Introduction to long range interactions: a theoretical physicist's view.

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Thanks: the very many people I have been working with on this subject!

La Promenade des Anglais, by Raoul Dufy

Why this title?

A theoretical physicist's view on Long Range Interactions (LRI):

- My main interest: LRI induces common features in very different physical systems
  $\rightarrow$ it suits the natural tendency of the theoretical physicist's to look for "universality"
  $\rightarrow$ scope of this conference
- Similarities between different LRI systems are typically expressed through common underlying mathematical structure
  $\rightarrow$ there will be some hints to mathematics
- I will try to keep emphasis on various physical systems. However: I do not claim to be competent in all the fields with LRI!

I. INTRODUCTION

1. On the definition
2. A lot of examples
3. Some basic remarks

On the definition of Long Range Interactions

One finds many definitions in the literature; usually criteria can be expressed through the 2-body interaction potential $V(r)$:

1. $V(r) \propto 1/r^{\alpha}$, with $\alpha < d$ =dimension. Then energy is *not additive* (see later).
2. $V(r) \propto 1/r^{d+\sigma}$, $0 < \sigma < \sigma_c(d)$. The long-range character then modifies the critical exponents.
3. $V(r)$ falls off slower than exponentially. Correlations are then qualitatively different. E.g. : Van der Waals interactions.
4. *One can propose the definition of long range on the nanoscale starting with "extending beyond a single bond".* R.H. French et al. , Long range interactions in nanoscale science (**Rev. Mod. Phys.** 2010).

On the definition of Long Range Interactions, 2

One conclusion: *"What constitutes a long range as opposed to short range interaction depends primarily on the specific problem under investigation."* R.H. French et al.

I will concentrate on definition 1:

1. $V(r) \propto 1/r^{\alpha}$, with $\alpha < d$ =dimension. Energy *not additive*.

However some ideas are relevant beyond these strong LRI.

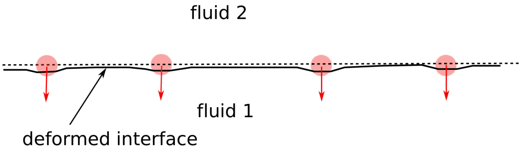
NB: I have used the potential in the definition; one could think of using the force...

Some important examples

- Fundamental interactions
  - Newtonian gravity $V(r) \propto -\frac{1}{r}$: paradigmatic example.
    $\rightarrow$ galactic dynamics, globular clusters, cosmology...
    Clearly: controlled experiments difficult!
  - Coulomb interaction $V(r) \propto \frac{1}{r}$.
    -Non neutral plasma, systems of trapped charged particles: different experimental realizations.
    -Neutral plasmas: huge importance of course.
- Effective interactions
  - Vortex-vortex in 2D fluids: $H \propto \ln r$.
  - Wave-particles: the wave acts as a global degree of freedom interacting with all particles.
    E.g.: single wave model in plasma and fluid dynamics; free electron laser; cold atoms in cavity...

More examples

- colloids at interface + capillarity (A. Dominguez et al.)



Colloids (size $\sim \mu m$) trapped at a fluid interface, subjected to an external vertical force.
$\rightarrow$ an effective long range attraction (or repulsion, depending on the external force)
For $r \leq \lambda$ and not too small

$$V_{\text{eff}}(r) \propto \ln \frac{r}{\lambda} \quad : \quad \sim \text{ 2D gravity!}$$

λ=capillary length, $\sim mm$.
NB: Overdamped dynamics

More examples

- Chemotaxis

$\rho =$ concentration of bacteria; $c =$ concentration of a chemical substance (chemo-attractant).

Bacterial dynamics:

$$\partial_t \rho = D_1 \Delta \rho + \nabla \left(-\sigma \rho \nabla c \right) \ : \ \text{drift up the gradient of } c$$

Chemo-attractant dynamics:

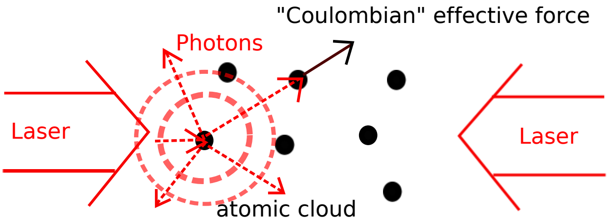
$$\partial_t c = D_2 \Delta c - \lambda c + \alpha \rho \ : \ \text{bacteries} = \text{source for } c$$

$\rightarrow$ again models similar to overdamped 2D gravity.

Huge related activity in mathematical biology.

More examples

- Cold atoms in a magneto-optical trap: multiple diffusion of light



Multiple diffusion and effective force

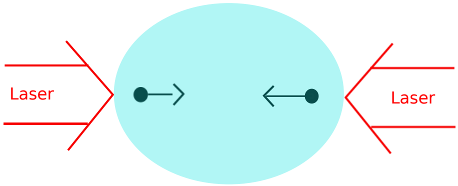
$$\rightarrow \quad \vec{F}_i \propto \sum_j \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3}$$

The $1/r^2$ dependence of the force comes from the solid angle in 3D.

This is an oversimplification; more or less a "standard model" (Sesko, Walker, Wieman 1990).

More examples

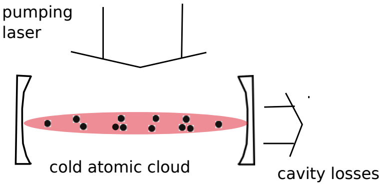
- Cold atoms in a magneto-optical trap: shadow effect



Laser intensities decrease while propagating into the cloud
→ effective force towards the center
Weak absorption approximation: $\nabla \cdot \mathbf{F}_{Shadow} \propto -\rho$ (Dalibard 1988)
→ Just like gravitation... but it does not derive from a potential!

More examples

- Self-organization in optical cavities (G. Morigi et al.)

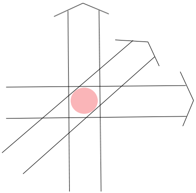


Laser = far from atomic resonance → conservative system as a first approximation.

Integrate over cavity degrees of freedom → effective long-range interaction, of mean-field type, between atoms.

