Introduction to long range interactions: a theoretical physicist’s view.

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Thanks: the very many people I have been working with on this subject!
Why this title?

A theoretical physicist’s view on Long Range Interactions (LRI):

- My main interest: LRI induces common features in very different physical systems
  → it suits the natural tendency of the theoretical physicist’s to look for ”universality”
  → scope of this conference

- Similarities between different LRI systems are typically expressed through common underlying mathematical structure
  → there will be some hints to mathematics

- I will try to keep emphasis on various physical systems. However: I do not claim to be competent in all the fields with LRI!
I. INTRODUCTION

1. On the definition

2. A lot of examples

3. Some basic remarks
On the definition of Long Range Interactions

One finds many definitions in the literature; usually criteria can be expressed through the 2-body interaction potential $V(r)$:

1. $V(r) \propto 1/r^\alpha$, with $\alpha < d = \text{dimension}$. Then energy is not additive (see later).
2. $V(r) \propto 1/r^{d+\sigma}$, $0 < \sigma < \sigma_c(d)$. The long-range character then modifies the critical exponents.
3. $V(r)$ falls off slower than exponentially. Correlations are then qualitatively different. E.g. : Van der Waals interactions.
4. One can propose the definition of long range on the nanoscale starting with “extending beyond a single bond”. R.H. French et al. , Long range interactions in nanoscale science (Rev. Mod. Phys. 2010).
One conclusion: "What constitutes a long range as opposed to short range interaction depends primarily on the specific problem under investigation." R.H. French et al.

I will concentrate on definition 1:

1. $V(r) \propto 1/r^\alpha$, with $\alpha < d =$dimension. Energy not additive.

However some ideas are relevant beyond these strong LRI.

**NB:** I have used the potential in the definition; one could think of using the force...
Some important examples

- **Fundamental interactions**
  - Newtonian gravity $V(r) \propto -\frac{1}{r}$: paradigmatic example.
    → galactic dynamics, globular clusters, cosmology...
    Clearly: controlled experiments difficult!
  - Coulomb interaction $V(r) \propto \frac{1}{r}$.
    - Non neutral plasma, systems of trapped charged particles: different experimental realizations.
    - Neutral plasmas: huge importance of course.

- **Effective interactions**
  - Vortex-vortex in 2D fluids: $H \propto \ln r$.
  - Wave-particles: the wave acts as a global degree of freedom interacting with all particles.
    E.g.: single wave model in plasma and fluid dynamics; free electron laser; cold atoms in cavity...
More examples

• colloids at interface + capillarity (A. Dominguez et al.)

![Diagram of colloids at fluid interface with capillarity](image)

Colloids (size $\sim \mu m$) trapped at a fluid interface, subjected to an external vertical force.

$\rightarrow$ an effective long range attraction (or repulsion, depending on the external force)

For $r \leq \lambda$ and not too small

$$V_{\text{eff}}(r) \propto \ln \frac{r}{\lambda} : \sim 2\text{D gravity!}$$

$\lambda =$ capillary length, $\sim mm$.

NB: Overdamped dynamics
More examples

- Chemotaxis
  \( \rho = \) concentration of bacteria;  \( c = \) concentration of a chemical substance (chemo-attractant).
  Bacterial dynamics:
  \[
  \partial_t \rho = D_1 \Delta \rho + \nabla (-\sigma \rho \nabla c) \quad : \text{drift up the gradient of } c
  \]

  Chemo-attractant dynamics:
  \[
  \partial_t c = D_2 \Delta c - \lambda c + \alpha \rho \quad : \text{bacteria = source for } c
  \]

  → again models similar to overdamped 2D gravity.
  Huge related activity in mathematical biology.
More examples

- Cold atoms in a magneto-optical trap: multiple diffusion of light

"Coulombian" effective force

\[ \vec{F}_i \propto \sum_j \frac{\vec{r}_i - \vec{r}_j}{|\vec{r}_i - \vec{r}_j|^3} \]

The $1/r^2$ dependence of the force comes from the solid angle in 3D.

This is an oversimplification; more or less a "standard model" (Sesko, Walker, Wieman 1990).
More examples

- Cold atoms in a magneto-optical trap: shadow effect

Laser intensities decrease while propagating into the cloud → effective force towards the center
Weak absorption approximation: $\nabla \cdot \mathbf{F}_{\text{Shadow}} \propto -\rho$ (Dalibard 1988)
→ Just like gravitation... but it does not derive from a potential!
More examples

- Self-organization in optical cavities (G. Morigi et al.)

Laser = far from atomic resonance → conservative system as a first approximation.

Integrate over cavity degrees of freedom → effective long-range interaction, of mean-field type, between atoms.
More examples

- Dipolar interactions in a Bose-Einstein condensate (O’Dell et al., 2000). BEC irradiated with intense off-resonant lasers → dipolar interactions between atoms

Size of the cloud \ll \lambda \ (laser’s wavelength) \rightarrow near field approximation.
Averaging over lasers → the \(1/r^3\) dominant term is suppressed. The remaining term is \(\propto 1/r\), attractive (possibly anisotropic). → gravity-like force

NB: quantum system; description by Gross-Pitaevskii equation. NB: extra oscillating terms due to interferences neglected here...
More examples

- Active particles and thermophoresis (R. Golestanian 2012)
  Colloidal particles with (partial) metal coating

Thermophoretic effect $\rightarrow$ move up (or down) the temperature gradient
Metal absorbs laser light $\rightarrow$ particles are ”temperature sources”
$\rightarrow$ Again, an overdamped ”gravity-like” dynamics
More examples

- Eigenvalues of random matrices. Eg. complex Ginibre ensemble. Each entry of $A$ (size $n \times n$) is $A_{kl} = X_{kl} + iY_{kl}$, $X$s and $Y$s are independent, law $\mathcal{N}(0, 1/2)$. The eigenvalues $z_k$ of $A$ have joint probability density:

$$P(Z_1, \ldots, z_n) \propto \prod_{k=1}^{n} e^{-|z_k|^2} \prod_{1 \leq k < l \leq n} |z_k - z_l|^2$$

$$\propto \exp \left[ - \left( \sum_{k} |z_k|^2 - 2 \sum_{k,l} \ln |z_k - z_l| \right) \right]$$

→ analogous to a 2D Coulomb gas confined in an harmonic trap!

