

Statistical Mechanics of Self-Gravitating Systems

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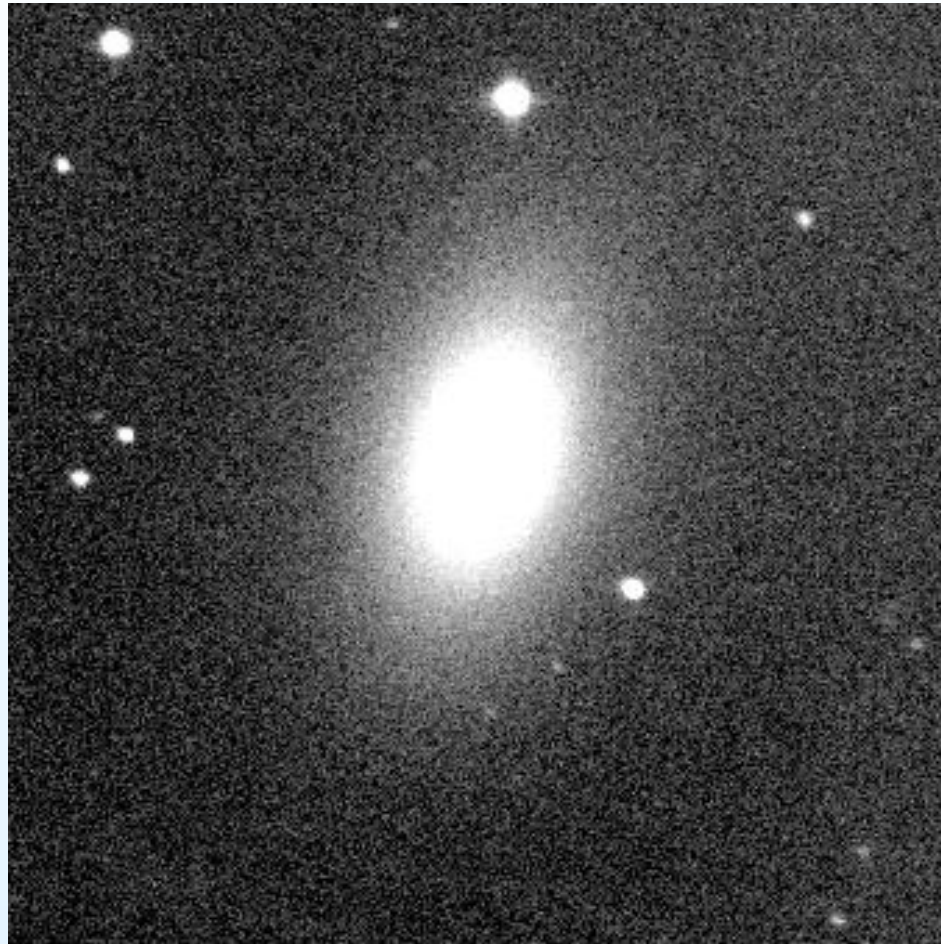
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Elliptical Galaxies



Levin et al. Phys. Rev. E 78, 021130 (2008); Mon. Not. R. As. Soc. 417, L21 (2011).

Vlasov Equation

$$\frac{Df}{Dt} = 0 \rightarrow \frac{\partial f}{\partial t} + \vec{v} \frac{\partial f}{\partial \vec{r}} - \nabla \psi \frac{\partial f}{\partial \vec{v}} = 0$$

