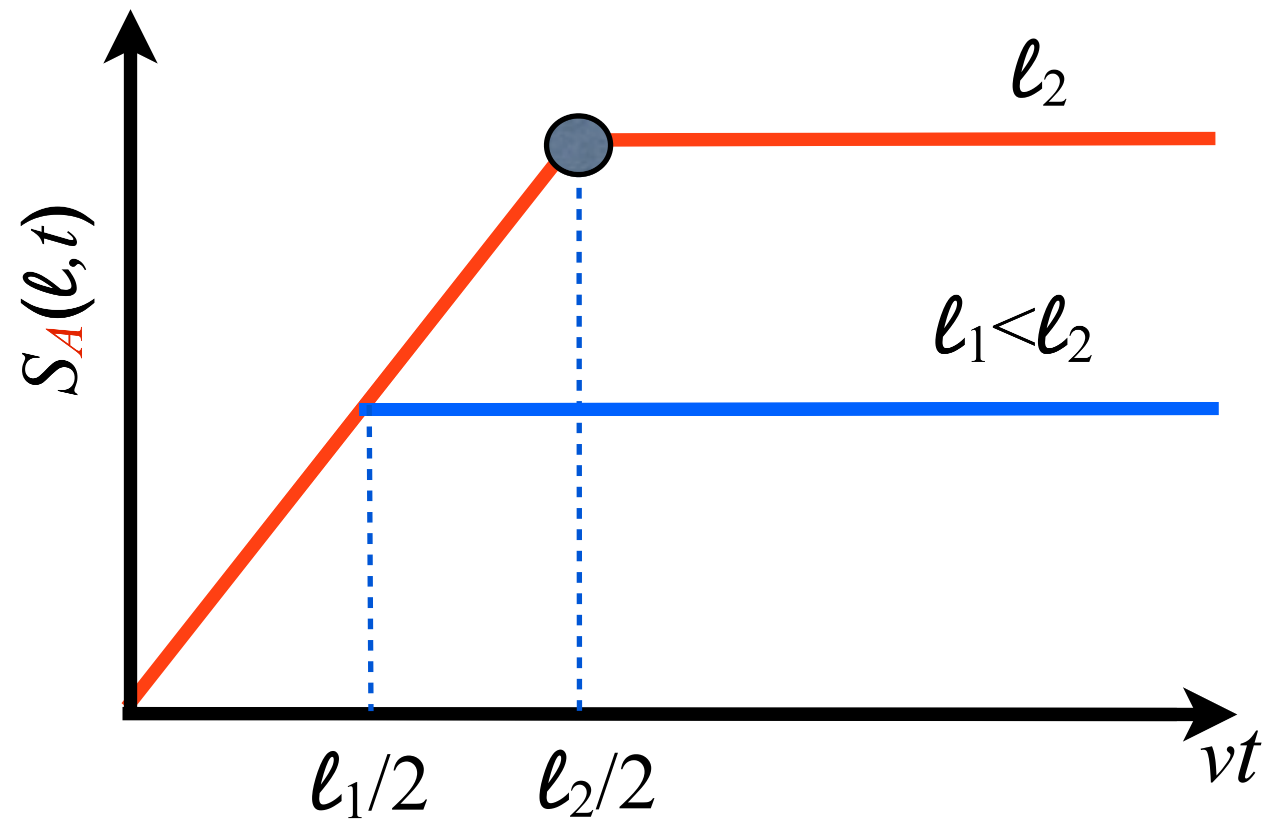
but this is another story/talk....

Entanglement after a quench

PC, Cardy 2005

In a CFT (i.e. exactly linear dispersion relation $E=vk$ up to a cutoff)

$$S_A(\ell,t) = \begin{cases} \frac{\pi c t \, \mathrm{v}}{6\epsilon} & 2\mathrm{v}t<\ell \\ \frac{\pi c \, \ell}{12\epsilon} & 2\mathrm{v}t>\ell \end{cases}$$

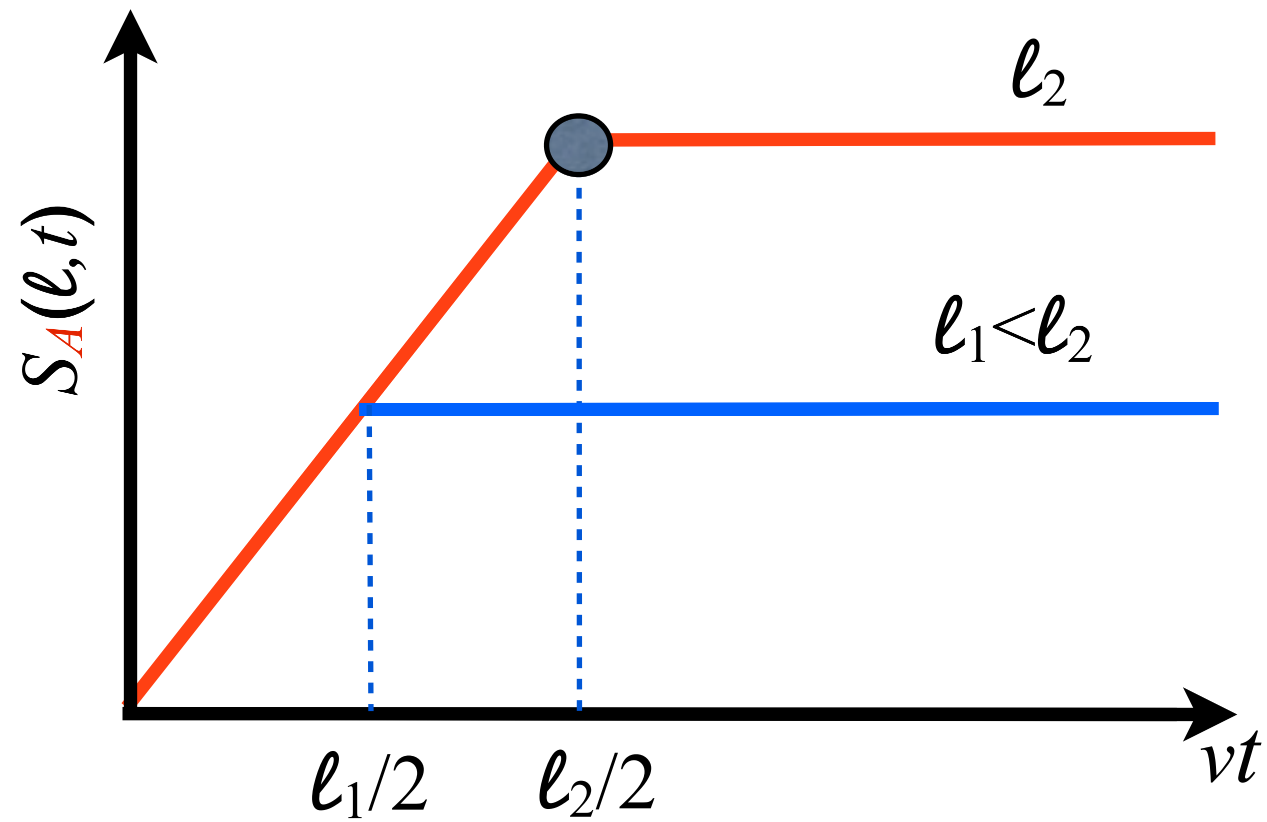


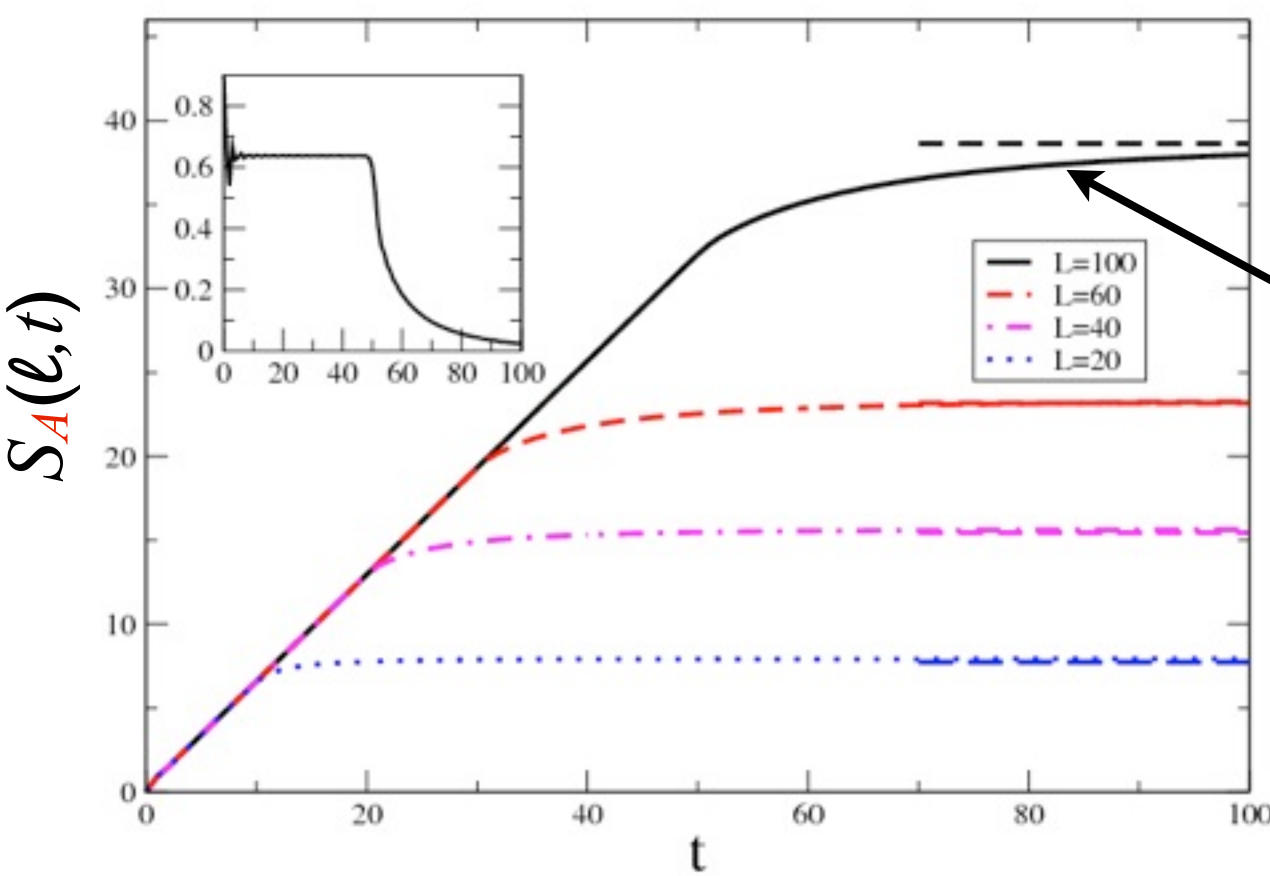
Entanglement after a quench

PC, Cardy 2005

In a CFT (i.e. exactly linear dispersion relation $E=vk$ up to a cutoff)

$$S_A(\ell,t)=\begin{cases} \dfrac{\pi c t\, \mathrm{v}}{6\epsilon} & 2\mathrm{v}t<\ell \\[2ex] \dfrac{\pi c\, \ell}{12\epsilon} & 2\mathrm{v}t>\ell \end{cases}$$