- There are similar laws for other random matrix ensembles. Intense mathematical activity related to determinantal processes.
More examples

- Trapped free fermions, 1D harmonic trap
  Pauli exclusion principle →

  \[ |\psi_0(x_1, \ldots, x_n)|^2 \propto e^{-\alpha^2 \sum_k x_k^2 \prod_{k<l}|x_k - x_l|^2} \]

  where \( \psi_0 \) = ground state wave function.

- Stellar dynamics around a massive black hole (Tremaine, Sridhar and Touma...)
  - Short time scales = Keplerian dynamics of stars around the black hole
  - Longer time scales: interaction between stars (+relativistic corrections+...)
  Averaging over short time scales → an effective system of ”interacting orbits”
Toy models

- Underlying idea: long range interactions have similar effects in different systems, leading to some "universal" properties → it makes sense to use toy models to illustrate, or study in details these properties more easily...

- Indeed has been used for a long-time: see Thirring’s models to illustrate peculiarities of equilibrium statistical mechanics.

- THE toy model: HMF (= mean-field XY model + kinetic term)

\[
H = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \frac{K}{N} \sum_{i=1}^{N} [1 - \cos(\theta_i - \theta_j)]
\]

- Very useful; of course it goes with all the caveats regarding toy models...
Some caveats

• I am of course ignorant in most of these fields!
→ Please react if I am not as accurate as I should when I say a few words about some of them
• Concentrate on classical physics; apologies to quantum physicists.
Some concepts might still be relevant for quantum systems.
Some basic remarks

Specific difficulties of LRI: one particle interacts with many others...

▶ impossible to cut the system in almost independent pieces.
  Related difficulty: no distinction bulk/boundary

▶ numerical problem: with a naive algorithm, each type step costs $\propto N^2$

Also specific advantages:
One particle interacts with many others
→ fluctuations suppressed; law of large numbers, Central Limit
Theorem, large deviations...
Related idea: ”mean-field” should be a very good approximation
II. EQUILIBRIUM STATISTICAL MECHANICS

1. On scaling, extensivity, (non) additivity

2. On the mean field approximation

3. Examples and discussions
Equilibrium statistical mechanics

\( N \) long-range interacting particles or \( N \) spins on a lattice:

\[
H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i \neq j} V(x_i - x_j) \text{ or } H = -J \sum_{i \neq j} \frac{S_i S_j}{|i - j|^{\alpha}}
\]

Microcanonical equilibrium, fixed energy \( E \):

\[
d\mu_M (\{x_i, p_i\}_{i=1,\ldots,N}) \propto \delta[E - H(\{x_i, p_i\})] \Pi_i dx_i dp_i
\]

Canonical equilibrium, fixed inverse temperature \( \beta = 1/T \):

\[
d\mu_C (\{x_i, p_i\}_{i=1,\ldots,N}) \propto \exp[-\beta H(\{x_i, p_i\})] \Pi_i dx_i dp_i
\]
Special features of LRI, scaling

- In the usual ”Thermodynamic limit” $N \to \infty$, fixed density $\to$ potential energy $\gg N$.

  Where whereas entropy $\propto N$.

- Potential energy always wins, at any $T > 0$, the system is in the ground state (possibly singular) when $N \to \infty$.

- True, but not very interesting. Rather than looking for any large $N$ limit, we should look for something independent of $N$ in the large $N$ limit.

  Compare

  $$\lim_{N \to \infty} u(N) = 0 \quad \text{and} \quad \lim_{N \to \infty} Nu(N) = c.$$
Special features of LRI, scaling

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- Potential energy always wins, at any $T > 0$, the system is in the ground state (possibly singular) when $N \to \infty$!
- True, but not very interesting. Rather than looking for any large $N$ limit, we should look for something independent of $N$ in the large $N$ limit.

Other example:

$$u(a, N) = \frac{a}{N} + \frac{1}{N^2} \to \text{scalings i) } a \text{ fixed, or ii) } \tilde{a} = Na \text{ fixed}$$
Special features of LRI, scaling

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  Whereas entropy $\propto N$.
- Potential energy always wins, at any $T > 0$, the system is in the ground state (possibly singular) when $N \to \infty$!
- True, but not very interesting. Rather than looking for any large $N$ limit, we should look for something independent of $N$ in the large $N$ limit.
  We will see several examples in the following.
Scaling, examples

- **Example**: scaling for spin systems (1D, $0 \leq \alpha < 1$)

\[
H = \frac{1}{2\tilde{N}_\alpha} \sum_{i \neq j} \frac{-S_i S_j}{|i - j|^\alpha} \quad \text{with} \quad \tilde{N}_\alpha \propto N^{1-\alpha}
\]

Or, equivalently, scale the temperature...

- **Example**: scaling for self gravitating systems.

Microcanonical: $V^{1/3}E/GM^2$ fixed ($V$ = volume, $M$ = total mass); there are several ways to enforce this scaling.