More examples

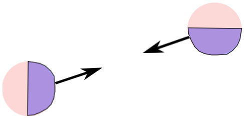
• Dipolar interactions in a Bose-Einstein condensate (O'Dell et al., 2000). BEC irradiated with intense off-resonant lasers $\rightarrow$ dipolar interactions between atoms



Size of the cloud $\ll \lambda$ (laser's wavelength) $\rightarrow$ near field approximation.
Averaging over lasers $\rightarrow$ the $1/r^3$ dominant term is suppressed.
The remaining term is $\propto 1/r$, attractive (possibly anisotropic).
$\rightarrow$ gravity-like force
NB: quantum system; description by Gross-Pitaevskii equation.
NB: extra oscillating terms due to interferences neglected here...

More examples

• Active particles and thermophoresis (R. Golestanian 2012)
Colloidal particles with (partial) metal coating



Thermophoretic effect $\rightarrow$ move up (or down) the temperature gradient
Metal absorbs laser light $\rightarrow$ particles are "temperature sources"
$\rightarrow$ Again, an overdamped "gravity-like" dynamics

More examples

• Eigenvalues of random matrices. Eg. complex Ginibre ensemble. Each entry of A (size $n \times n$) is $A_{kl} = X_{kl} + iY_{kl}$, Xs and Ys are independent, law $\mathcal{N}(0, 1/2)$. The eigenvalues z_k of A have joint probability density:

$$\begin{aligned} P(Z_1, \ldots, z_n) &\propto \Pi_{k=1}^{n} e^{-|z_k|^2} \Pi_{1 \leq k < l \leq n} |z_k - z_l|^2 \\ &\propto \exp\left[- \left(\sum_k |z_k|^2 - 2 \sum_{k,l} \ln |z_k - z_l| \right) \right] \end{aligned}$$

$\rightarrow$ analogous to a 2D Coulomb gas confined in an harmonic trap!
• There are similar laws for other random matrix ensembles.
Intense mathematical activity related to determinantal processes.

More examples

- Trapped free fermions, 1D harmonic trap
  Pauli exclusion principle $\rightarrow$

$$|\Psi_0(x_1, \ldots, x_n)|^2 \propto e^{-\alpha^2 \sum_k x_k^2} \Pi_{k<l} |x_k - x_l|^2$$

  where $\Psi_0 =$ ground state wave function.
- Stellar dynamics around a massive black hole (Tremaine, Sridhar and Touma...)
  -Short time scales = Keplerian dynamics of stars around the black hole
  -Longer time scales : interaction between stars (+relativistic corrections+...)
  Averaging over short time scales $\rightarrow$ an effective system of "interacting orbits"

Toy models

- Underlying idea: long range interactions have similar effects in different systems, leading to some "universal" properties
$\rightarrow$ it makes sense to use toy models to illustrate, or study in details these properties more easily...
- Indeed has been used for a long-time : see Thirring's models to illustrate peculiarities of equilibrium statistical mechanics.
- THE toy model: HMF (= mean-field XY model + kinetic term)

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{K}{N}\sum_{i=1}^{N}[1-\cos(\theta_i-\theta_j)]$$

- Very useful; of course it goes with all the caveats regarding toy models...

Some caveats

• I am of course ignorant in most of these fields!
$\rightarrow$ Please react if I am not as accurate as I should when I say a few words about some of them

• Concentrate on classical physics; apologies to quantum physicists.
Some concepts might still be relevant for quantum systems.

Some basic remarks

Specific difficulties of LRI: one particle interacts with many others...

- impossible to cut the system in almost independent pieces. Related difficulty: no distinction bulk/boundary
- numerical problem: with a naive algorithm, each type step costs $\propto N^2$

Also specific advantages:
One particle interacts with many others
$\rightarrow$ fluctuations suppressed; law of large numbers, Central Limit Theorem, large deviations...
Related idea: "mean-field" should be a very good approximation

II. EQUILIBRIUM STATISTICAL MECHANICS

1. On scaling, extensivity, (non) additivity

2. On the mean field approximation

3. Examples and discussions

Equilibrium statistical mechanics

N long-range interacting particles or N spins on a lattice:

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i \neq j} V(x_i - x_j) \text{ or } H = -J \sum_{i \neq j} \frac{S_i S_j}{|i - j|^{\alpha}}$$

Microcanonical equilibrium, fixed energy E:

$$d\mu_M \left(\{x_i, p_i\}_{i=1,\dots,N} \right) \propto \delta[E - H(\{x_i, p_i\})] \Pi_i dx_i dp_i$$

Canonical equilibrium, fixed inverse temperature $\beta = 1/T$:

$$d\mu_C \left(\{x_i, p_i\}_{i=1,\dots,N} \right) \propto \exp\left[-\beta H(\{x_i, p_i\})\right] \Pi_i dx_i dp_i$$

Special features of LRI, scaling

• In the usual "Thermodynamic limit" $N \to \infty$, fixed density $\to$ potential energy $\gg N$.
Whereas entropy $\propto N$.
$\to$ potential energy always wins, at any $T > 0$, the system is in the ground state (possibly singular) when $N \to \infty$!
• True, but not very interesting. Rather than looking for any large N limit, we should look for something independent of N in the large N limit.
Compare

$$\lim_{N\to\infty} u(N) = 0 \text{ and } \lim_{N\to\infty} Nu(N) = c.$$

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Other example:

$$u(a, N) = \frac{a}{N} + \frac{1}{N^2} \to \text{ scalings i)} a \text{ fixed, or ii)} \tilde{a} = Na \text{ fixed}$$

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• True, but not very interesting. Rather than looking for any large N limit, we should look for something independent of N in the large N limit.
We will see several examples in the folllowing.

Scaling, examples

- **Example:** scaling for spin systems (1D, $0 \leq \alpha < 1$)

$$H = \frac{1}{2\tilde{N}_\alpha} \sum_{i \neq j} \frac{-S_i S_j}{|i-j|^\alpha} \text{ with } \tilde{N}_\alpha \propto N^{1-\alpha}$$

Or, equivalently, scale the temperature...

- **Example:** scaling for self gravitating systems.

Microcanonical: $\frac{V^{1/3}E}{GM^2}$ fixed (V =volume, M = total mass); there are several ways to enforce this scaling

The short range singularity should be regularized...

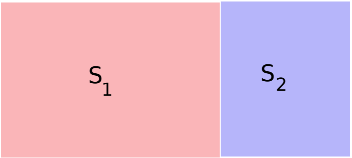
- **Example:** neutral plasmas.

Same short-range singularity. Once it is regularized, there is a well-defined thermodynamic limit! (Lebowitz, Lieb, Narnhofer)

Special features of LRI: about extensivity, additivity

Extensivity: energy proportional to N, or to the volume V. Does not make much sense without a specified scaling. Choosing a scaling may restore extensivity (good to compare with entropy).