Exact results for the Ising model

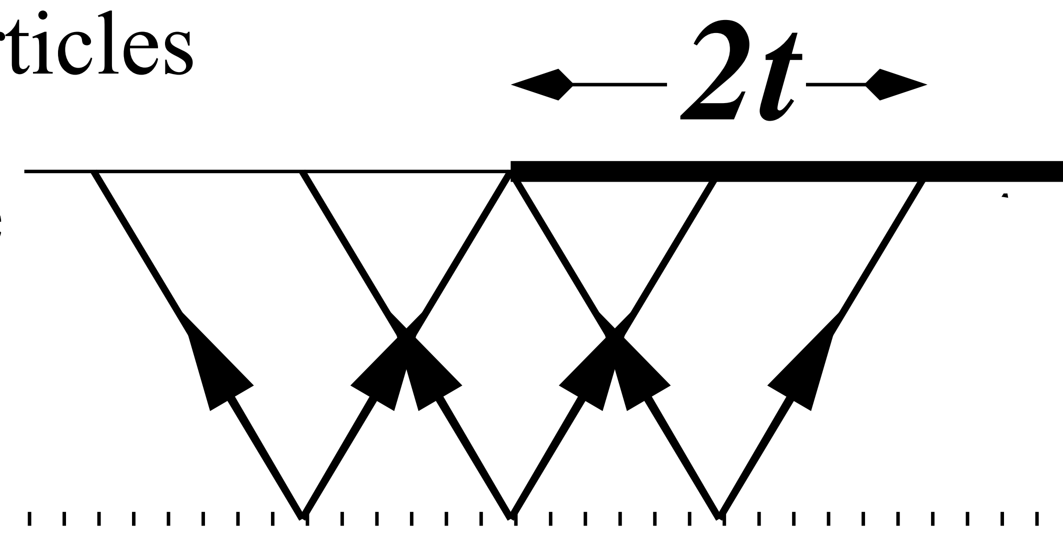
Fagotti, PC 2008

Curvature: effect of the non linear dispersion

Physical explanation

PC, Cardy 2005

- $|\psi_0\rangle$ has large energy: source of quasi-particles
- **Pairs** of quasi-particles move in opposite directions with velocity $\pm v_k$
- Particles emitted from the same point are entangled

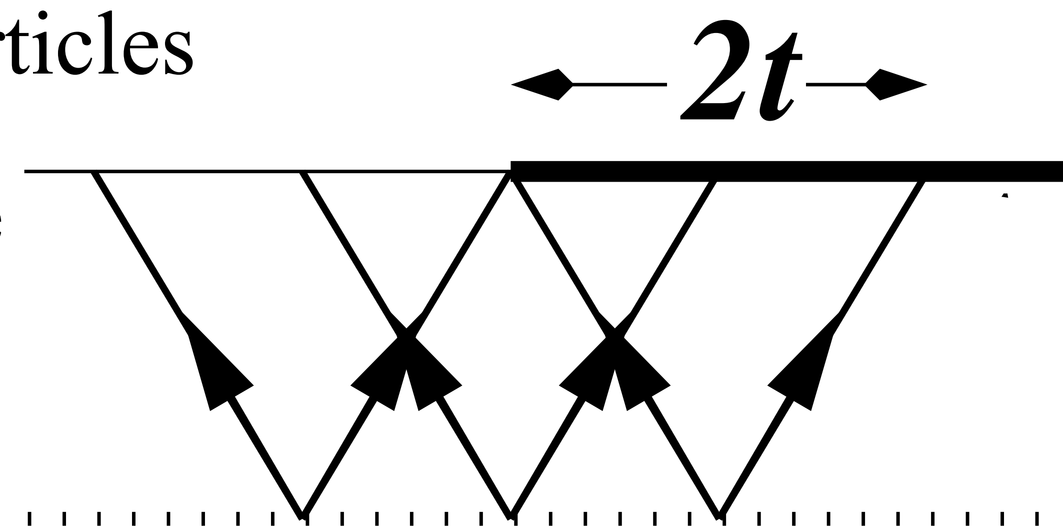


$v_k = \frac{d\Omega_k}{dk}$ v_{max} exists

Physical explanation

PC, Cardy 2005

- $|\psi_0\rangle$ has large energy: source of quasi-particles



- **Pairs** of quasi-particles move in opposite directions with velocity $\pm v_k$
- Particles emitted from the same point are entangled

$v_k = \frac{d\Omega_k}{dk}$ $\quad$ v_{max} exists

- **Light cone**: Points at separation ℓ become entangled when left- and right-movers originated from the same point reach them

correlations form at $t = \ell/2v_{max}$

- If all particles move at the same speed, entanglement and correlations are **frozen** for $t>\ell/2v$

Slower particles change entanglement and correlations after $t = \ell/2v_{max}$: large t is driven by slowest particles

Is it true?

M. Cheneau et al, Nature 2012

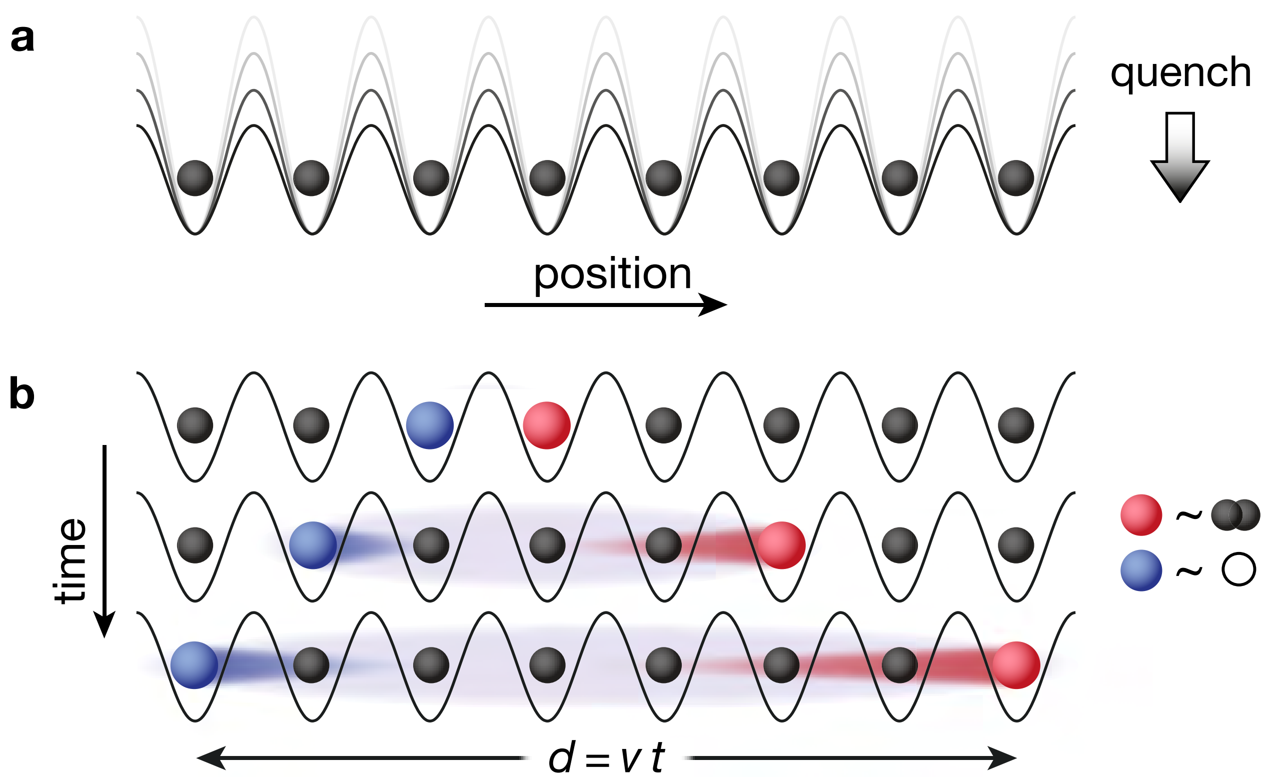
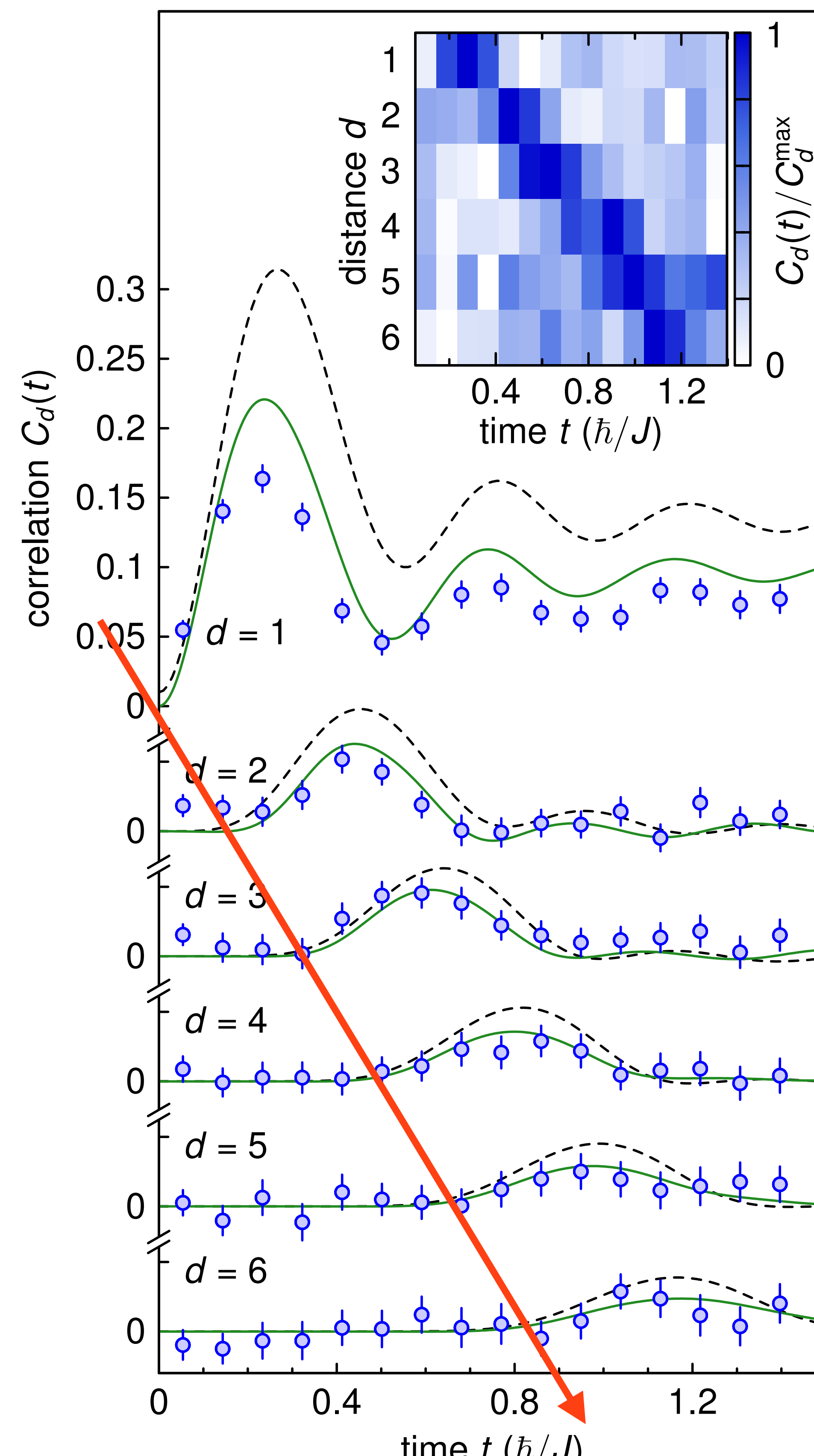