The short range singularity should be regularized...

- **Example**: neutral plasmas.

Same short-range singularity. Once it is regularized, there is a well-defined thermodynamic limit! (Lebowitz, Lieb, Narnhofer)
**Extensivity:** energy proportional to $N$, or to the volume $V$. Does not make much sense without a specified scaling. Choosing a scaling may restore extensivity (good to compare with entropy).

**Non additivity:**

$$E_{\text{tot}} \neq E_1 + E_2$$

→ no phase separation possible in the usual sense
Special features of LRI: about extensivity, additivity

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\[
E_{\text{tot}} \neq E_1 + E_2
\]

→ no phase separation possible in the usual sense

And: phase separation $\Rightarrow$ entropy concave

\[
e = xe_1 + (1 - x)e_2 \Rightarrow S(e) \geq xS(e_1) + (1 - x)S(e_2)
\]
Special features of LRI: about extensivity, additivity

**Extensivity:** energy proportional to $N$, or to the volume $V$. Does not make much sense without a specified scaling. Choosing a scaling may restore extensivity (good to compare with entropy).

**Non additivity:**

\[ E_{\text{tot}} \neq E_1 + E_2 \]

→ no phase separation possible in the usual sense
Furthermore: free energy = Legendre transform of entropy; this operation is invertible only if the entropy is concave...
Special features of LRI: about extensivity, additivity

**Extensivity:** energy proportional to $N$, or to the volume $V$. Does not make much sense without a specified scaling. Choosing a scaling may restore extensivity (good to compare with entropy).

**Non additivity:**

$S_1 + S_2 \neq E_{tot} \neq E_1 + E_2$

→ no phase separation possible in the usual sense
→ no reason for equivalence between canonical and microcanonical ensembles
cf Hugo Touchette’s talk.
Special features of LRI: about mean field approximation

• One particle interacts with many others
  → a mean field description should be very good, fluctuations small
Correct intuition: in a well chosen scaling limit, a mean-field theory often becomes exact.
  → for instance, always classical critical exponents

• A perfectly suited mathematical tool: large deviation theory.

• Caveats:
  - strong fluctuations close to second order phase transitions
  - sometimes a short-range singularity together with the long-range character (eg: gravitation)
  - short range interactions can bring additional correlations
  - more than one scaling may be relevant (see the non neutral plasma case).
Equilibrium statistical mechanics, examples (1)

- Self-gravitating systems, chief example. Regularities in the structures of galaxies → natural to think of a statistical physics argument (I am being naive here, see later!).
  - Difficulties with both the absence of confinement and the short range singularity.
  - Main features (microcanonical): beyond a certain central density, no equilibrium state any more, even metastable → ”gravothermal catastrophe”.
  - Beautiful theory, but seems difficult to find clear situations where it is applicable; I don’t know everything here! Some explanations later.

- Self-gravitating systems: models of interacting orbits (Tremaine, Sridhar, Touma...); may be a nice application of equilibrium statistical mechanics?
Equilibrium statistical mechanics, examples (2)

- Vortices (first study by Onsager); $x_i \in \mathbb{R}^2$

$$H^N = -\frac{1}{2\pi} \sum_{i<j} \ln |x_i - x_j|$$

Qualitatively very useful predictions: it may be statistically favorable to form large scale structures!

- Another effective model: wave + particles description of a plasma (Escande, Elskens, Firpo...)
  - Plasma + Langmuir wave $\sim$ non resonant bulk + resonant particles
    $\rightarrow$ effective description: wave + resonant particles
  - Classical question: when does the wave damps completely?
    - Elskens-Firpo: a statistical mechanics answer. Not sure it is quantitatively accurate...
Equilibrium statistical mechanics, examples (3)

- Non neutral plasmas (in Penning traps for instance). Simplified version:

\[ H^N = \frac{1}{2} \sum_{i \neq j} \frac{1}{|x_i - x_j|} + \frac{1}{2} \sum_i x_i^2 \]

Balance between trap and interaction $\rightarrow$ typical size $R \propto N^{1/3}$.

*Ground state at mean-field level* = uniformly charged sphere, radius $R$.

*Absolute ground state* = ordered configuration.

Leading order: mean density

Next order: local correlations
• Non neutral plasmas (in Penning traps for instance). Simplified version:

\[ H^N = \frac{1}{2} \sum_{i \neq j} \frac{1}{|x_i - x_j|} + \frac{1}{2} \sum_i x_i^2 \]

Balance between trap and interaction \( \rightarrow \) typical size \( R \propto N^{1/3} \).

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**First scaling:** \( \beta N^2 / L \) fixed

\( \rightarrow \) Describes the cloud’s shape, cross-over from gaussian to mean-field ground state; no phase transition.
Equilibrium statistical mechanics, examples (3)

- Non neutral plasmas (in Penning traps for instance). Simplified version:
  \[
  H^N = \frac{1}{2} \sum_{i \neq j} \frac{1}{|x_i - x_j|} + \frac{1}{2} \sum_i x_i^2
  \]

  Balance between trap and interaction → typical size \( R \propto N^{1/3} \).

  *Ground state at mean-field level* = uniformly charged sphere, radius \( R \).

  *Absolute ground state* = ordered configuration.

  **Second scaling:** \( \beta \) fixed.

  → The system is in its ground state at mean-field level; \( \beta \) controls the non trivial local correlations; phase transition possible.
Equilibrium statistical mechanics, examples (3)

• Non neutral plasmas (in Penning traps for instance). Simplified version:

$$H^N = \frac{1}{2} \sum_{i \neq j} \frac{1}{|x_i - x_j|} + \frac{1}{2} \sum_i x_i^2$$

Balance between trap and interaction $\rightarrow$ typical size $R \propto N^{1/3}$.