Non additivity:



$$E_{tot} \neq E_1 + E_2$$

$\rightarrow$ no phase separation possible in the usual sense

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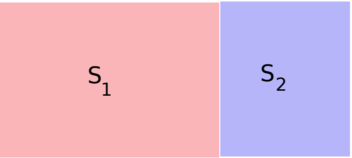

$$E_{tot} \neq E_1 + E_2$$

$\rightarrow$ no phase separation possible in the usual sense
And: phase separation $\Rightarrow$ entropy concave

$$e = xe_1 + (1-x)e_2 \Rightarrow S(e) \geq xS(e_1) + (1-x)S(e_2)$$

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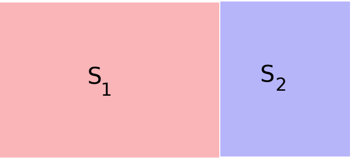
Non additivity:



$$E_{tot} \neq E_1 + E_2$$

$\rightarrow$ no phase separation possible in the usual sense

Furthermore: free energy = Legendre transform of entropy; this operation is invertible only if the entropy is concave...

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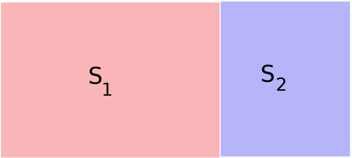
Non additivity:



$$E_{tot} \neq E_1 + E_2$$

$\rightarrow$ no phase separation possible in the usual sense
$\rightarrow$ no reason for equivalence between canonical and microcanonical ensembles
cf Hugo Touchette's talk.

Special features of LRI: about mean field approximation

- One particle interacts with many others

$\rightarrow$ a mean field description should be very good, fluctuations small
Correct intuition: in a well chosen scaling limit, a mean-field theory often becomes exact.
$\rightarrow$ for instance, always classical critical exponents

- A perfectly suited mathematical tool: large deviation theory.

- Caveats:

-strong fluctuations close to second order phase transitions
-sometimes a short-range singularity together with the long-range character (eg: gravitation)
-short range interactions can bring additional correlations
-more than one scaling may be relevant (see the non neutral plasma case).

Equilibrium statistical mechanics, examples (1)

- Self gravitating systems, chief example. Regularities in the structures of galaxies $\rightarrow$ natural to think of a statistical physics argument (I am being naive here, see later!).
  -Difficulties with both the absence of confinement and the short range singularity.
  -Main features (microcanonical): beyond a certain central density, no equilibrium state any more, even metastable $\rightarrow$ "gravothermal catastrophe".
  -Beautiful theory, but seems difficult to find clear situations where it is applicable; I don't know everything here! Some explanations later.
- Self-gravitating systems: models of interacting orbits (Tremaine, Sridhar, Touma...); may be a nice application of equilibrium statistical mechanics?

Equilibrium statistical mechanics, examples (2)

- Vortices (first study by Onsager); $x_i \in \mathbb{R}^2$

$$H^N = -\frac{1}{2\pi} \sum_{i<j} \ln |x_i - x_j|$$

Qualitatively very useful predictions: it may be statistically favorable to form large scale structures!

- Another effective model: wave + particles description of a plasma (Escande, Elskens, Firpo...)

-Plasma + Langmuir wave $\sim$ non resonant bulk + resonant particles

$\rightarrow$ effective description: wave + resonant particles

-Classical question: when does the wave damps completely?

-Elskens-Firpo: a statistical mechanics answer. Not sure it is quantitatively accurate...

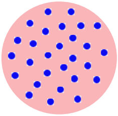
Equilibrium statistical mechanics, examples (3)

• Non neutral plasmas (in Penning traps for instance). Simplified version:

$$H^N = \frac{1}{2}\sum_{i \neq j} \frac{1}{|x_i - x_j|} + \frac{1}{2}\sum_i x_i^2$$

Balance between trap and interaction $\rightarrow$ typical size $R \propto N^{1/3}$.
Ground state at mean-field level $=$ uniformly charged sphere, radius R.
Absolute ground state $=$ ordered configuration.

Leading order:
mean density



Next order:
local correlations

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Balance between trap and interaction $\rightarrow$ typical size $R \propto N^{1/3}$.
Ground state at mean-field level = uniformly charged sphere, radius R.
Absolute ground state = ordered configuration.
First scaling: $\beta N^2/L$ fixed
$\rightarrow$ Describes the cloud's shape, cross-over from gaussian to mean-field ground state; no phase transition.

Equilibrium statistical mechanics, examples (3)

• Non neutral plasmas (in Penning traps for instance). Simplified version:

$$H^N = \frac{1}{2}\sum_{i \neq j} \frac{1}{|x_i - x_j|} + \frac{1}{2}\sum_i x_i^2$$

Balance between trap and interaction $\rightarrow$ typical size $R \propto N^{1/3}$.
Ground state at mean-field level = uniformly charged sphere, radius R.
Absolute ground state = ordered configuration.
Second scaling: β fixed.
$\rightarrow$ The system is in its ground state at mean-field level; β controls the non trivial local correlations; phase transition possible.

Equilibrium statistical mechanics, examples (3)

• Non neutral plasmas (in Penning traps for instance). Simplified version:

$$H^N = \frac{1}{2}\sum_{i\neq j}\frac{1}{|x_i - x_j|} + \frac{1}{2}\sum_i x_i^2$$

Balance between trap and interaction $\rightarrow$ typical size $R \propto N^{1/3}$.
Ground state at mean-field level = uniformly charged sphere, radius R.
Absolute ground state = ordered configuration.
$\rightarrow$ Example where two different scalings are interesting!
Side note: understanding this type of "absolute ground state" -and the phase transition- is a long-standing mathematical problem.

Some conclusions

- Many universal features related to the long range character of the interactions (see also Hugo Touchette's talk): non additivity, inequivalence between statistical ensembles ($\rightarrow$ peculiar phase transitions), negative specific heat,...
- Beautiful theory, but... *(my opinion)* there are not that many experimentally meaningful applications of equilibrium statistical mechanics with long-range interactions.
- There is a good reason for this: very slow relaxation times! $\rightarrow$ kinetic theory

III. KINETIC THEORY

1. Hamiltonian case
   - Collisionless equations and their properties
   - Secular evolution and collisional equations
2. Non Hamiltonian case

Kinetic theory, Hamiltonian case

• Boltzmann picture (short range interaction): rare collisions that have a strong impact
$f(x, v, t) =$ one-point distribution function

$$\partial_t f + v \cdot \nabla_x f = \mathcal{C}(f, f)$$

This is again obtained in a specific scaling when $N \to \infty$: Boltzmann-Grad scaling.
• Long-range interactions: "collisions" not rare! Instead: law of large numbers $\to$ a dynamical mean field equation, in a well chosen scaling limit.