Casimir invariants

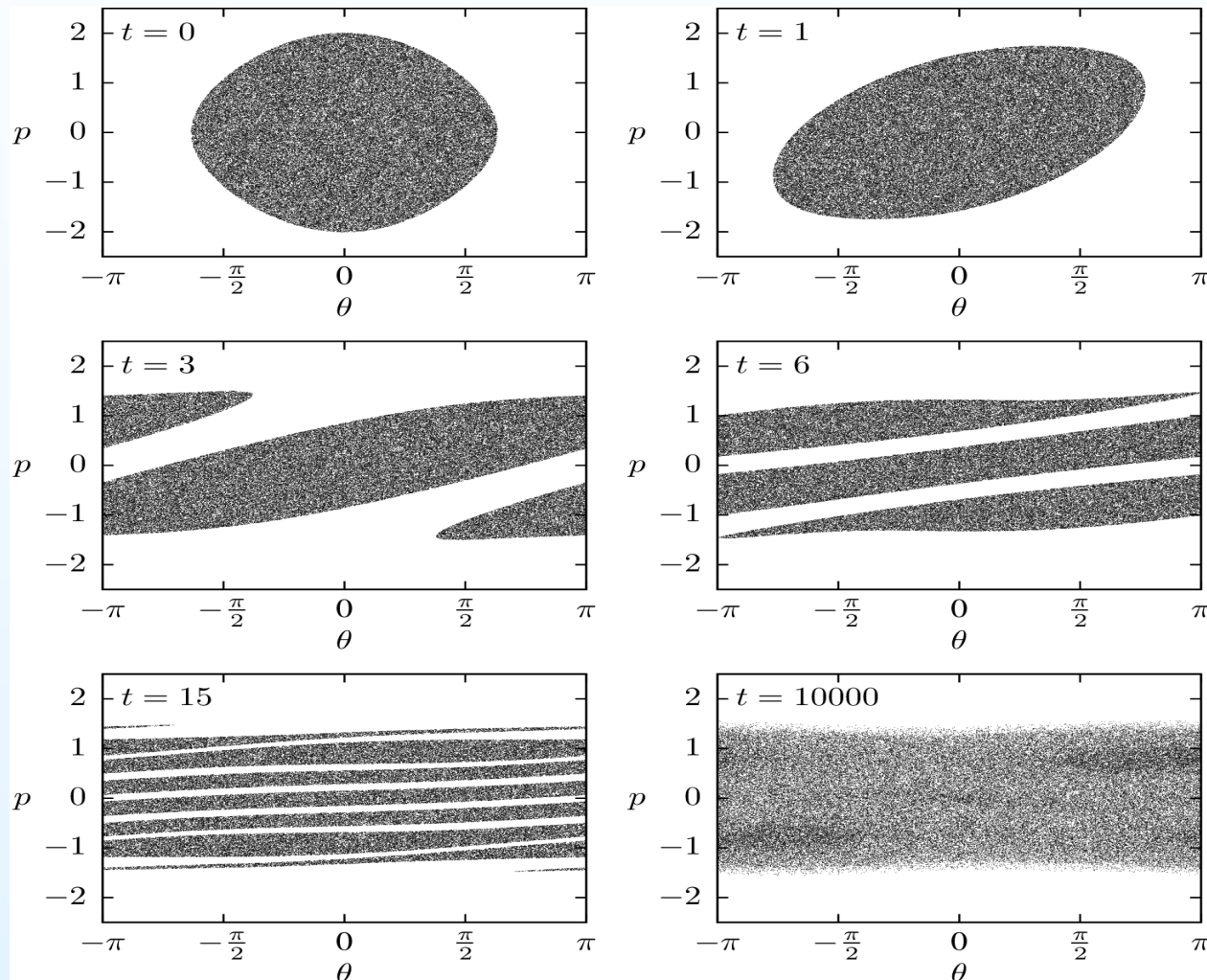
$$C_s[f] = \int s(f(\mathbf{r}, \mathbf{v}, t)) d^d \mathbf{r} d^d \mathbf{v}$$

$s(f)$ is an arbitrary function of f

$$s(f) = f, s(f) = f^N, s(f) = f \ln f, \dots$$

The Boltzmann entropy is constant!