*Ground state at mean-field level* = uniformly charged sphere, radius $R$.

*Absolute ground state* = ordered configuration.

$\rightarrow$ Example where two different scalings are interesting!

Side note: understanding this type of ”absolute ground state” -and the phase transition- is a long-standing mathematical problem.
Some conclusions

▶ Many universal features related to the long range character of the interactions (see also Hugo Touchette’s talk): non additivity, inequivalence between statistical ensembles (→ peculiar phase transitions), negative specific heat,...

▶ Beautiful theory, but... \textit{(my opinion)} there are not that many experimentally meaningful applications of equilibrium statistical mechanics with long-range interactions.

▶ There is a good reason for this: very slow relaxation times! → kinetic theory
III. KINETIC THEORY

1. Hamiltonian case
   - Collisionless equations and their properties
   - Secular evolution and collisional equations

2. Non Hamiltonian case
Kinetic theory, Hamiltonian case

- Boltzmann picture (short range interaction): rare collisions that have a strong impact
  \[ f(x, v, t) = \text{one-point distribution function} \]

  \[ \partial_t f + v \cdot \nabla_x f = C(f, f) \]

  This is again obtained in a specific scaling when \( N \to \infty \): Boltzmann-Grad scaling.

- Long-range interactions: "collisions" not rare! Instead: law of large numbers \( \to \) a dynamical mean field equation, in a well chosen scaling limit.
Examples of collisionless kinetic equations

- Point charged particles $\rightarrow$ Vlasov-Poisson equation
  \[
  \partial_t f + v \cdot \nabla_x f - \nabla_x \Phi \nabla_v f = 0, \text{ with } \Delta \Phi = 1 - \rho
  \]

- Point masses $\rightarrow$ Vlasov-Newton (collisionless Boltzmann)
  \[
  \partial_t f + v \cdot \nabla_x f - \nabla_x \Phi \nabla_v f = 0, \text{ with } \Delta \Phi = \rho
  \]

- Point vortices $\rightarrow$ 2D Euler equation
  \[
  \partial_t \omega + (\bar{u} \cdot \nabla) \omega = 0, \text{ with } \omega = -\Delta \psi, \; \bar{u} = -\nabla^\perp \psi.
  \]

- Particles + wave $\rightarrow$ Vlasov + wave
Examples of collisionless kinetic equations

- Point charged particles $\rightarrow$ Vlasov-Poisson equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \Phi \nabla_v f = 0, \text{ with } \Delta \Phi = 1 - \rho$$

- Point masses $\rightarrow$ Vlasov-Newton (collisionless Boltzmann)

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \Phi \nabla_v f = 0, \text{ with } \Delta \Phi = \rho$$

- Point vortices $\rightarrow$ 2D Euler equation

$$\partial_t \omega + (\vec{u} \cdot \nabla) \omega = 0, \text{ with } \omega = -\Delta \Psi , \vec{u} = -\nabla_\perp \Psi.$$  

- Other example: light propagation in a non linear non local medium $\rightarrow$ a Vlasov regime starting from Non Linear Schrödinger (Picozzi et al.)!

**Conclusion:** These different collisionless kinetic equations have similar properties $\rightarrow$ another striking example of universality induced by LRI.
On the mathematical status of these equations, 1

- Formal derivation easy: "mean-field approximation"; + hints that mean-field should be "good", and in fact one would like to say something like "Vlasov equation becomes exact in the \( N \to \infty \) limit". Is it true, and in which sense? Starting point:

\[
\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= \frac{1}{N} \sum_{j \neq i} K(x_i - x_j)
\end{align*}
\]

Central quantity: empirical density \( \hat{f}^N \)

\[
\hat{f}^N(x, v, t) = \frac{1}{N} \sum_i \delta(x - x_i(t))\delta(v - v_i(t))
\]

Can we say that \( \hat{f}^N(x, v, t) \) is close to \( f(x, v, t) \), solution of the Vlasov equation with initial condition \( f(x, v, t = 0) \) close to \( \hat{f}^N(x, v, t = 0) \)?
Key observations:
i) $\hat{f}^N$ is itself a solution of Vlasov equation

ii) Take $f_1$ and $f_2$ two solutions of Vlasov equation, then for some constant $C$ ($C$ depends on the interaction force $K$), and some well chosen distance $d$

$$d(f_1(t), f_2(t)) \leq d(f_1(t = 0), f_2(t = 0)) e^{Ct}$$

$\rightarrow$ a theorem for regular interactions (Neunzert, Dobrushin, Braun and Hepp 70’s)
A theorem

Main hypothesis: $K$ and its derivative are assumed bounded.
Then: Take a sequence of initial condition for the $N$ particles that tends to $f_0$ when $N \to \infty$, any fixed time $T$ and any $\varepsilon > 0$. Call $f(t)$ the solution of Vlasov equation with initial condition $f_0$. Then for any $N > N_c(T, \varepsilon)$, and any time $t \leq T$

$$d(\hat{f}^N(t), f(t)) \leq \varepsilon$$

Remarks:

i) No average needed: take any initial condition close to $f_0$, the empirical density follows closely Vlasov equation, for any realization.

ii) Vlasov dynamics OK for large $N$ for a fixed time horizon $\to$ the asymptotic behavior of the particles’ dynamics may not be given by making $t \to \infty$ in the Vlasov dynamics!

iii) From the proof, it appears that $N_c$ may increase very fast with $T$...
On singular interactions