Examples of collisionless kinetic equations

- Point charged particles → Vlasov-Poisson equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \Phi \nabla_v f = 0 \ , \ \text{with } \Delta\Phi = 1 - \rho$$

- Point masses → Vlasov-Newton (collisionless Boltzmann)

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \Phi \nabla_v f = 0 \ , \ \text{with } \Delta\Phi = \rho$$

- Point vortices → 2D Euler equation

$$\partial_t \omega + (\vec{u} \cdot \nabla)\omega = 0 \ , \text{with } \omega = -\Delta\Psi \ , \ \vec{u} = -\nabla^{\perp}\Psi.$$

- Particles + wave → Vlasov + wave

Examples of collisionless kinetic equations

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- Point vortices $\rightarrow$ 2D Euler equation

$$\partial_t \omega + (\vec{u} \cdot \nabla)\omega = 0 \ , \text{with } \omega = -\Delta\Psi \ , \ \vec{u} = -\nabla^{\perp}\Psi.$$

- Other example: light propagation in a non linear non local medium $\rightarrow$ a Vlasov regime starting from Non Linear Schrödinger (Picozzi et al.)!

Conclusion: These different collisionless kinetic equations have similar properties

$\rightarrow$ another striking example of universality induced by LRI.

On the mathematical status of these equations, 1

• Formal derivation easy: "mean-field approximation"; + hints that mean-field should be "good", and in fact one would like to say something like "Vlasov equation becomes exact in the $N \to \infty$ limit". Is it true, and in which sense? Starting point:

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= \frac{1}{N} \sum_{j \neq i} K(x_i - x_j) \end{aligned} \tag{1}$$

Central quantity: empirical density $\hat{f}^N$

$$\hat{f}^N(x, v, t) = \frac{1}{N} \sum_i \delta(x - x_i(t)) \delta(v - v_i(t))$$

Can we say that $\hat{f}^N(x, v, t)$ is close to $f(x, v, t)$, solution of the Vlasov equation with initial condition $f(x, v, t = 0)$ close to $\hat{f}^N(x, v, t = 0)$?

On the mathematical status of these equations, 2

Key observations:
i) $\hat{f}^N$ is itself a solution of Vlasov equation
ii) Take f_1 and f_2 two solutions of Vlasov equation, then for some constant C (C depends on the interaction force K), and some well chosen distance d

$$d(f_1(t), f_2(t)) \leq d(f_1(t=0), f_2(t=0))e^{Ct}$$

$\rightarrow$ a theorem for regular interactions (Neunzert, Dobrushin, Braun and Hepp 70's)

A theorem

Main hypothesis: K and its derivative are assumed bounded.
Then: Take a sequence of initial condition for the N particles that tends to f_0 when $N \to \infty$, any fixed time T and any $\varepsilon > 0$. Call $f(t)$ the solution of Vlasov equation with initial condition f_0. Then for any $N > N_c(T, \varepsilon)$, and any time $t \leq T$

$$d(\hat{f}^N(t), f(t)) \leq \varepsilon$$

Remarks:
i) No average needed: take any initial condition close to f_0, the empirical density follows closely Vlasov equation, for any realization.
ii) Vlasov dynamics OK for large N *for a fixed time horizon* $\to$ the asymptotic behavior of the particles' dynamics may not be given by making $t \to \infty$ in the Vlasov dynamics!
iii) From the proof, it appears that N_c may increase very fast with T...

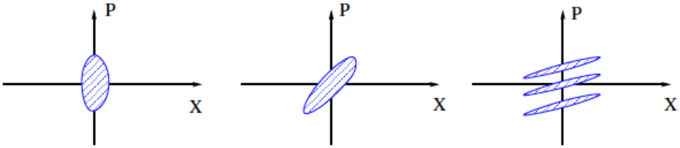
On singular interactions

Many interesting interactions are actually singular... $\to$ a mathematical problem, and also a numerical one for people trying to approximate Vlasov equation with particles. Some contributions:

- From point vortices to 2D Euler (Goodman et al.): logarithmic singularity still "acceptable"
- Singular forces with $K(x) \sim 1/|x|^\alpha$, $\alpha < 1$ (averaging techniques, Hauray and Jabin); Coulomb not included!
- Kiessling: a kind of "if theorem" for the Coulomb case. If some quantity is bounded uniformly in N, then...
- Pickl, Boers, Lazarovici (2015): up to the Coulomb case (with small N dependent cut-off), making use of "probabilistic" degrees of freedom.

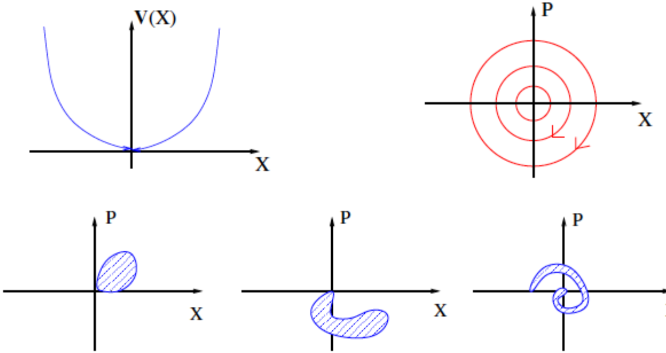
Qualitative features

Transport $\rightarrow$ phase space filamentation (phase mixing). Example with periodic boundary conditions:



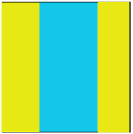
Qualitative features

Example with a non trivial potential



Qualitative features

Example of 2D Euler evolution (perturbation of a shear flow, simulation H. Morita).



Vlasov-Poisson equation is a world in itself; it is of course crucial for plasma physics. I will discuss some generic properties of Vlasov or related equations.

Some properties of Vlasov-like equations

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \left(\iint V(x-y) f(y,v,t) dy dv \right) \cdot \nabla_v f = 0$$

- Inherited from the particles: conservation of energy, momentum...
- Many more conserved quantities (Casimirs)

$$\frac{d}{dt} \int C(f) dx dv = 0 \ , \ \text{for any function } C.$$

  Not directly inherited from conserved quantities for the particles.
- In particular, the volume of each level set of f is conserved $\rightarrow$ Vlasov dynamics $=$ mixing of these level sets, involves finer and finer scales.