FIG. 1. **Spreading of correlations in a quenched atomic Mott insulator.** **a**, A 1D ultracold gas of bosonic atoms (black balls) in an optical lattice is initially prepared deep in the Mott-insulating phase with unity filling. The lattice depth is then abruptly lowered, bringing the system out of equilibrium. **b**, Following the quench, entangled quasiparticle pairs emerge at all sites. Each of these pairs consists of a doublon (red ball) and a holon (blue ball) on top of the unity-filling background, which propagate ballistically in opposite directions. It follows that a correlation in the parity of the site occupancy builds up at time t between any pair of sites separated by a distance $d = vt$, where v is the relative velocity of the doublons and holons.



Light-cone spreading of entanglement entropy

PC, J Cardy 2005

- The entanglement entropy of an interval A of length ℓ is proportional to the total number of pairs of particles emitted from arbitrary points such that at time t, $x \in A$ and $x' \in B$
- Denoting with $f(p)$ the rate of production of pairs of momenta $\pm p$ and their contribution to the entanglement entropy, this implies

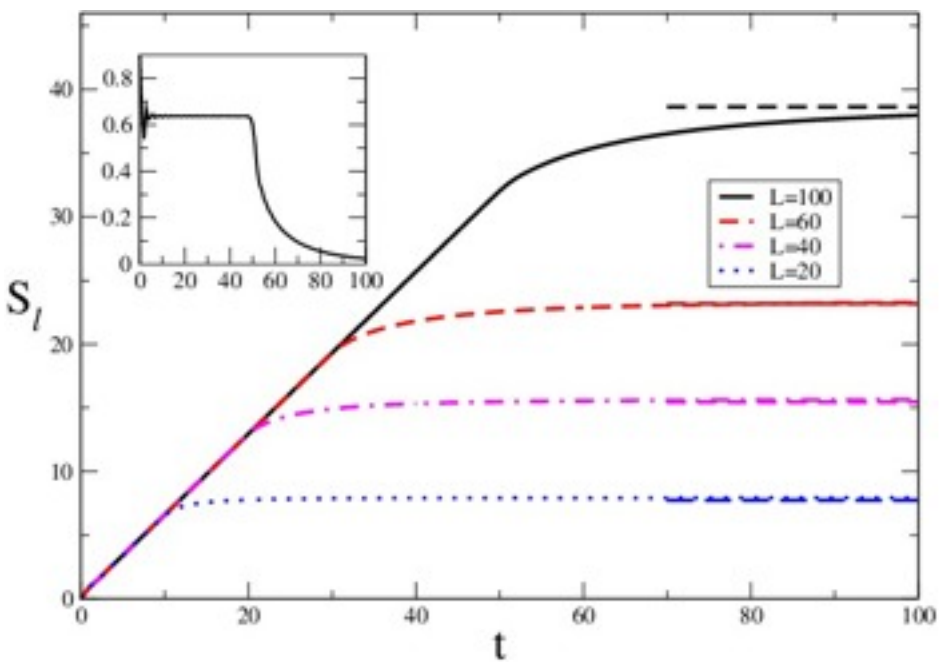
$$
\begin{aligned}
S_A(t) \approx & \int_{x' \in A} dx' \int_{x'' \in B} dx'' \int_{-\infty}^{\infty} dx \int f(p) dp \delta\big(x' - x - v_p t\big) \delta\big(x'' - x + v_p t\big) \\
\propto & \; t \int_0^{\infty} dp f(p) 2 v_p \theta(\ell - 2 v_p t) + \ell \int_0^{\infty} dp f(p) \theta(2 v_p t - \ell)
\end{aligned}
$$

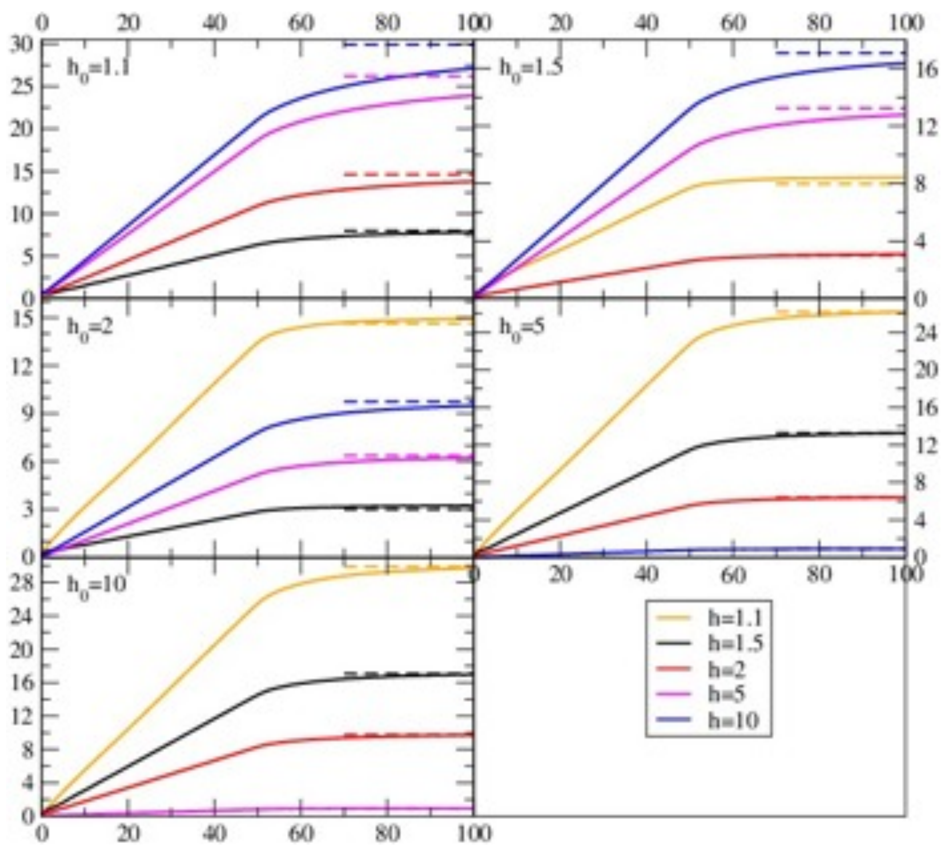
- When v_p is bounded (e.g. Lieb-Robinson bounds) $|v_p| < v_{\max}$, the second term is vanishing for $2 v_{\max} t < \ell$ and the entanglement entropy grows linearly with time up to a value linear in ℓ

One example

Transverse field Ising chain

PC, J Cardy 2005





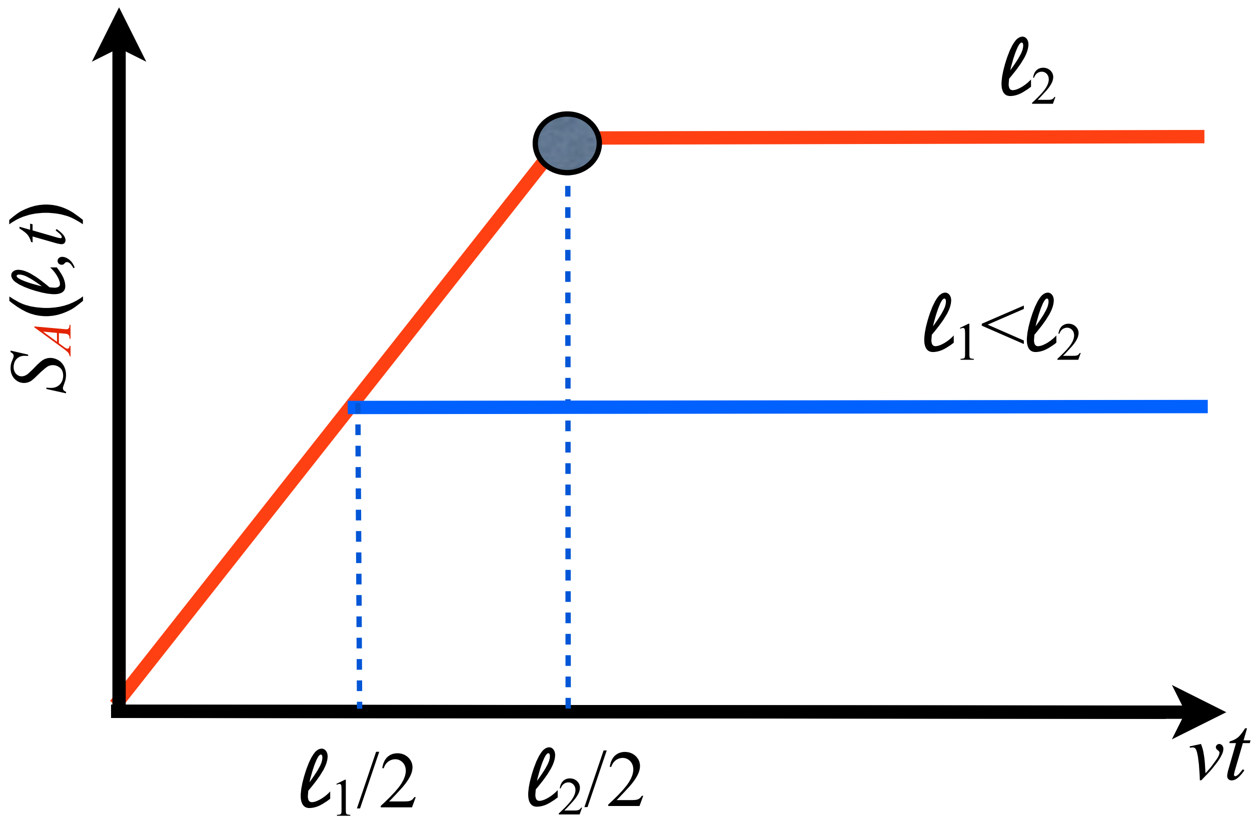
Analytically for t, $\ell \gg 1$ with t/ℓ constant

M Fagotti, PC 2008

$$S(t) = t \int_{2|\epsilon'|t<\ell} \frac{d\varphi}{2\pi} 2|\epsilon'| H(\cos\Delta_\varphi) + \ell \int_{2|\epsilon'|t>\ell} \frac{d\varphi}{2\pi} H(\cos\Delta_\varphi)$$

$$\cos\Delta_\varphi = \frac{1-\cos\varphi(h+h_0)+hh_0}{\epsilon_\varphi\epsilon^0_\varphi} \qquad H(x) = -\frac{1+x}{2}\log\frac{1+x}{2} - \frac{1-x}{2}\log\frac{1-x}{2}$$

Physical interpretation at $t=\infty$



The extensive value at $t=\infty$ is the **thermodynamic entropy** in the mixed state because

$$\lim_{t\to\infty} \rho_A(t) = \rho_A(\infty)$$

For large time the entanglement entropy becomes thermodynamic entropy

Understood even in more complicated situations

What about experiments?