Many interesting interactions are actually singular... → a mathematical problem, and also a numerical one for people trying to approximate Vlasov equation with particles. Some contributions:

- From point vortices to 2D Euler (Goodman et al.): logarithmic singularity still ”acceptable”
- Singular forces with $K(x) \sim 1/|x|^\alpha$, $\alpha < 1$ (averaging techniques, Hauray and Jabin); Coulomb not included!
- Kiessling: a kind of ”if theorem” for the Coulomb case. If some quantity is bounded uniformly in $N$, then...
- Pickl, Boers, Lazarovici (2015): up to the Coulomb case (with small $N$ dependent cut-off), making use of ”probabilistic” degrees of freedom.
Qualitative features

Transport $\rightarrow$ phase space filamentation (phase mixing). Example with periodic boundary conditions:
Qualitative features

Example with a non trivial potential
Qualitative features

Example of 2D Euler evolution (perturbation of a shear flow, simulation H. Morita).

Vlasov-Poisson equation is a world in itself; it is of course crucial for plasma physics. I will discuss some generic properties of Vlasov or related equations.
Some properties of Vlasov-like equations

\[ \partial_t f + v \cdot \nabla_x f - \nabla_x \left( \int \int V(x - y)f(y, v, t)dydv \right) \cdot \nabla_v f = 0 \]

- Inherited from the particles: conservation of energy, momentum...
- Many more conserved quantities (Casimirs)

\[ \frac{d}{dt} \int C(f)dxdv = 0, \text{ for any function } C. \]

Not directly inherited from conserved quantities for the particles.

- In particular, the volume of each level set of \( f \) is conserved → Vlasov dynamics = mixing of these level sets, involves finer and finer scales.
On stationary solutions

- Many stationary solutions; statistical equilibrium = only one of these. Ex.: $f(v)$, homogeneous in space, constant potential → stationary for any $f$.

- Constructing stationary solutions from conserved quantities: critical points of conserved quantities are stationary! → look for extrema of

$$\iint C(f) + \beta H[f] + \alpha \iint f$$

May be a useful point of view to investigate stability.

- Clearly: No approach to statistical equilibrium.

- → Important question: what is the asymptotic behavior of a Vlasov-like equation? Difficult problem...
Asymptotic behavior of Vlasov equation

• Linearize around stationary solution. 
\( \lambda \) eigenvalue \( \rightarrow -\lambda, \lambda^*, -\lambda^* \) also eigenvalues... \( \rightarrow \) no asymptotic stability in the usual sense.

• Yet, for stable stationary states, a kind of exponential stability: **Landau damping**.
  
  ▶ discovered in plasma physics (1946)
  ▶ now a fundamental concept in galactic dynamics
  ▶ related to the inviscid damping in 2D fluids (known before Landau)
  ▶ + many other instances, including non Hamiltonian ones (synchronization models, bubbly fluids...)

\( \rightarrow \) again, a universal concept.
Question: Take an initial condition $f(t = 0)$; what can we say about $f(t \to \infty)$?
An old question in physics; recently a hot mathematical topic.

i) **Dynamical system approach:** perturbation theory, builds on linear theory. Ideas from non linear dynamical systems. Drawback: validity a priori limited to neighborhoods of stationary states.

ii) **Stat. mech. approach:** an equilibrium statistical mechanics that would take into account the dynamical constraints of Vlasov equation (pioneered by Lynden-Bell in astrophysics).

iii) **Other ideas:** mix the previous ones; try to take into account as much dynamics as possible.
Dynamical system approach, 1

- Non linear stability (starting with Antonov): uses a variational approach, stationary states seen as critical points of a conserved functional.

Typical result: criteria for stability (if \( f(t = 0) \) is close to some \( f_{\text{stat}} \), then \( f(t) \) remains close to \( f_{\text{stat}} \))

Example: take a stationary solution of Vlasov-Newton equation, of the form

\[
f = F_0(E) = \varphi \frac{v^2}{2} + \phi(x), \text{ with } \Delta \Phi(x) = 4\pi G \int f dv \text{ and } F_0' < 0;
\]

then \( f \) is stable.

**NB**: no precise information on the dynamics, filamentation process (and Landau damping) overlooked; mathematically: involves norms without derivative.
• Non linear Landau damping: Landau damping = comes from the linearized Vlasov equation. Example, close to a homogeneous stationary state $f_0(v)$, write $f = f_0 + \delta f$

\[
\partial_t \delta f + v \partial_x \delta f - \partial_x \left( \int V(x - y) \delta f(y, v') dv' \right) f'_0(v) = \\
\partial_x \left( \int V(x - y) \delta f(y, v') dv' \right) \partial_v \delta f
\]

Linearized Vlasov equation: should be OK for "small" $\delta f$. The non linear term becomes larger and larger because of filamentation $\rightarrow$ ??

Important remark: the mathematical meaning of "close" and "small" is crucial!
Mouhot-Villani theorem (2010): if the perturbation is small enough (in a very strong manner), the perturbed potential tends to 0 exponentially, with Landau rate. NB: $\delta f$ does not tend to 0.

Lin-Zheng (2011): if one measures the smallness of $\delta f$ in a less demanding way, there are undamped solutions arbitrarily close to $f_0$ (there is a precise regularity threshold).
Dynamical system approach, 4

- Stable stationary state, beyond Landau damping: when the perturbation exceeds a certain threshold, damping is incomplete; excitation of non linear solutions known as Bernstein-Greene-Kruskal modes (Manfredi, Lancellotti-Dorning).

Stable stationary state, beyond Landau damping: when the perturbation exceeds a certain threshold, damping is incomplete; excitation of non linear solutions known as Bernstein-Greene-Kruskal modes (Manfredi, Lancellotti-Dorning).