On stationary solutions

- Many stationary solutions; statistical equilibrium = only one of these. Ex.: $f(v)$, homogeneous in space, constant potential $\rightarrow$ stationary for any f.
- Constructing stationary solutions from conserved quantities: critical points of conserved quantities are stationary! $\rightarrow$ look for extrema of

$$\iint C(f) + \beta H[f] + \alpha \iint f$$

  May be a useful point of view to investigate stability.
- Clearly: No approach to statistical equilibrium.
- $\rightarrow$ Important question: what is the asymptotic behavior of a Vlasov-like equation? Difficult problem...

Asymptotic behavior of Vlasov equation

• Linearize around stationary solution.
λ eigenvalue $\rightarrow -\lambda, \lambda^\star, -\lambda^\star$ also eigenvalues... $\rightarrow$ no asymptotic stability in the usual sense.
• Yet, for stable stationary states, a kind of exponential stability: **Landau damping**.

- discovered in plasma physics (1946)
- now a fundamental concept in galactic dynamics
- related to the inviscid damping in 2D fluids (known before Landau)
- $+$ many other instances, including non Hamiltonian ones (synchronization models, bubbly fluids...)

$\rightarrow$ again, a universal concept.

Asymptotic behavior of Vlasov equation

Question: Take an initial condition $f(t=0)$; what can we say about $f(t \to \infty)$?
An old question in physics; recently a hot mathematical topic.
i) Dynamical system approach: perturbation theory, builds on linear theory. Ideas from non linear dynamical systems.
Drawback: validity a priori limited to neighborhoods of stationary states.
ii) Stat. mech. approach: an equilibrium statistical mechanics that would take into account the dynamical constraints of Vlasov equation (pioneered by Lynden-Bell in astrophyics).
iii) Other ideas: mix the previous ones; try to take into account as much dynamics as possible.

Dynamical system approach,1

• Non linear stability (starting with Antonov): uses a variational approach, stationary states seen as critical points of a conserved functional.
Typical result: criteria for stability (if $f(t=0)$ is close to some $f_{\rm stat}$, then $f(t)$ remains close to $f_{\rm stat}$)
Example: take a stationary solution of Vlasov-Newton equation, of the form

$$f = F_0(E) = \varphi\frac{v^2}{2} + \phi(x)\,,\ \text{with}\ \Delta\Phi(x) = 4\pi G\int f dv\ \text{and}\ F_0' < 0\,;$$

then f is stable.
NB: no precise information on the dynamics, filamentation process (and Landau damping) overlooked; mathematically: involves norms without derivative.

Dynamical system approach, 2

• *Non linear Landau damping:* Landau damping = comes from the linearized Vlasov equation. Example, close to a homogeneous stationary state $f_0(v)$, write $f = f_0 + \delta f$:

$$\partial_t \delta f + v\partial_x \delta f - \partial_x \left(\int V(x-y)\delta f(y,v')dv' \right) f_0'(v) = \partial_x \left(\int V(x-y)\delta f(y,v')dv' \right) \partial_v \delta f$$

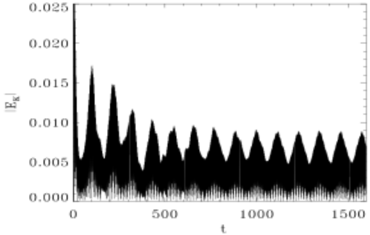
Linearized Vlasov equation: should be OK for "small" δf.
The non linear term becomes larger and larger because of filamentation → ??
Important remark: the mathematical meaning of "close" and "small" is crucial!

Dynamical system approach, 3

- Mouhot-Villani theorem (2010): if the perturbation is small enough (in a very strong manner), the perturbed potential tends to 0 exponentially, with Landau rate.
  NB: δf does not tend to 0.
- Lin-Zheng (2011): if one measures the smallness of δf in a less demanding way, there are undamped solutions arbitrarily close to f_0 (there is a precise regularity threshold).

Dynamical system approach, 4

- Stable stationary state, beyond Landau damping: when the perturbation exceeds a certain threshold, damping is incomplete; excitation of non linear solutions known as Bernstein-Greene-Kruskal modes (Manfredi, Lancellotti-Dorning).



Simulations by G. Manfredi (1997).

Dynamical system approach, 4

- Stable stationary state, beyond Landau damping: when the perturbation exceeds a certain threshold, damping is incomplete; excitation of non linear solutions known as Bernstein-Greene-Kruskal modes (Manfredi, Lancellotti-Dorning).

- Weakly unstable stationary state: does the instability saturate, and how? An old question, which is actually a complicated bifurcation problem. For homogeneous stationary state, many contributions (O'Neil, Crawford, Del-Castillo-Negrete...) **One conclusion:** a universal weakly non linear dynamics, governed by the "Single Wave Model".

Side remark: Yet, the "Single Wave Model" is less universal than Landau damping... (eg: Kuramoto model).
$\rightarrow$ Question: could one classify more precisely these bifurcations with continuous spectrum?

Dynamical system approach, 4

- Stable stationary state, beyond Landau damping: when the perturbation exceeds a certain threshold, damping is incomplete; excitation of non linear solutions known as Bernstein-Greene-Kruskal modes (Manfredi, Lancellotti-Dorning).

- Weakly unstable stationary state: does the instability saturate, and how? An old question, which is actually a complicated bifurcation problem. For homogeneous stationary state, many contributions (O'Neil, Crawford, Del-Castillo-Negrete...) **One conclusion:** a universal weakly non linear dynamics, governed by the "Single Wave Model".

- Non homogeneous stationary state: different physics, technical difficulties (PhD thesis of David Métivier, with Y. Yamaguchi).
- Response theories (Ogawa-Yamaguchi, Patelli et al.)

$\rightarrow$ a very rich problem, with still plenty to explore.

Statistical mechanics approach

Far from linear regime: out of reach for dynamical systems techniques.
Another approach: statistical mechanics.
Rationale: regularities in the structure of galaxies; it is natural to think of a statistical mechanics argument. Yet, we know that the equilibrium stat. mech. of the N particles is irrelevant...
Idea (Lynden-Bell, 68): could one define an equilibrium for Vlasov dynamics?
Basic ingredient: Vlasov dynamics preserves all level volumes of f.
Basic assumption: we have to look for the "most disordered" state compatible with all constraints.
$\rightarrow$ describe the state by a probability distribution on the levels at each point (x, v), and maximize the entropy of this "field of pdf", under constraints.