Quantum thermalization through entanglement in an isolated many-body system

Adam M. Kaufman, M. Eric Tai, Alexander Lukin, Matthew Rispoli, Robert Schittko, Philipp M. Preiss, Markus Greiner*

Science 353, 794 (2016)

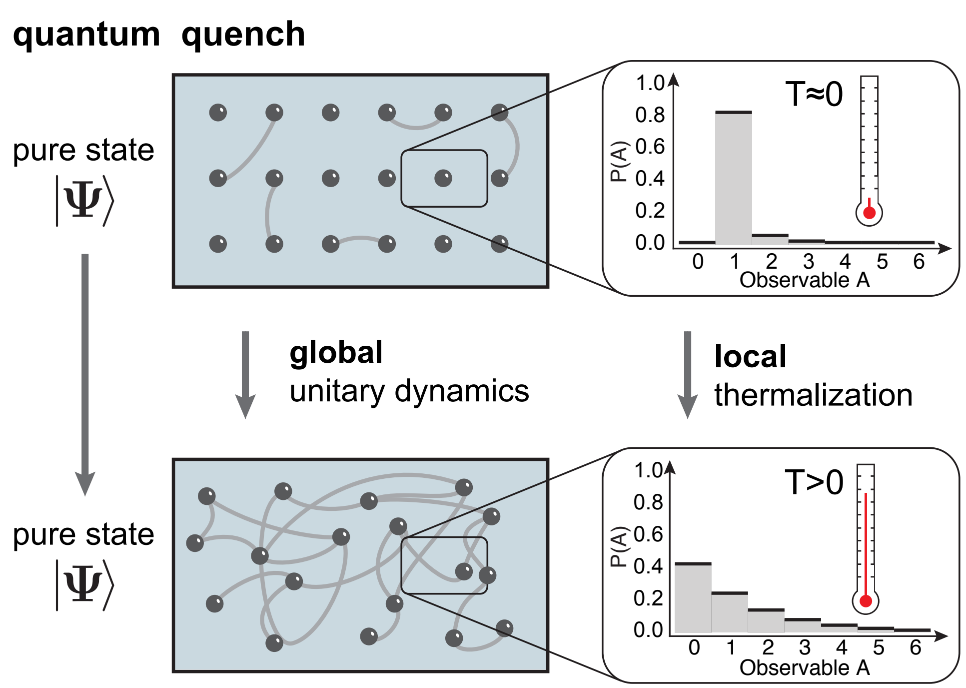
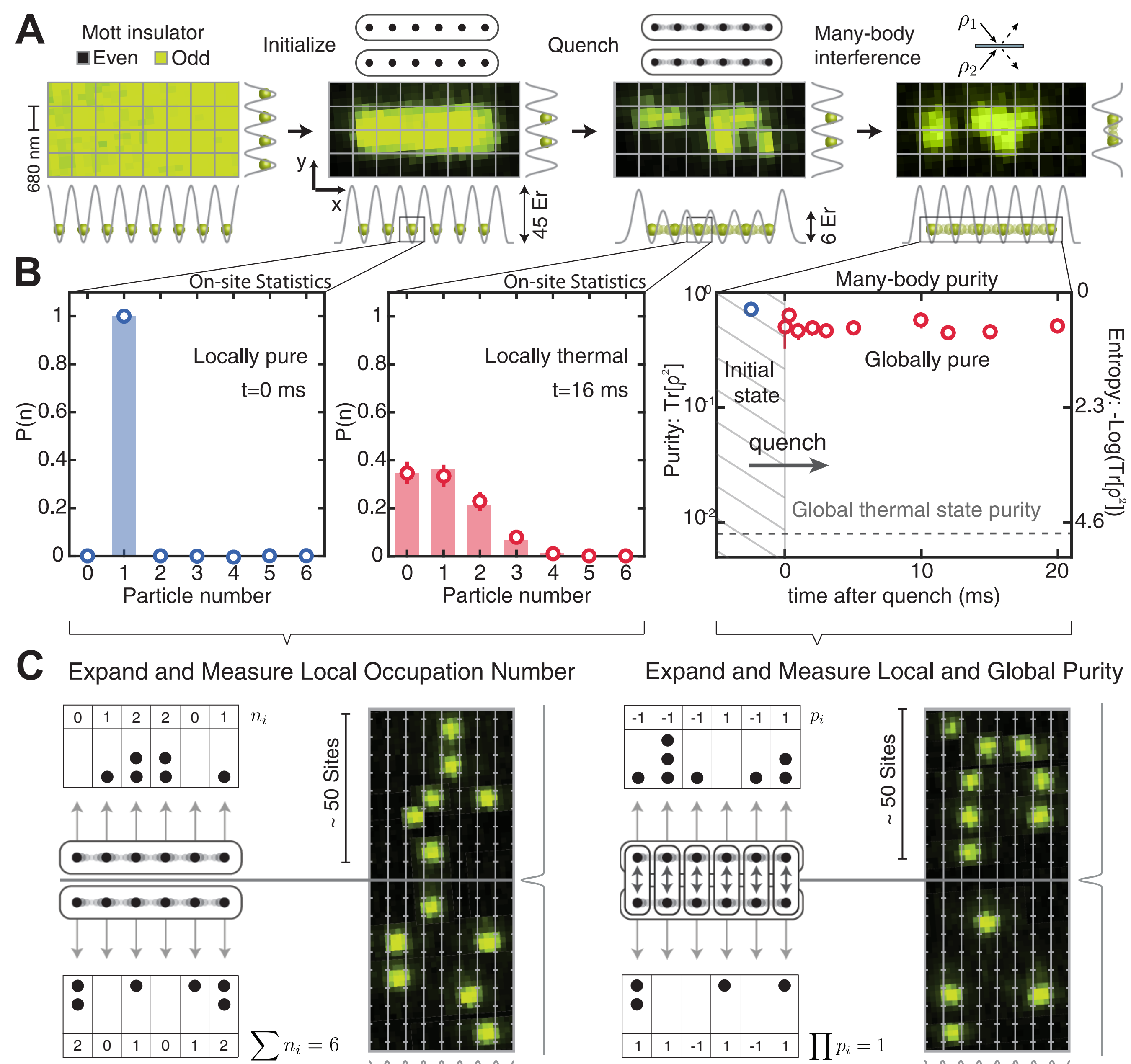


FIG. 1. **Schematic of thermalization dynamics in closed systems**. An isolated quantum system at zero temperature can be described by a single pure wavefunction $|\Psi\rangle$. Subsystems of the full quantum state appear pure, as long as the entanglement (indicated by grey lines) between subsystems is negligible. If suddenly perturbed, the full system evolves unitarily, developing significant entanglement between all parts of the system. While the full system remains in a pure, zero-entropy state, the entropy of entanglement causes the subsystems to equilibrate, and local, thermal mixed states appear to emerge within a globally pure quantum state.



What about experiments?