Weakly unstable stationary state: does the instability saturate, and how? An old question, which is actually a complicated bifurcation problem. For homogeneous stationary state, many contributions (O’Neil, Crawford, Del-Castillo-Negrete...)

One conclusion: a universal weakly non linear dynamics, governed by the ”Single Wave Model”.

Side remark: Yet, the ”Single Wave Model” is less universal than Landau damping... (eg: Kuramoto model).
→ Question: could one classify more precisely these bifurcations with continuous spectrum?
Stable stationary state, beyond Landau damping: when the perturbation exceeds a certain threshold, damping is incomplete; excitation of non linear solutions known as Bernstein-Greene-Kruskal modes (Manfredi, Lancellotti-Dorning).

Weakly unstable stationary state: does the instability saturate, and how? An old question, which is actually a complicated bifurcation problem. For homogeneous stationary state, many contributions (O’Neil, Crawford, Del-Castillo-Negrete...)

**One conclusion:** a universal weakly non linear dynamics, governed by the "Single Wave Model".

Non homogeneous stationary state: different physics, technical difficulties (PhD thesis of David Métivier, with Y. Yamaguchi).

Response theories (Ogawa-Yamaguchi, Patelli et al.)

→ a very rich problem, with still plenty to explore.
Statistical mechanics approach

Far from linear regime: out of reach for dynamical systems techniques.

**Another approach:** statistical mechanics.
Rationale: regularities in the structure of galaxies; it is natural to think of a statistical mechanics argument. Yet, we know that the equilibrium stat. mech. of the $N$ particles is irrelevant...

*Idea* (Lynden-Bell, 68): could one define an equilibrium for Vlasov dynamics?

*Basic ingredient:* Vlasov dynamics preserves all level volumes of $f$.

*Basic assumption:* we have to look for the "most disordered" state compatible with all constraints.

→ describe the state by a probability distribution on the levels at each point $(x, v)$, and maximize the entropy of this "field of pdf", under constraints.
Statistical mechanics approach

Far from linear regime: out of reach for dynamical systems techniques.

**Another approach**: statistical mechanics.

Rationale: regularities in the structure of galaxies; it is natural to think of a statistical mechanics argument. Yet, we know that the equilibrium stat. mech. of the $N$ particles is irrelevant...

**Idea** (Lynden-Bell, 68): could one define an equilibrium for Vlasov dynamics?

Some comments:
- A beautiful idea, which sometimes gives qualitatively useful predictions.
- The assumption of a maximum mixing is far from verified in general.
- A similar approach has been developed in 2D fluid dynamics.
**Statistical mechanics approach**

Far from linear regime: out of reach for dynamical systems techniques.

**Another approach:** statistical mechanics.

Rationale: regularities in the structure of galaxies; it is natural to think of a statistical mechanics argument. Yet, we know that the equilibrium stat. mech. of the $N$ particles is irrelevant...

*Idea* (Lynden-Bell, 68): could one define an equilibrium for Vlasov dynamics?

**Mixed approaches:** try to take into account as much dynamics as possible...

Relate initial conditions and final state by assuming a ”not too violent” transient (Ex: De Buyl et al., Pakter-Levin).

A parametric resonance during the transient dynamics (Levin, Pakter et al.) → a successful theory of core-halo structures (if not ”universal” feature, commonly observed...)
Beyond Vlasov equation

Particles: should approach statistical equilibrium when $N \to \infty$.

**Questions:** How to describe this approach to equilibrium? On which timescale?

Vlasov equation = mean field dynamics; particles dynamics = mean-field + fluctuations

Formal analysis of these fluctuations $\rightarrow$ Balescu-Lenard equation (plasma physics)

\[
\partial_t f = \frac{C}{N} \int d^3k \, k \cdot \nabla_v \int d^3v' \, \frac{\tilde{V}^2(k)}{|\epsilon(k, k \cdot v)|^2} \delta(k \cdot v - k \cdot v')k \cdot (f(v')\nabla_v f - f(v)\nabla_v f)
\]

"Collisions" $\rightarrow$ approach to equilibrium on a long time scale

No mathematical proof: much more difficult than Vlasov, because it encodes the passage time reversible/ irreversible!
About Balescu-Lenard equation

- Timescale: $\sim N\tau_{\text{dyn}}; \sim (N/\ln N)\tau_{\text{dyn}}$ for 3D Coulomb or Newton cases.
  - a very important piece of information, to decide whether to describe a system with Vlasov equation, or equilibrium statistical mechanics.
- Basic physical mechanism: resonances $\rightarrow$ fluxes in velocity (or actions) space.
- For a homogeneous background: Balescu-Lenard equation well established.
- Non homogeneous backgrounds (crucial in astrophysics!): technical difficulties; subject of current research (Luciani-Pellat 1987, Heyvaerts, Pichon, Fouvry, Chavanis, Tremaine, Bennetti, Marcos...)
- Question: standard techniques rely on the integrability of the background potential; what can we say when it is not integrable?
Dynamical evolution, summary

1. Initial conditions (out of equilibrium)
2. Fast evolution, on Vlasov timescale → "Quasi-stationary state"
3. Slow "collisionnal" relaxation (Balescu-Lenard)
4. Statistical equilibrium
Kinetic theory - Non Hamiltonian systems

With a friction $-\gamma \nu$ and a noise $\eta(t)$:

$$
\dot{x}_i = \nu_i \\
\dot{\nu}_i = \frac{1}{N} \sum_{j \neq i} K(x_i - x_j) - \gamma \nu_i + \sqrt{2D} \eta_i(t)
$$

"Kinetic" equation: Vlasov-Fokker-Planck

$$
\partial_t f + \nu \cdot \nabla_x f - (K \star \int fd\nu) \cdot \nabla_v f = \nabla_v \cdot (\gamma \nu f + D \nabla_v f)
$$

- Side remark: what is really $f$?
  - Limit of the empirical density $\frac{1}{N} \sum_i \delta(x - x_i)\delta(\nu - \nu_i)$?
  - Limit of the one-particle distribution function?