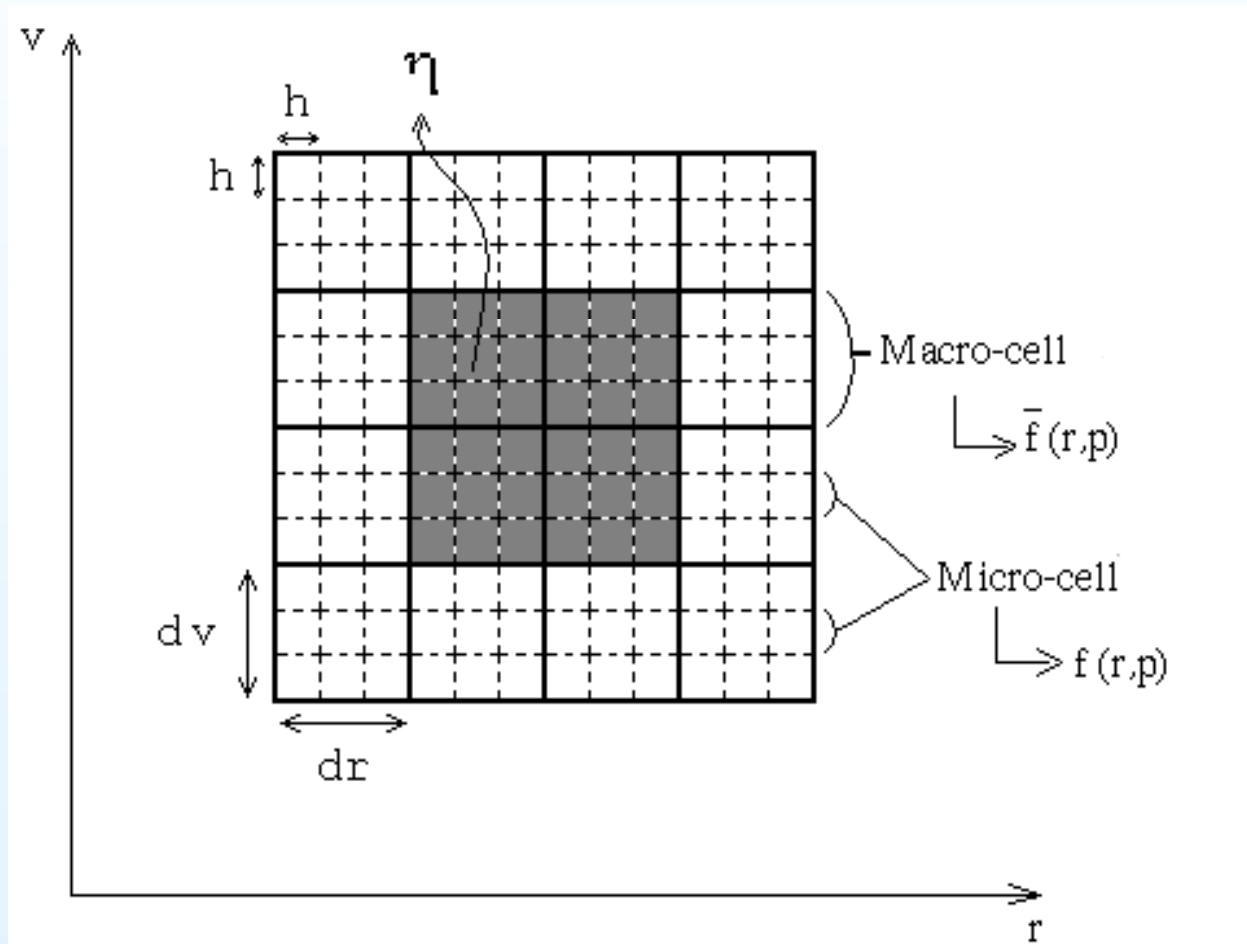
The Phase Space



*Levin et al., Physics Reports 535 1 (2014)