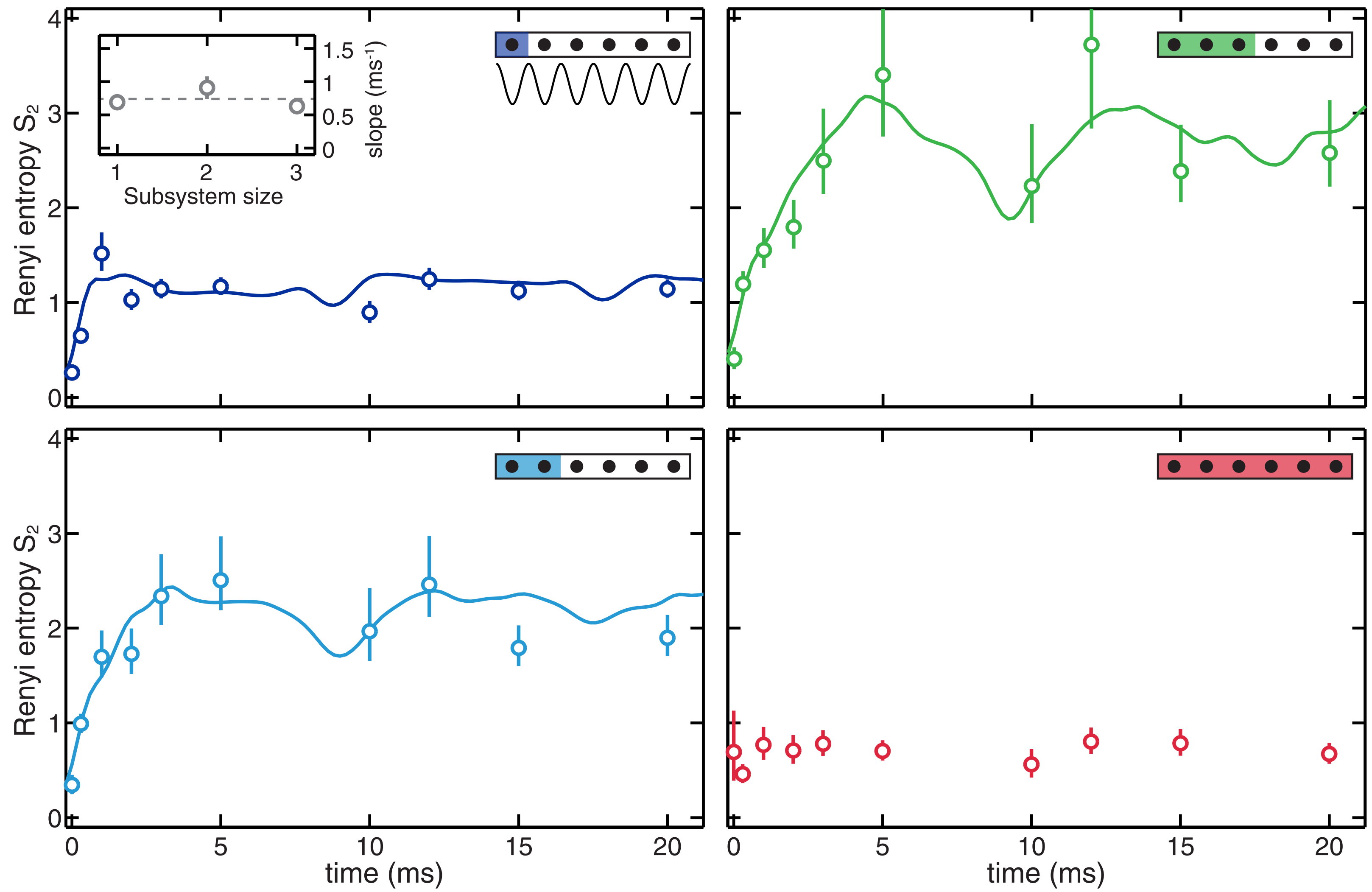


FIG. 3. **Dynamics of entanglement entropy.** Starting from a low-entanglement ground state, a global quantum quench leads to the development of large-scale entanglement between all subsystems. We quench a six-site system from the Mott insulating product state ($J/U \ll 1$) with one atom per site to the weakly interacting regime of $J/U = 0.64$ and measure the dynamics of the entanglement entropy. As it equilibrates, the system acquires local entropy while the full system entropy remains constant and at a value given by measurement imperfections. The dynamics agree with exact numerical simulations with no free parameters (solid lines). Error bars are the standard error of the mean (S.E.M.). For the largest entropies encountered in the three-site system, the large number of populated microstates leads to a significant statistical uncertainty in the entropy, which is reflected in the upper error bar extending to large entropies or being unbounded. Inset: slope of the early time dynamics, extracted with a piecewise linear fit (see Supplementary Material). The dashed line is the mean of these measurements.

For large time the entanglement entropy becomes thermodynamic entropy

Idea: We could use the knowledge of the entropy in the stationary state to go backward in time for the entanglement entropy.

Alba & PC, 2016

Making a long story very short: after a quench in a Bethe ansatz integrable model, the TD entropy has the Yang-Yang form:

$$S_{YY} = L \sum_{n=1}^{\infty} \int d\lambda [\rho_{n,t}(\lambda) \ln \rho_{n,t}(\lambda) - \rho_{n,p}(\lambda) \ln \rho_{n,p}(\lambda) - \rho_{n,h}(\lambda) \ln \rho_{n,h}(\lambda)]$$

$s_n(\lambda)$

Assuming that the Bethe excitations are the entangling quasi-particles:

conjecture:

$$S(t) = \sum_n \left[2t \int_{2|v_n|t<\ell} d\lambda v_n(\lambda) s_n(\lambda) + \ell \int_{2|v_n|t>\ell} d\lambda s_n(\lambda) \right],$$

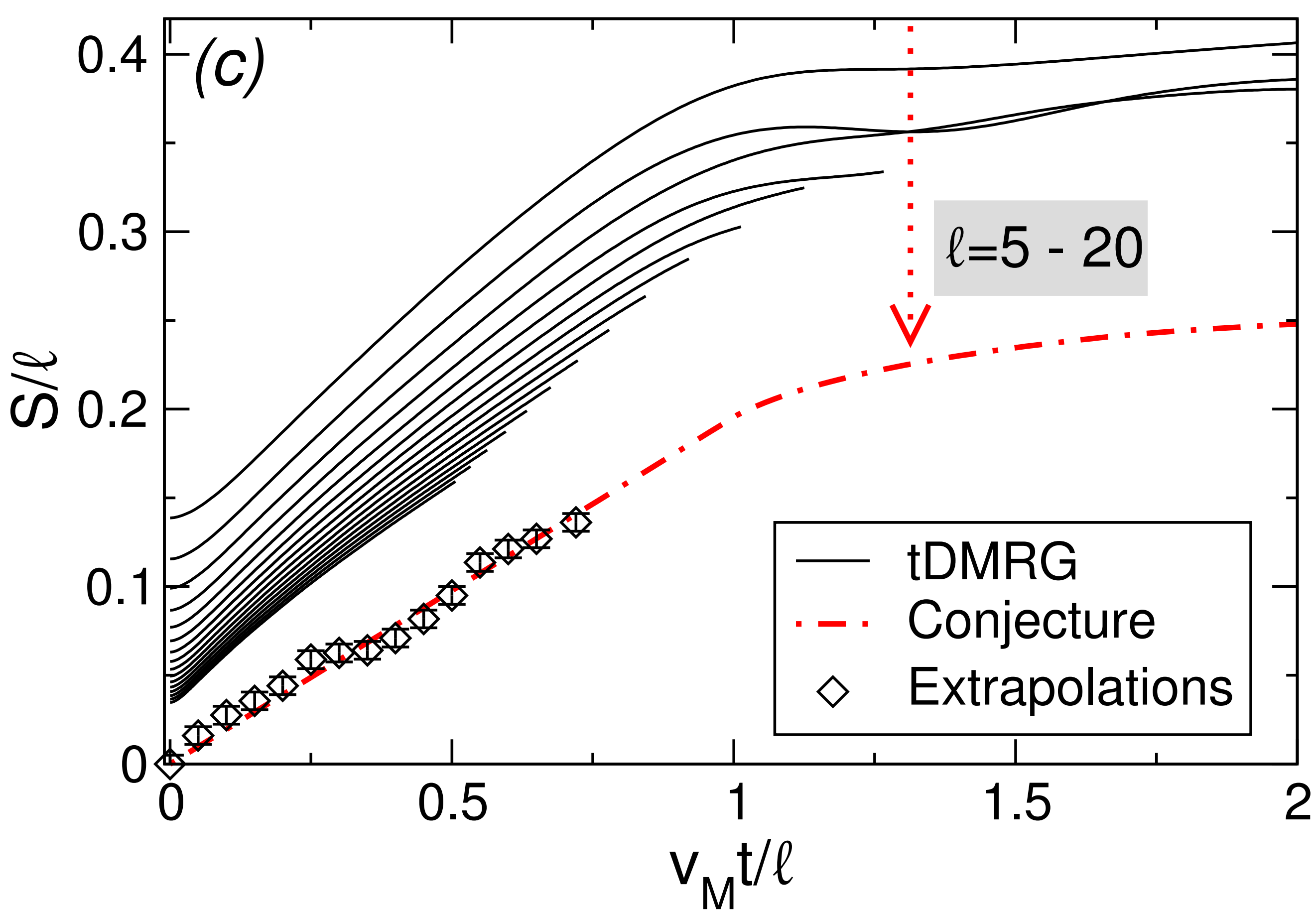
Warning: Determining $v_n(\lambda)$ is non-trivial

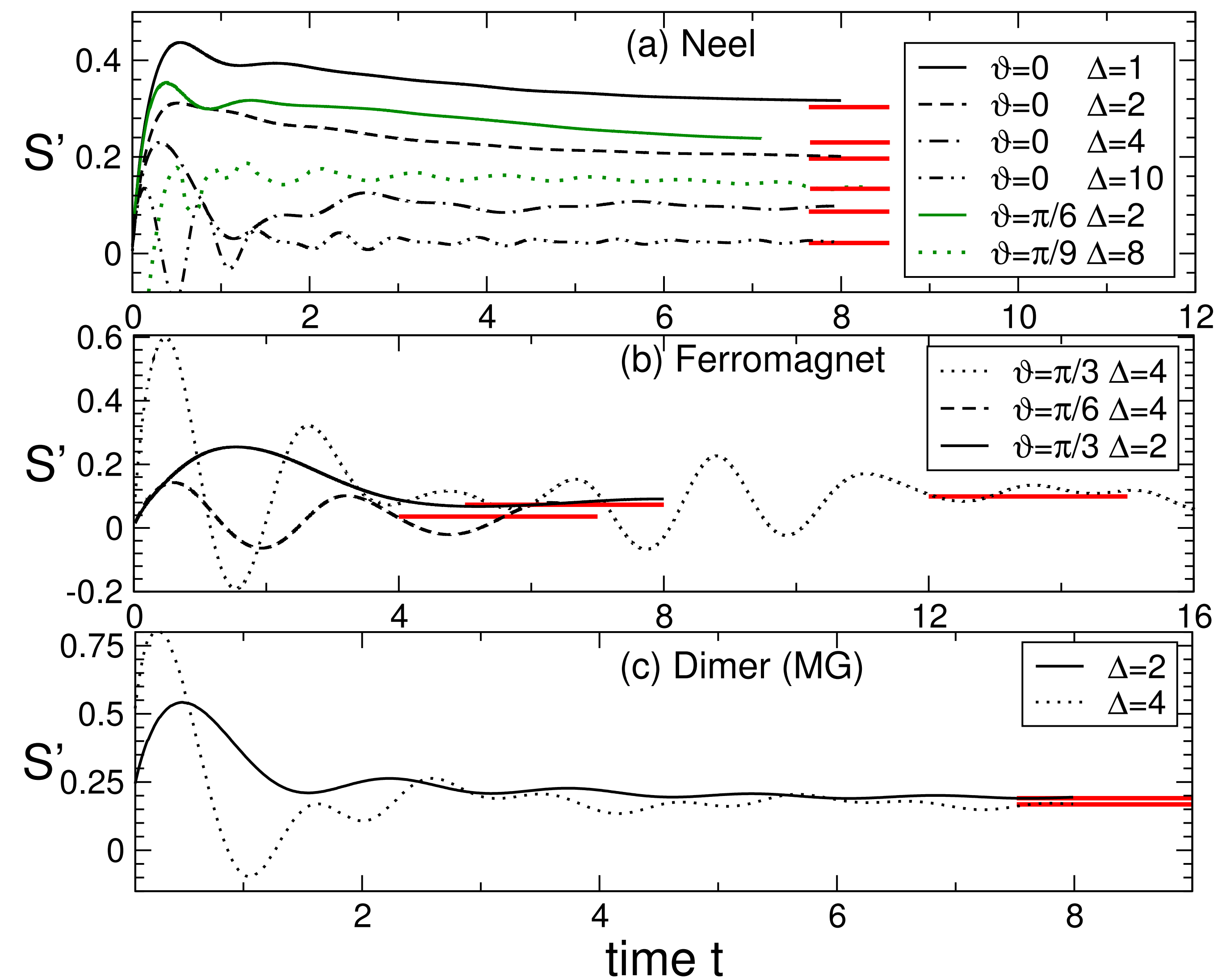
For large time the entanglement entropy becomes thermodynamic entropy

conjecture:

$$S(t) = \sum_n \left[2t \int_{2|v_n|t<\ell} d\lambda v_n(\lambda) s_n(\lambda) + \ell \int_{2|v_n|t>\ell} d\lambda s_n(\lambda) \right],$$