Same thing if the particles distribution is "chaotic", ie $f^{(2)}(z_1, z_2) \rightarrow f(z_1)f(z_2)$
Kinetic theory - Non Hamiltonian systems

With a friction $-\gamma v$ and a noise $\eta(t)$:

$$
\dot{x}_i = v_i \\
\dot{v}_i = \frac{1}{N} \sum_{j \neq i} K(x_i - x_j) - \gamma v_i + \sqrt{2D} \eta_i(t)
$$

"Kinetic" equation: Vlasov-Fokker-Planck

$$
\partial_t f + v \cdot \nabla_x f - (K * \int f dv) \cdot \nabla_v f = \nabla_v \cdot (\gamma vf + D \nabla_v f)
$$

- Mathematical status of VFP equation:
  Empirical density $\hat{f}^N$ not a solution of VFP... but not far
  $\rightarrow$ convergence to a solution of VFP when $N \rightarrow \infty$, under
  regularity hypotheses for the force again.
Dynamical evolution, summary (2)

Friction $\rightarrow$ new time scale.
$\rightarrow$ competition between dynamical $\tau_{dyn}$, relaxation $\tau_{rel} \gg \tau_{dyn}$ and friction $\tau_{fric}$ time scales.

![Diagram showing dynamical evolution process]

- Initial condition
- Timescale $\tau_{dyn}$
- Asymptotic state - Vlasov
- Vlasov dynamics
- Timescale $\tau_{coll}$
- Statistical equilibrium
- Collisional dynamics
Dynamical evolution, summary (2)

Friction $\rightarrow$ new time scale.
$\rightarrow$ competition between dynamical $\tau_{dyn}$, relaxation $\tau_{rel} \gg \tau_{dyn}$
and friction $\tau_{fric}$ time scales.

i) $\tau_{fric} \gg \tau_{rel}$: no change to the Hamiltonian phenomenology, until $t \sim \tau_{fric}$.
Possible physical example: some globular clusters

ii) $\tau_{dyn} \ll \tau_{fric} \ll \tau_{rel}$: Quasi-stationary state driven towards equilibrium (or other) by Fokker-Planck operator.
Possible physical example: galactic evolution? (external actions are much more complicated than friction + noise though!))

iii) $\tau_{fric} \ll \tau_{dyn}$: Fokker-Planck operator hides Vlasov dynamics.
Possible physical example: a dynamical regime of Magneto-optical traps.
Beyond Vlasov-Fokker-Planck, 1

- Vlasov-Fokker-Planck equation $\simeq$ law of large numbers.
  $\rightarrow$ finite $N$ fluctuations?
- For simplicity, I will consider the Mac-Kean-Vlasov (overdamped) setting

  $$\dot{x}_i = \frac{1}{N} \sum_{j=1}^{N} K(x_i - x_j) + \sqrt{2D}\eta_i(t)$$

  Central object: empirical density

  $$\hat{\rho}_N = \frac{1}{N} \sum_{i} \delta(x - x_i(t))$$

  Law of large numbers: with high probability, $\hat{\rho}_N(t)$ is close to $\rho(t, x)$, solution of Mac-Kean-Vlasov equation

  $$\partial_t \rho = \nabla \cdot (- (K * \rho) \rho + D \nabla \rho)$$
Large deviations: what is the probability that $\hat{\rho}_N(t)$ is close to some $\rho$ that is not solution of Mac-Kean-Vlasov equation?

$$\mathbb{P}(\hat{\rho}_N \approx \rho) \approx e^{-NI_{[0,\tau]}[\rho]}$$, with

$$I_{[0,\tau]}[\rho] = \frac{1}{4D} \int_0^T \left[ \inf_{j, \partial_t \rho + \nabla \cdot j = 0} \int \left[ \frac{j - (K \ast \rho)\rho + D\nabla \rho}{\rho} \right]^2 dx \right] dt$$

Formal noisy PDE version:

$$\partial_t \hat{\rho}_N + \nabla \left( -D\nabla \hat{\rho}_N + (K \ast \hat{\rho}_N)\hat{\rho}_N \right) = \nabla \left( \sqrt{\frac{\hat{\rho}_N}{N}} \eta(x, t) \right)$$

→ we are ready for ”macroscopic fluctuation theory” (Bertini et al.)
Conclusions

- This was a personal view on long-range interactions. There are probably many others.
- Guiding idea: common features due to long-range interactions
  Of course, there are many caveats when comparing systems as different as galaxies, colloids and cold atoms...
- Nevertheless: we all have a lot to share and to learn by mixing people from different fields with long range interactions, such as in this conference!
More specialized section

Perturbing a non homogeneous stationary state of the Vlasov equation

Co-authors: David Métivier (U. of Nice, France) and Yoshiyuki Yamaguchi (U. of Kyoto, Japan)

**Question:** Start close to a stationary state, stable, or weakly unstable. What can we say about the dynamics, using dynamical systems methods?
Vlasov equation:

\[ \partial_t f + \mathbf{v} \cdot \nabla_x f - \nabla_x \Phi \cdot \nabla_v f = 0 \ , \ \Phi = \int V(x - y)f(y, v)dy \ dv. \]

- Long-range interacting systems described by Vlasov equation over time scales that diverge with \( N \) → the asymptotic dynamics of Vlasov equation may be relevant for some particles systems
- Approach followed here: "dynamical systems"; ie: study stationary state, linear and non linear stability, weakly non linear dynamics...
- Weakly non linear dynamics close to a homogeneous stationary state \( F_0(v) \): a long story, now relatively well understood.
  This work: non homogeneous \( F_0(x, v) \).
An astrophysical motivation

**Radial Orbit Instability:** take a family of spherically symmetric stationary state of the gravitational Vlasov-Poisson equation, depending on a parameter $\alpha$.