Statistical mechanics approach

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Some comments:
-A beautiful idea, which sometimes gives qualitatively useful predictions.
-The assumption of a maximum mixing is far from verified in general.
-A similar approach has been developed in 2D fluid dynamics

Statistical mechanics approach

Far from linear regime: out of reach for dynamical systems techniques.
Another approach: statistical mechanics.
Rationale: regularities in the structure of galaxies; it is natural to think of a statistical mechanics argument. Yet, we know that the equilibrium stat. mech. of the N particles is irrelevant...
Idea (Lynden-Bell, 68): could one define an equilibrium for Vlasov dynamics?
Mixed approaches: try to take into account as much dynamics as possible...
Relate initial conditions and final state by assuming a "not too violent" transient (Ex: De Buyl et al., Pakter-Levin).
A parametric resonance during the transient dynamics (Levin, Pakter et al.) $\rightarrow$ a succesful theory of core-halo structures (if not "universal" feature, commonly observed...)

Beyond Vlasov equation

Particles: should approach statistical equilibrium when $N \to \infty$.
Questions: How to describe this approach to equilibrium? On which timescale?
Vlasov equation = mean field dynamics; particles dynamics = mean-field + fluctuations
Formal analysis of these fluctuations $\to$ Balescu-Lenard equation (plasma physics)

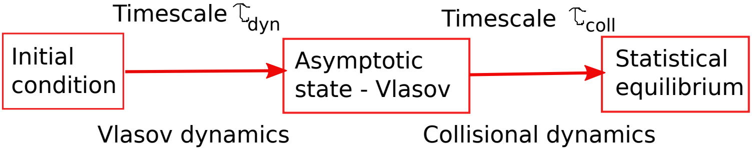
$$\partial_t f = \frac{C}{N} \int d^3\mathbf{k}\, \mathbf{k}\cdot\nabla_{\mathbf{v}} \int d\mathbf{v}' \frac{\tilde{V}^2(\mathbf{k})}{|\epsilon(\mathbf{k}, \mathbf{k}\cdot\mathbf{v})|^2} \delta(\mathbf{k}\cdot\mathbf{v} - \mathbf{k}\cdot\mathbf{v}')\mathbf{k}\cdot(f(\mathbf{v}')\nabla_{\mathbf{v}} f - f(\mathbf{v})\nabla_{\mathbf{v}'} f)$$

"Collisions" $\to$ approach to equilibrium on a long time scale
No mathematical proof: much more difficult than Vlasov, because it encodes the passage time reversible/ irreversible!

About Balescu-Lenard equation

- Timescale: $\sim N\tau_{\rm dyn}$; $\sim (N/\ln N)\tau_{\rm dyn}$ for 3D Coulomb or Newton cases.
  $\rightarrow$ a very important piece of information, to decide whether to describe a system with Vlasov equation, or equilibrium statistical mechanics.
- Basic physical mechanism: resonances $\rightarrow$ fluxes in velocity (or actions) space.
- For a homogeneous background: Balescu-Lenard equation well established.
  Non homogeneous backgrounds (crucial in astrophysics!): technical difficulties; subject of current research (Luciani-Pellat 1987, Heyvaerts, Pichon, Fouvry, Chavanis, Tremaine, Bennetti, Marcos...)
- Question: standard techniques rely on the integrability of the background potential; what can we say when it is not integrable?

Dynamical evolution, summary



1. Initial conditions (out of equilibrium)
2. Fast evolution, on Vlasov timescale $\rightarrow$ "Quasi-stationary state"
3. Slow "collisionnal" relaxation (Balescu-Lenard)
4. Statistical equilibrium

Kinetic theory - Non Hamiltonian systems

With a friction $-\gamma v$ and a noise $\eta(t)$:

$$\begin{aligned}\dot{x}_i &= v_i \\ \dot{v}_i &= \frac{1}{N}\sum_{j\neq i} K(x_i - x_j) - \gamma v_i + \sqrt{2D}\eta_i(t)\end{aligned}$$

"Kinetic" equation: Vlasov-Fokker-Planck

$$\partial_t f + v\cdot\nabla_x f - \left(K \star \int f dv\right)\cdot\nabla_v f = \nabla_v\cdot(\gamma v f + D\nabla_v f)$$

- Side remark: what is really f?

-Limit of the empirical density $\frac{1}{N}\sum_i \delta(x-x_i)\delta(v-v_i)$?
-Limit of the one-particle distribution function?
Same thing if the particles distribution is "chaotic", ie
$f^{(2)}(z_1,z_2)\to f(z_1)f(z_2)$

Kinetic theory - Non Hamiltonian systems

With a friction $-\gamma v$ and a noise $\eta(t)$:

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\dot{x}_i &= v_i \\
\dot{v}_i &= \frac{1}{N}\sum_{j\neq i} K(x_i - x_j) - \gamma v_i + \sqrt{2D}\eta_i(t)
\end{aligned}$$

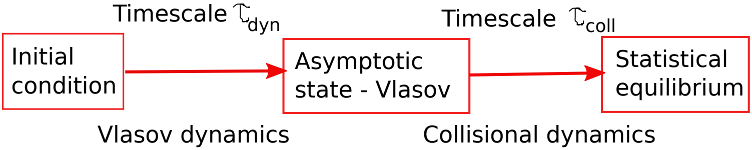
"Kinetic" equation: Vlasov-Fokker-Planck

$$\partial_t f + v\cdot\nabla_x f - \left(K \star \int f dv\right)\cdot\nabla_v f = \nabla_v\cdot(\gamma v f + D\nabla_v f)$$

• Mathematical status of VFP equation:
Empirical density $\hat{f}^N$ not a solution of VFP... but not far
$\rightarrow$ convergence to a solution of VFP when $N\rightarrow\infty$, under regularity hypotheses for the force again.

Dynamical evolution, summary (2)

Friction $\rightarrow$ new time scale.
$\rightarrow$ competition between dynamical τ_{dyn}, relaxation $\tau_{rel} \gg \tau dyn$ and friction τ_{fric} time scales.