Check for quenches in the XXZ spin-chain



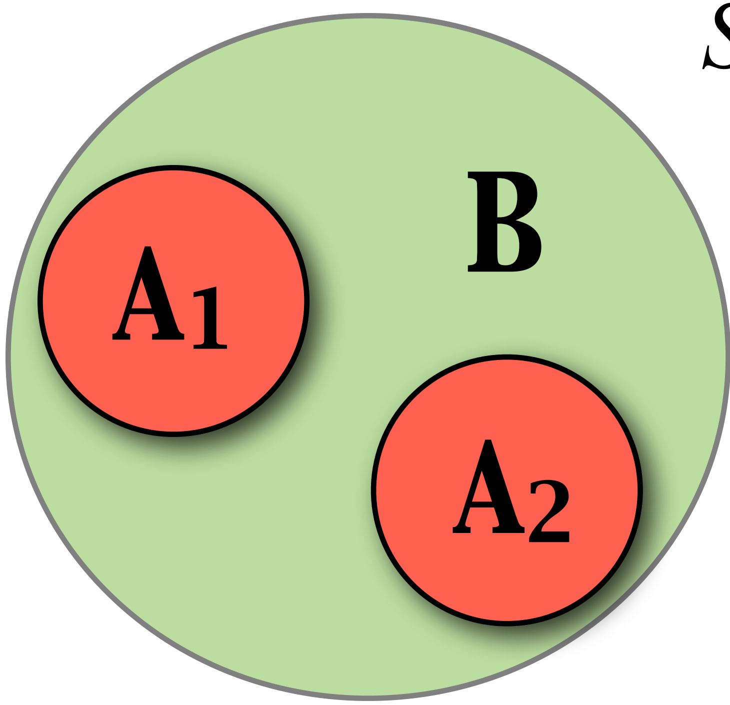


Conclusions

The entanglement entropy is a very useful concept for many-body systems:

- It encodes all universal properties of critical 1D many-body systems (i.e. central charge, operator content, etc.).
- Note: the entanglement in the vacuum encodes all this. There is a lot in the vacuum (ground-state)
- It is a tool to design better performing numerical algorithms
- It provides a mechanism for thermalization
- Many other things that do not fit in one hour

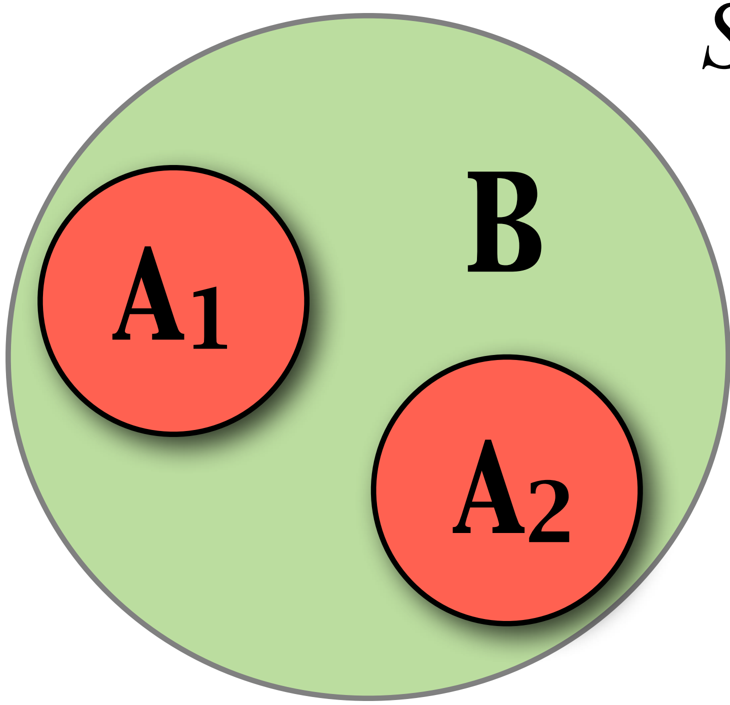
Entanglement of non-complementary parts



$S_{A_1 \cup A_2}$ gives the entanglement between A and B

The mutual information $S_{A_1} + S_{A_2} - S_{A_1 \cup A_2}$ gives an upper bound on the entanglement between A_1 and A_2

Entanglement of non-complementary parts



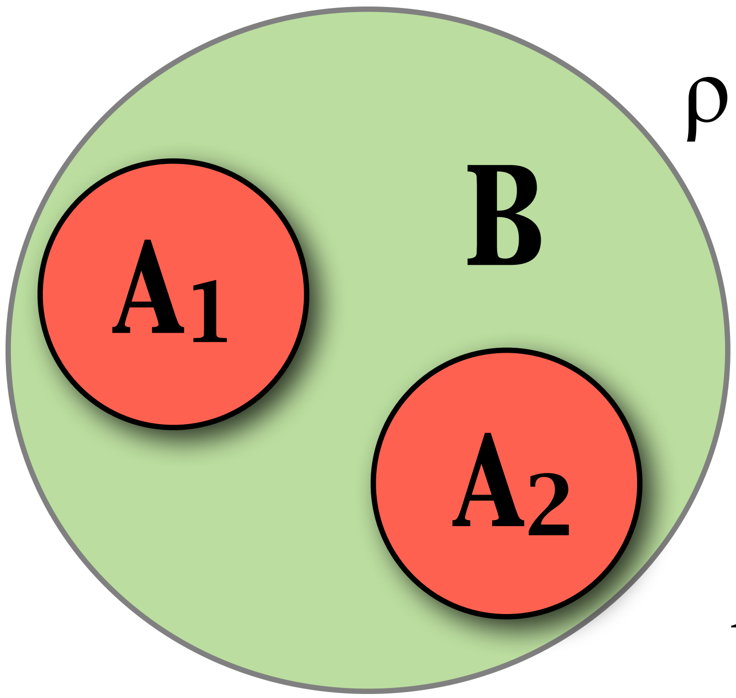
$S_{A_1 \cup A_2}$ gives the entanglement between A and B

The mutual information $S_{A_1} + S_{A_2} - S_{A_1 \cup A_2}$ gives an upper bound on the entanglement between A_1 and A_2

What is the entanglement between the two non-complementary parts A_1 and A_2?

A **computable** measure of entanglement exists:
the logarithmic negativity [Vidal-Werner 02]

Entanglement negativity



Let us denote with $|e_i^{(1)}\rangle$ and $|e_j^{(2)}\rangle$ two bases in A_1 and A_2

ρ is the density matrix of $A_1 \cup A_2$, not pure

The **partial transpose** is

$$\langle e_i^{(1)} e_j^{(2)} | \rho^{T_2} | e_k^{(1)} e_l^{(2)} \rangle = \langle e_i^{(1)} e_l^{(2)} | \rho | e_k^{(1)} e_j^{(2)} \rangle$$

And the **logarithmic negativity**

$$\mathcal{E} \equiv \ln || \rho^{T_2} || = \ln \mathrm{Tr}\, |\rho^{T_2}|$$

$$\mathrm{Tr}|\rho^{T_2}| = \sum_i |\lambda_i| = \sum_{\lambda_i > 0} \lambda_i - \sum_{\lambda_i < 0} \lambda_i$$

It measures “how much” the eigenvalues of ρ^{T_2} are negative because $\mathrm{Tr}\,(\rho^{T_2})=1$

A replica approach to negativity

PC, J Cardy, E Tonni 2012

- Let us consider traces of integer powers of ρ^{T_2}

$$\mathrm{Tr}(\rho^{T_2})^{n_e} = \sum_i \lambda_i^{n_e} = \sum_{\lambda_i>0} |\lambda_i|^{n_e} + \sum_{\lambda_i<0} |\lambda_i|^{n_e} \qquad n_e \text{ even}$$

$$\mathrm{Tr}(\rho^{T_2})^{n_o} = \sum_i \lambda_i^{n_o} = \sum_{\lambda_i>0} |\lambda_i|^{n_o} - \sum_{\lambda_i<0} |\lambda_i|^{n_o} \qquad n_o \text{ odd}$$

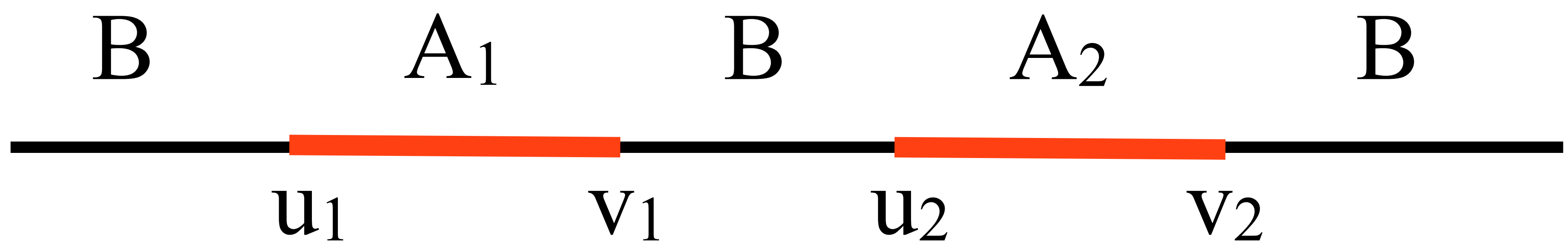
- The analytic continuations from n_e and n_o are different

$$\mathcal{E} = \lim_{n_e\to 1} \ln \mathrm{Tr}(\rho^{T_2})^{n_e} \qquad\qquad \lim_{n_o\to 1} \mathrm{Tr}(\rho^{T_2})^{n_o} = \mathrm{Tr}\rho^{T_2} = 1$$