Few low angular momentum stars ($\text{large } \alpha \rightarrow \text{stable}$)

Many low angular momentum stars ($\text{small } \alpha \rightarrow \text{unstable, real eigenvalue}$)

*What happens when the instability develops?* Supposed to play an important role in determining the shape of some galaxies.

Palmer et al. (1990): detailed numerics and approximate computations. Ex:

$$f(E, L) \propto \frac{1}{L^2 + \alpha^2}$$
An astrophysical motivation, 2

Scenario according to Palmer et al.:

- Axisymmetric dynamics
- Unconstrained dynamics

Spherical = stable

A nearby oblate solution + a far away stable prolate solution

Prolate solution unstable

How general is it? Can we quantify this (what does “nearby” means)?
An astrophysical motivation, 2

Scenario according to Palmer et al.:

- Axisymmetric dynamics
- Unconstrained dynamics

spherical = stable

a nearby oblate solution + a far away stable prolate solution

prolate solution unstable

**Strategy:** Use of asymptotic expansions (backed by numerical simulations), trying to control the errors → results currently limited to 1D
Bifurcations, standard case

- A family of stationary states. Varying a parameter, stable $\rightarrow$ unstable.
- **General strategy:** look at the linearized equation, identify the "slow modes", and taking advantage of the time-scale separation, find a reduced dynamics

→ a finite dimensional reduced dynamics
Bifurcations with continuous spectrum

A typical bifurcation for a Vlasov equation:

→ no slow manifold!
Bifurcations with continuous spectrum (2)

Continuous spectrum ↔ resonances between the growing perturbation and some particles

Reference state: free flowing particles

With a perturbation at zero frequency

Homogeneous background: old problem in plasma physics, extensive literature (Baldwin, O’Neil 60’s ... Crawford, Del Castillo Negrete 90’s).

Messages: strong non linear effects, divergences in standard expansions; yet: there is an universal reduced dynamics.
Continuous spectrum, inhomogeneous case

Reference state: particles in a stationary potential.

→ weak or no resonance for frequency $\omega = 0$.
→ differences with the plasma case expected.
• 3D gravitational Vlasov-Poisson: technical difficulties, even at linear level.
→ use simpler 1D models, for which explicit computations can be carried out, and numerics is easy.

**Hope:** the weakly non linear dynamics may be "universal"
Outline of the computations: unstable manifold expansion

**JD Crawford’s idea** (plasma): construct the *unstable manifold*

Expansion around the reference stationary state $f_0(x, v)$:

$$f(x, v, t) = f_0(x, v) + A(t)u(x, v) + R[A](x, v, t)$$

**Reduced dynamics** ($\varepsilon =$ instability rate):

$$\dot{A} = \varepsilon A + C(\varepsilon)A^2 + \ldots$$

with $C(\varepsilon) \sim c/\varepsilon$ (lengthy computations here).

$\rightarrow 1/\varepsilon$ singularities appear! Origin = the double eigenvalue at the instability threshold; different from homogeneous case.
Result of the computations

\[ \dot{A} = \varepsilon A + C(\varepsilon)A^2 + \ldots \]

Conclusions:

- There is an attractive (on the unstable manifold) stationary state \( A^* \propto \varepsilon^2 \)
- Asymmetry between the two directions on the unstable manifold: one direction goes to a ”nearby stationary state”, the other one goes far away, out of range for the present theory
- All this can be directly checked numerically. On a 1D model with a cosine potential (HMF model), it works nicely!
Numerics

- Standard semi-lagrangian method; uses GPU (cf Rocha Filho 2013)
- Simple cosine potential, periodic box (so called HMF model) + 1 spatial dimension
  → possible to reach good resolution (at least 1024x1024)
- Order of magnitude of the unstable eigenvalue $\varepsilon \simeq 0.05$
  → confirms predictions, including the scaling $A(t \to \infty) \propto \varepsilon^2$

perturbation $+\varepsilon$

perturbation $-\varepsilon$
Back to Radial Orbit Instability

**NB:** Radial Orbit Instability associated with a real eigenvalue → consistent with the present theory

Some of the findings in Palmer et al. 1990 are recovered; new information gained; some of their predictions are inaccessible with our method.
Back to Radial Orbit Instability

**NB:** Radial Orbit Instability associated with a real eigenvalue → consistent with the present theory

Some of the findings in Palmer et al. 1990 are recovered; new information gained; some of their predictions are inaccessible with our method.

- Existence of a nearby stationary state, attractive at least for a restricted dynamics
- We have a prediction for the *distance* of this state from the reference stationary state
- The system can go far away from the original reference stationary state
Conclusions

- The truncated reduced dynamics on the unstable manifold provides a good qualitative description, even for initial conditions that are not on the unstable manifold. More numerical investigations are needed.
- Higher dimensions: the structure of resonances is more complicated. → Universality of this scenario?
- Exploring the case of complex eigenvalues... Again resonances appear.