Dynamical evolution, summary (2)

Friction $\rightarrow$ new time scale.
$\rightarrow$ competition between dynamical τ_{dyn}, relaxation $\tau_{rel} \gg \tau dyn$ and friction τ_{fric} time scales.
i) $\tau_{fric} \gg \tau_{rel}$: no change to the Hamiltonian phenomenology, until $t \sim \tau_{fric}$.
Possible physical example: some globular clusters
ii) $\tau_{dyn} \ll \tau_{fric} \ll \tau_{rel}$ Quasi-stationary state driven towards equilibrium (or other) by Fokker-Planck operator.
Possible physical example: galactic evolution? (external actions are much more complicated than friction $+$ noise though!))
iii) $\tau_{fric} \ll \tau_{dyn}$ Fokker-Planck operator hides Vlasov dynamics.
Possible physical example: a dynamical regime of Magneto-optical traps.

Beyond Vlasov-Fokker-Planck, 1

• Vlasov-Fokker-Planck equation $\simeq$ law of large numbers.
$\rightarrow$ finite N fluctuations?

• For simplicity, I will consider the Mac-Kean-Vlasov (overdamped) setting

$$\dot{x}_i = \frac{1}{N} \sum_{i=1}^{N} K(x_i - x_j) + \sqrt{2D}\eta_i(t)$$

Central object: empirical density

$$\hat{\rho}_N = \frac{1}{N} \sum_i \delta(x - x_i(t))$$

Law of large numbers: with high probability, $\hat{\rho}_N(t)$ is close to $\rho(t,x)$, solution of Mac-Kean-Vlasov equation

$$\partial_t \rho = \nabla \cdot (-(K \star \rho)\rho + D\nabla\rho)$$

Beyond Vlasov-Fokker-Planck, 2

Large deviations: what is the probability that $\hat{\rho}_N(t)$ is close to some ρ that is **not** solution of Mac-Kean-Vlasov equation?

$$\mathbb{P}\left(\hat{\rho}_N \approx \rho\right) \asymp e^{-N I_{[0,T]}[\rho]}\ , \ \text{with}$$

$$I_{[0,T]}[\rho] = \frac{1}{4D}\int_0^T \left[\inf_{j,\ \partial_t \rho + \nabla\cdot j = 0} \int \frac{[j - (K\star\rho)\rho + D\nabla\rho]^2}{\rho} dx\right] dt$$

Formal noisy PDE version:

$$\partial_t \hat{\rho}_N + \nabla\left(-D\nabla\hat{\rho}_N + (K\star\hat{\rho}_N)\hat{\rho}_N\right) = \nabla\left(\sqrt{\frac{\hat{\rho}_N}{N}}\eta(x,t)\right)$$

$\rightarrow$ we are ready for "macroscopic fluctuation theory" (Bertini et al.)

Conclusions

- This was a personal view on long-range interactions. There are probably many others.
- Guiding idea: common features due to long-range interactions Of course, there are many caveats when comparing systems as different as galaxies, colloids and cold atoms...
- Nevertheless: we all have a lot to share and to learn by mixing people from different fields with long range interactions, such as in this conference!

More specialized section

Perturbing a non homogeneous stationary state of the Vlasov equation

Co-authors: David Métivier (U. of Nice, France) and Yoshiyuki Yamaguchi (U. of Kyoto, Japan)

Question: Start close to a stationary state, stable, or weakly unstable. What can we say about the dynamics, using dynamical systems methods?

Context

Vlasov equation:

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} \Phi \cdot \nabla_{\mathbf{v}} f = 0 \ , \ \Phi = \int V(x-y) f(y,v) dy \ dv.$$

- Long-range interacting systems described by Vlasov equation over time scales that diverge with N
  $\rightarrow$ the asymptotic dynamics of Vlasov equation may be relevant for some particles systems
- Approach followed here: "dynamical systems"; ie: study stationary state, linear and non linear stability, weakly non linear dynamics...
- Weakly non linear dynamics close to a **homogeneous** stationary state $F_0(v)$: a long story, now relatively well understood.
  This work: non homogeneous $F_0(x,v)$.

An astrophysical motivation

Radial Orbit Instability: take a family of spherically symmetric stationary state of the gravitational Vlasov-Poisson equation, depending on a parameter α.
Few low angular momentum stars (large α) $\rightarrow$ stable
Many low angular momentum stars (small α) $\rightarrow$ unstable, real eigenvalue

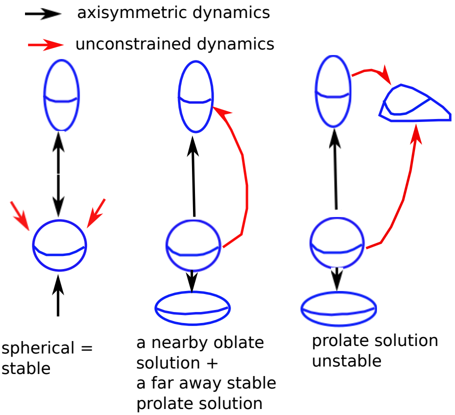
What happens when the instability develops? Supposed to play an important role in determining the shape of some galaxies.

Palmer et al. (1990): detailed numerics and approximate computations. Ex:

$$f(E, L) \propto \frac{1}{L^2 + \alpha^2}$$

An astrophysical motivation, 2

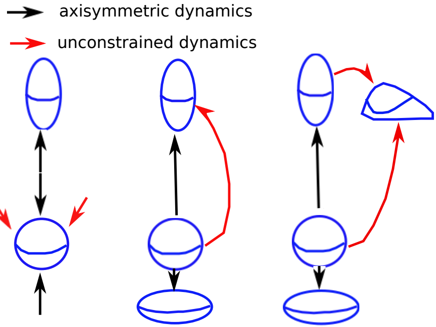
Scenario according to Palmer et al.:



How general is it? Can we quantify this (what does "nearby" means)?

An astrophysical motivation, 2

Scenario according to Palmer et al.:



spherical = stable

a nearby oblate solution + a far away stable prolate solution

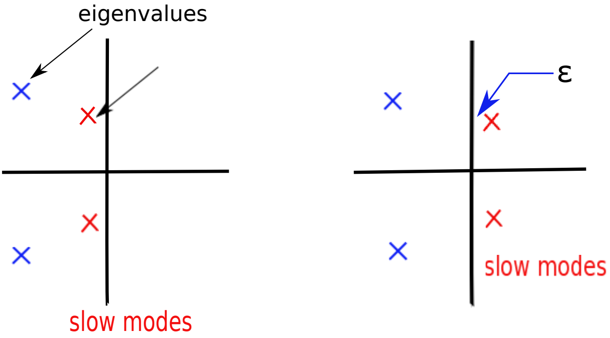
prolate solution unstable

Strategy: Use of asymptotic expansions (backed by numerical simulations), trying to control the errors $\rightarrow$ results currently limited to 1D

Bifurcations, standard case

• A family of stationary states.
Varying a parameter, stable → unstable.