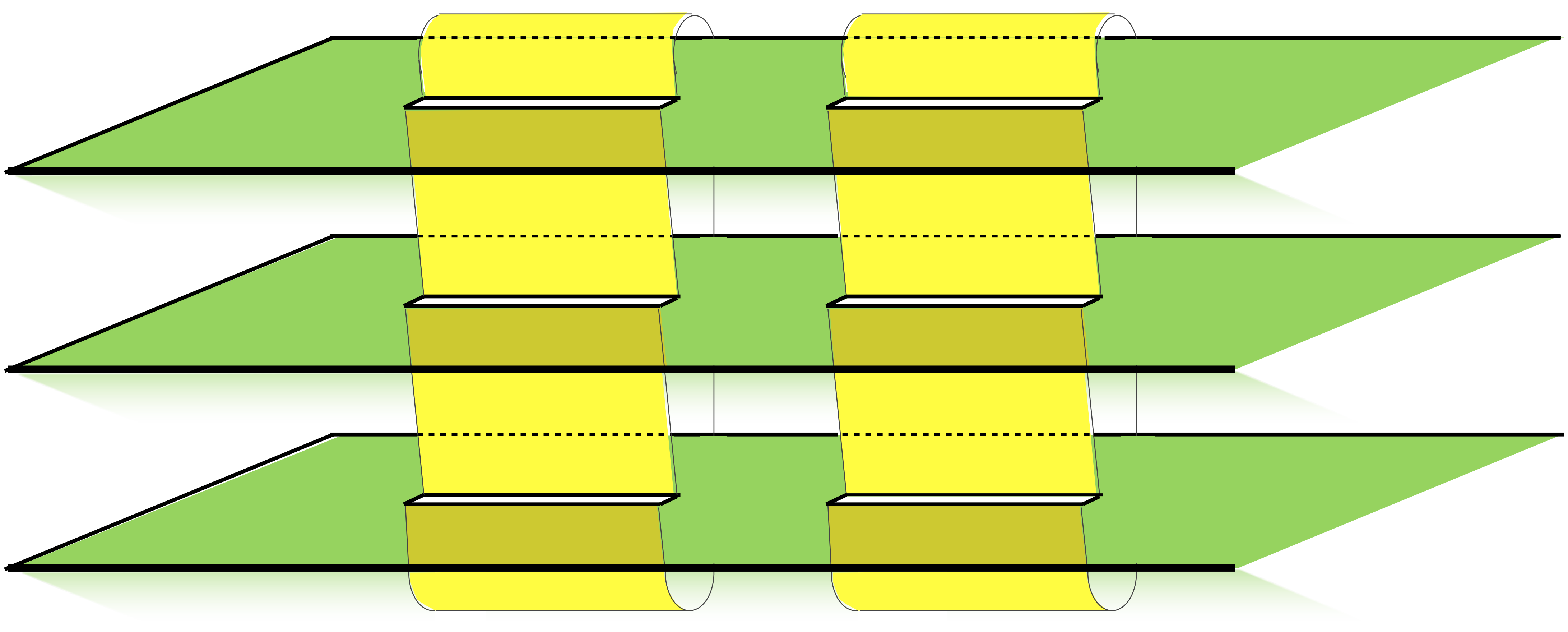
- For a pure state $\rho=|\psi\rangle\langle\psi|$ $\quad \mathrm{Tr}\,(\rho^{T_2})^n = \begin{cases} \mathrm{Tr}\,\rho_2^n & n = n_o \text{ odd} \\ (\mathrm{Tr}\,\rho_2^{n/2})^2 & n = n_e \text{ even} \end{cases}$

Negativity and QFT

Tripartion:

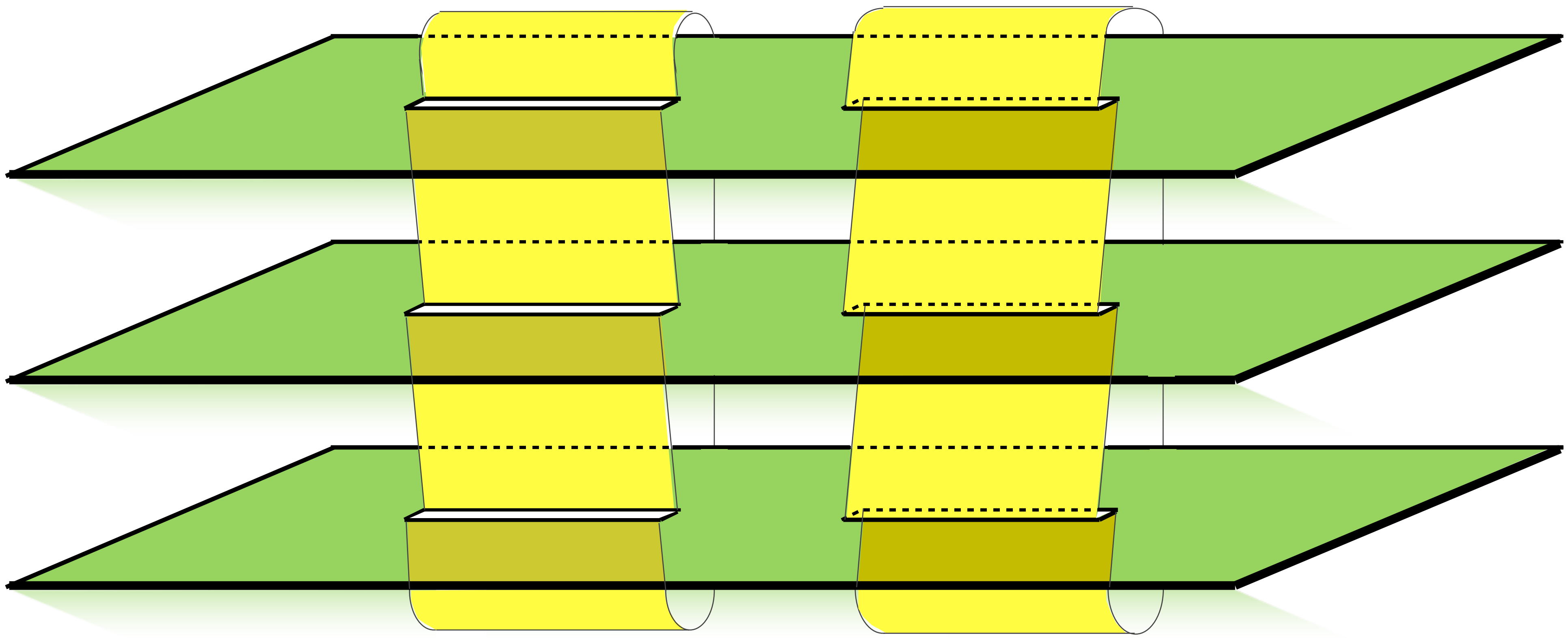


$$\mathrm{Tr}\rho_A^n = \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\mathcal{T}_n(u_2)\bar{\mathcal{T}}_n(v_2)\rangle$$



Negativity and QFT

$$\mathrm{Tr}(\rho_A^{T_2})^n =$$

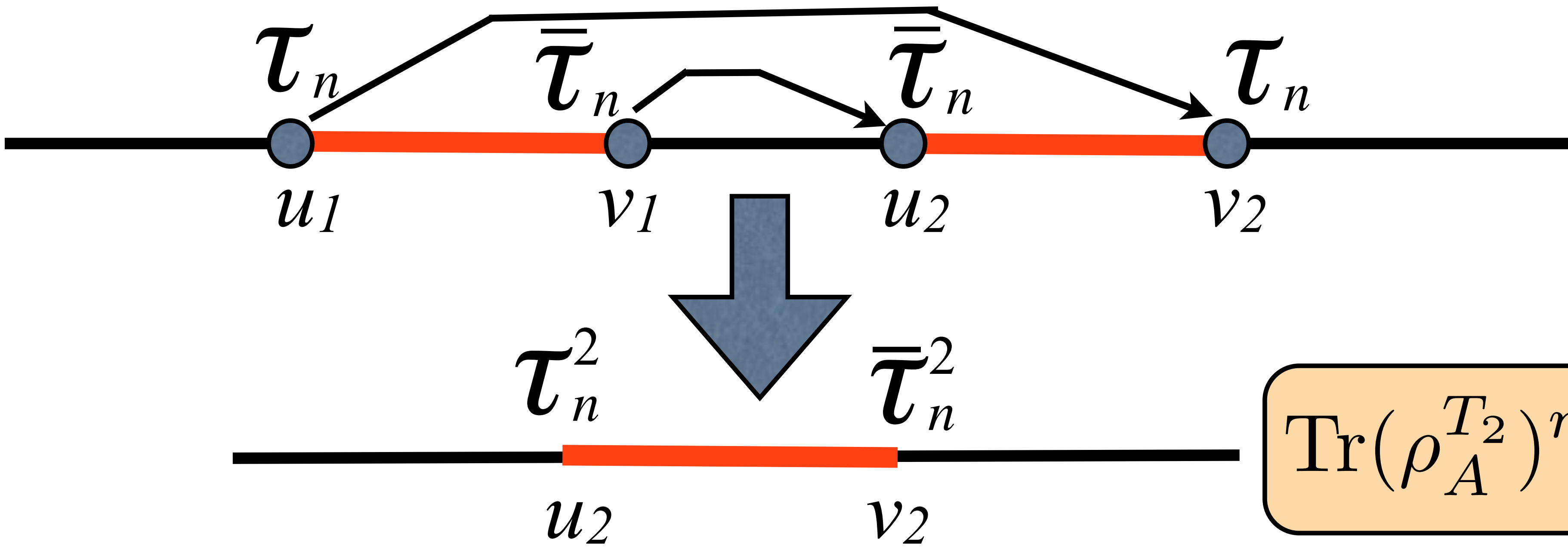


$$= \langle \mathcal{T}_n(u_1)\bar{\mathcal{T}}_n(v_1)\bar{\mathcal{T}}_n(u_2)\mathcal{T}_n(v_2)\rangle$$

The partial transposition exchanges two twist operators

Pure States in QFT

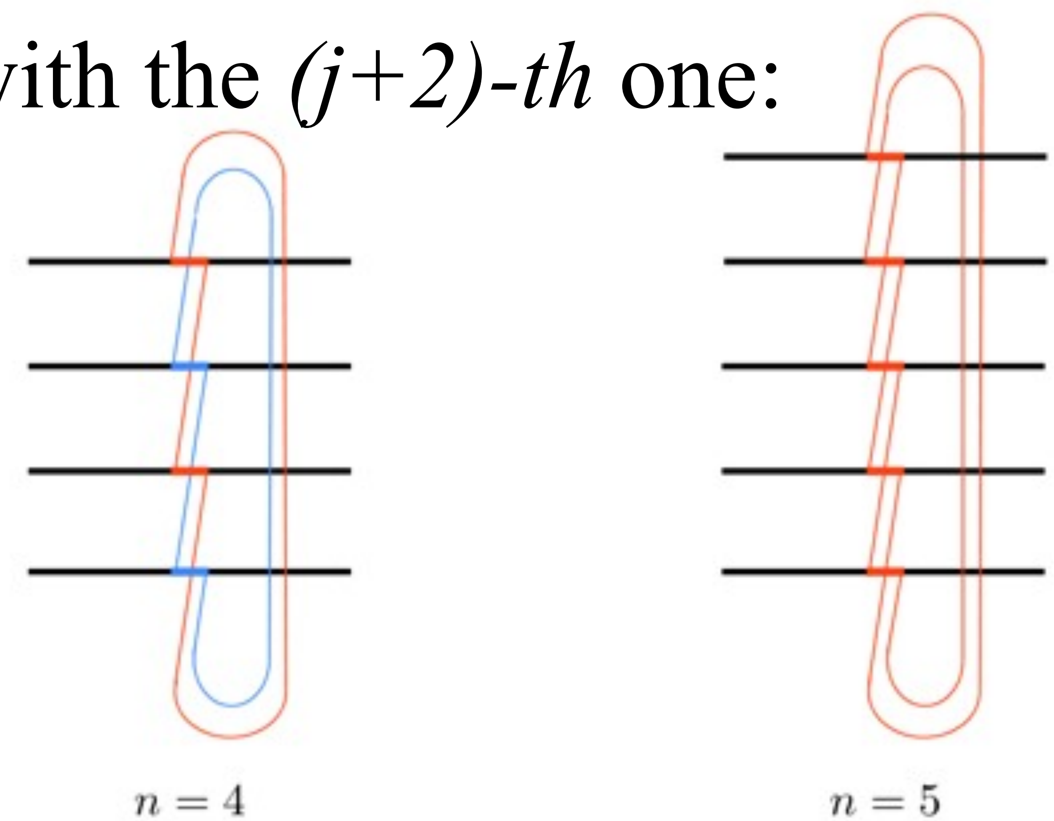
PC, J Cardy, E Tonni 2012



$$\mathrm{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n^2(u_2)\bar{\mathcal{T}}_n^2(v_2)\rangle$$