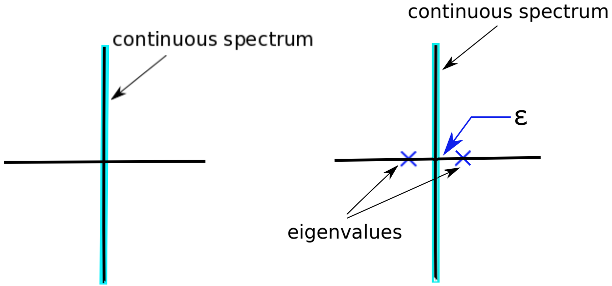
• **General strategy:** look at the linearized equation, identify the "slow modes", and taking advantage of the time-scale separation, find a reduced dynamics



→ a finite dimensional reduced dynamics

Bifurcations with continuous spectrum

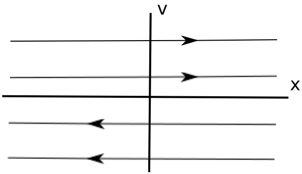
A typical bifurcation for a Vlasov equation:



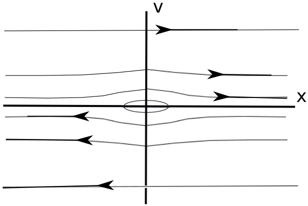
$\rightarrow$ no slow manifold!

Bifurcations with continuous spectrum (2)

Continuous spectrum $\leftrightarrow$ resonances between the growing perturbation and some particles



Reference state: free flowing particles



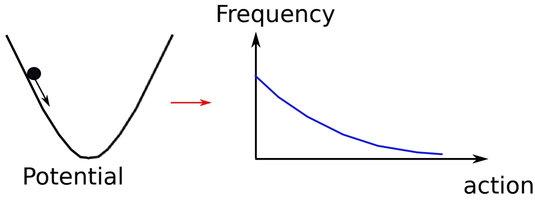
With a perturbation at zero frequency

Homogeneous background: old problem in plasma physics, extensive literature (Baldwin, O'Neil 60's . . . Crawford, Del Castillo Negrete 90's).

Messages: strong non linear effects, divergences in standard expansions; yet: there is an universal reduced dynamics.

Continuous spectrum, inhomogeneous case

Reference state: particles in a stationary potential.

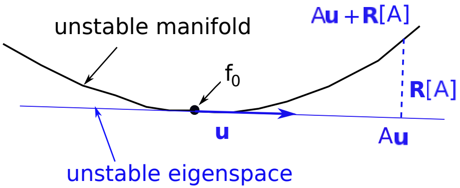


$\rightarrow$ weak or no resonance for frequency $\omega = 0$.
$\rightarrow$ differences with the plasma case expected.
• 3D gravitational Vlasov-Poisson: technical difficulties, even at linear level.
$\rightarrow$ use simpler 1D models, for which explicit computations can be carried out, and numerics is easy.

Hope: the weakly non linear dynamics may be "universal"

Outline of the computations: unstable manifold expansion

JD Crawford's idea (plasma): construct the *unstable manifold*



Expansion around the reference stationary state $f_0(x, v)$:

$$f(x, v, t) = f_0(x, v) + A(t)\mathbf{u}(x, v) + R[A](x, v, t)$$

Reduced dynamics ($\varepsilon =$ instability rate):

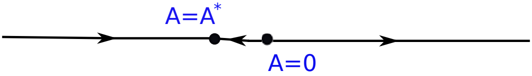
$$\dot{A} = \varepsilon A + C(\varepsilon)A^2 + \dots$$

with $C(\varepsilon) \sim c/\varepsilon$ *(lengthy computations here)*.

$\rightarrow 1/\varepsilon$ singularities appear! Origin = the double eigenvalue at the instability threshold; different from homogeneous case.

Result of the computations

$$\dot{A} = \varepsilon A + C(\varepsilon)A^2 + \ldots$$



Conclusions:

- There is an attractive (*on the unstable manifold*) *stationary state* $A^* \propto \varepsilon^2$
- *Asymmetry between the two directions on the unstable manifold: one direction goes to a "nearby stationary state", the other one goes far away, out of range for the present theory*
- *All this can be directly checked numerically. On a 1D model with a cosine potential (HMF model), it works nicely!*

Numerics

• Standard semi-lagrangian method; uses GPU (cf Rocha Filho 2013)

• Simple cosine potential, periodic box (so called HMF model) + 1 spatial dimension
$\rightarrow$ possible to reach good resolution (at least 1024x1024)

• Order of magnitude of the unstable eigenvalue $\varepsilon \simeq 0.05$
$\rightarrow$ confirms predictions, including the scaling $A(t \rightarrow \infty) \propto \varepsilon^2$

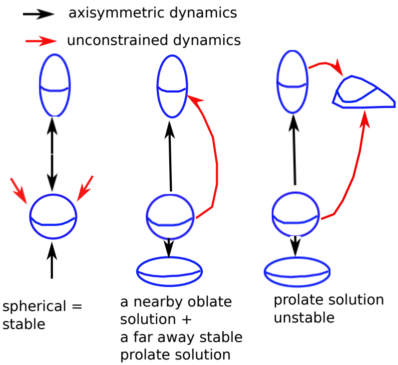
perturbation $+\varepsilon$

perturbation $-\varepsilon$

Back to Radial Orbit Instability

NB: Radial Orbit Instability associated with a real eigenvalue $\rightarrow$ consistent with the present theory

Some of the findings in Palmer et al. 1990 are recovered; new information gained; some of their predictions are inaccessible with our method.



Back to Radial Orbit Instability

NB: Radial Orbit Instability associated with a real eigenvalue $\rightarrow$ consistent with the present theory

Some of the findings in Palmer et al. 1990 are recovered; new information gained; some of their predictions are inaccessible with our method.

- Existence of a nearby stationary state, attractive at least for a restricted dynamics
- We have a prediction for the *distance* of this state from the reference stationary state
- The system can go far away from the original reference stationary state

Conclusions

- The truncated reduced dynamics on the unstable manifold provides a good qualitative description, even for initial conditions that are not on the unstable manifold. More numerical investigations are needed
- Higher dimensions: the structure of resonances is more complicated. $\rightarrow$ Universality of this scenario?
- Exploring the case of complex eigenvalues... Again resonances appear.