$\mathcal{T}_n^2$ connects the *j-th* sheet with the *(j+2)-th* one:

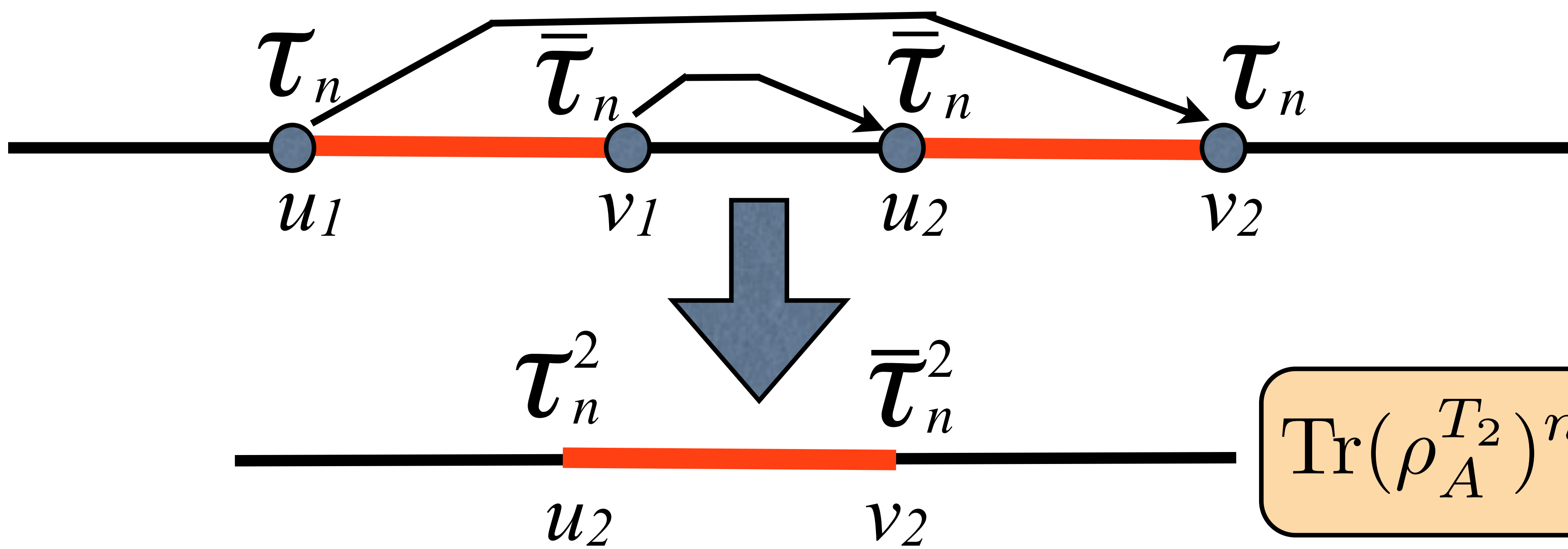
- For $n=n_e$ even, the R-surface decouples in two $n_e/2$ surface
- For $n=n_o$ odd, the n_o-sheeted surface remains n_o-sheeted



$$\mathrm{Tr}(\rho_A^{T_2})^{n_e} = (\langle \mathcal{T}_{n_e/2}(u_2)\bar{\mathcal{T}}_{n_e/2}(v_2)\rangle)^2 = (\mathrm{Tr}\rho_{A_2}^{n_e/2})^2$$

$$\mathrm{Tr}(\rho_A^{T_2})^{n_o} = \langle \mathcal{T}_{n_o}(u_2)\bar{\mathcal{T}}_{n_o}(v_2)\rangle = \mathrm{Tr}\rho_{A_2}^{n_o} ,$$

Pure States in CFT



$$\mathrm{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n^2(u_2)\bar{\mathcal{T}}_n^2(v_2)\rangle$$

$$\begin{aligned}\mathrm{Tr}(\rho_A^{T_2})^{n_e} &= (\langle \mathcal{T}_{n_e/2}(u_2)\bar{\mathcal{T}}_{n_e/2}(v_2)\rangle)^2 = (\mathrm{Tr}\rho_{A_2}^{n_e/2})^2 \\ \mathrm{Tr}(\rho_A^{T_2})^{n_o} &= \langle \mathcal{T}_{n_o}(u_2)\bar{\mathcal{T}}_{n_o}(v_2)\rangle = \mathrm{Tr}\rho_{A_2}^{n_o}\,,\end{aligned}$$

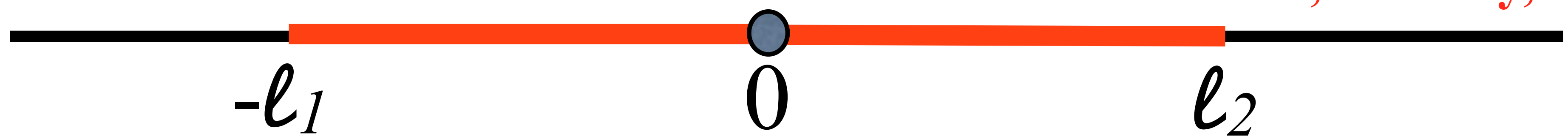
Thus in a CFT:

$\mathcal{T}_{n_o}^2$ has dimension $\Delta_{\mathcal{T}_{n_o}^2} = \frac{c}{12}\left(n_o - \frac{1}{n_o}\right)$, the same as $\mathcal{T}_{n_o}$

$\mathcal{T}_{n_e}^2$ has dimension $\Delta_{\mathcal{T}_{n_e}^2} = \frac{c}{6}\left(\frac{n_e}{2} - \frac{2}{n_e}\right)$

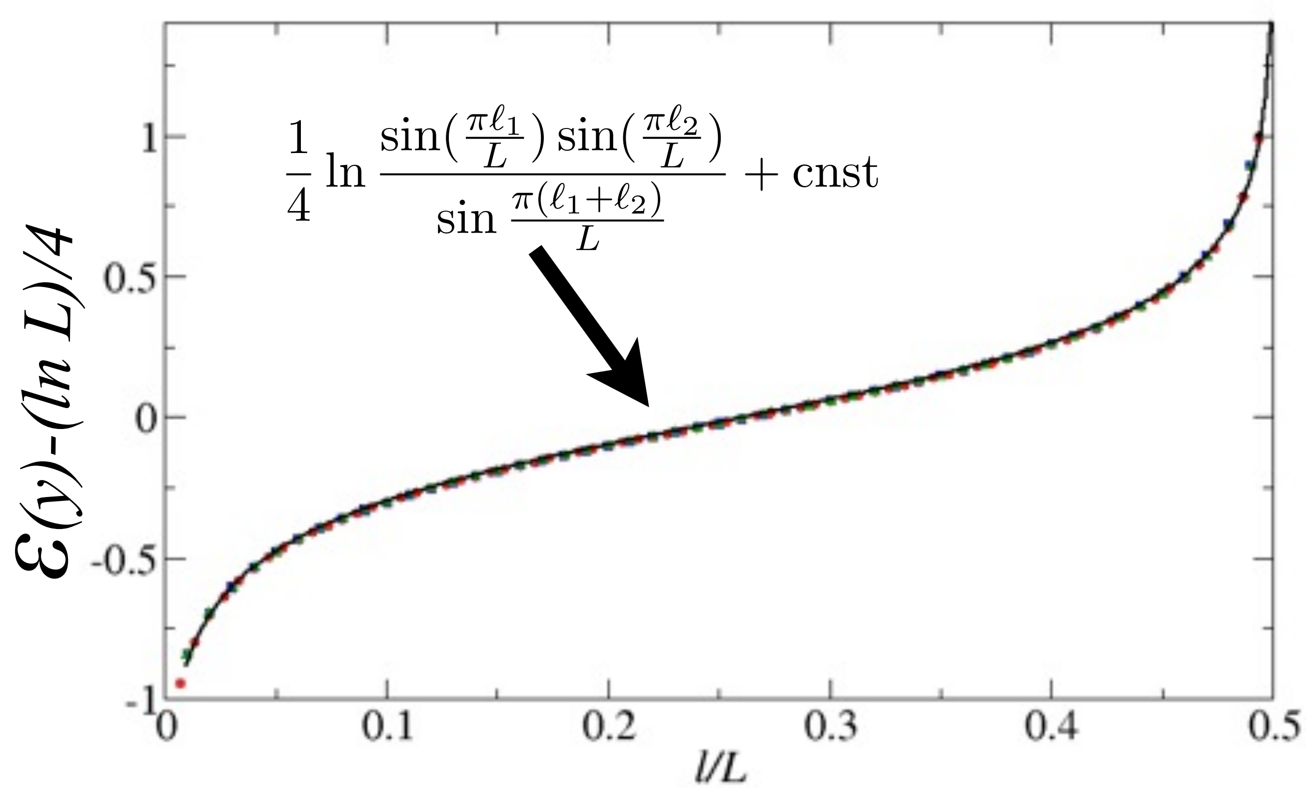
Example: Two adjacent intervals

PC, J Cardy, E Tonni 2012



$$\mathrm{Tr}(\rho_A^{T_2})^n = \langle \mathcal{T}_n(-\ell_1)\bar{\mathcal{T}}_n^2(0)\mathcal{T}_n(\ell_2)\rangle$$

$$||\rho_A^{T_2}|| \propto \left(\frac{\ell_1\ell_2}{\ell_1+\ell_2}\right)^{\frac{c}{4}} \Rightarrow \boxed{\mathcal{E} = \frac{c}{4}\ln\frac{\ell_1\ell_2}{\ell_1+\ell_2} + \mathrm{cnst}}$$



Several other universal results can be similarly derived