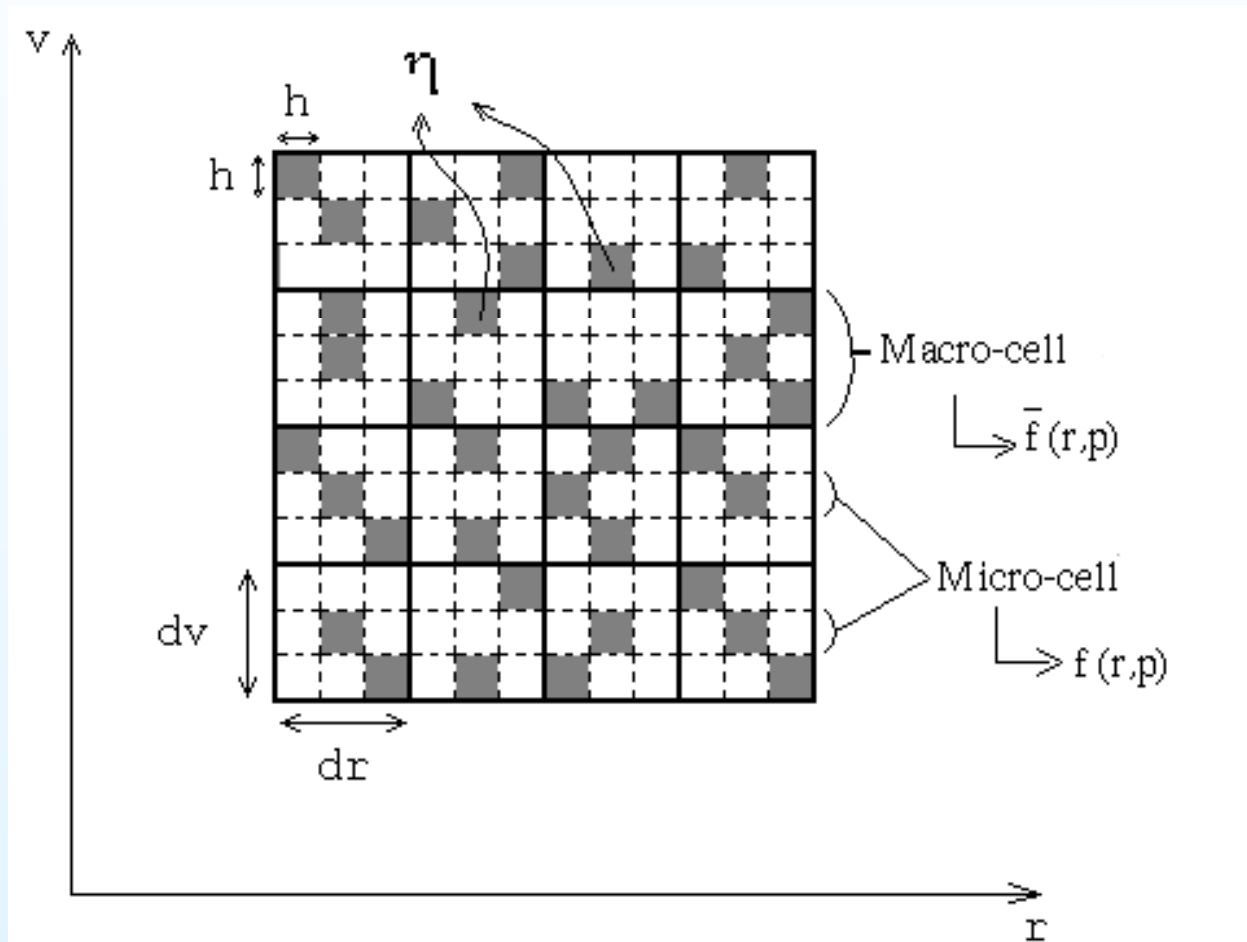
The Model

- Initial Distribution: $f_0(\mathbf{r}, \mathbf{v}) = \eta \Theta(r_m - r) \Theta(v_m - v)$



Incompressible Flow

- Final Distribution $f(\vec{r}, \vec{v}) = ?$

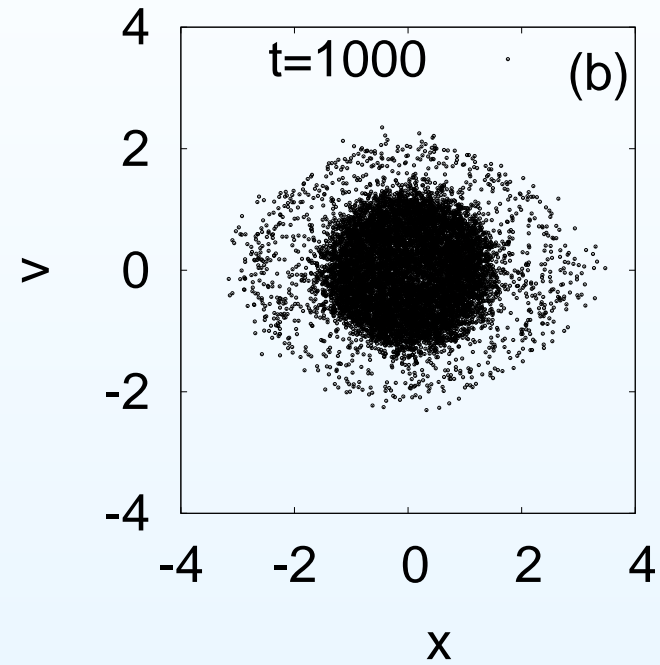
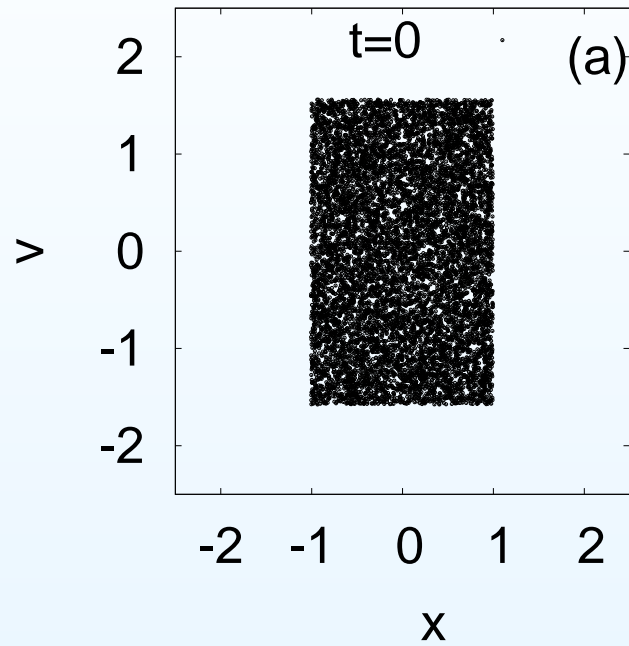


The Theory

The coarse-grained particle distribution must satisfy:

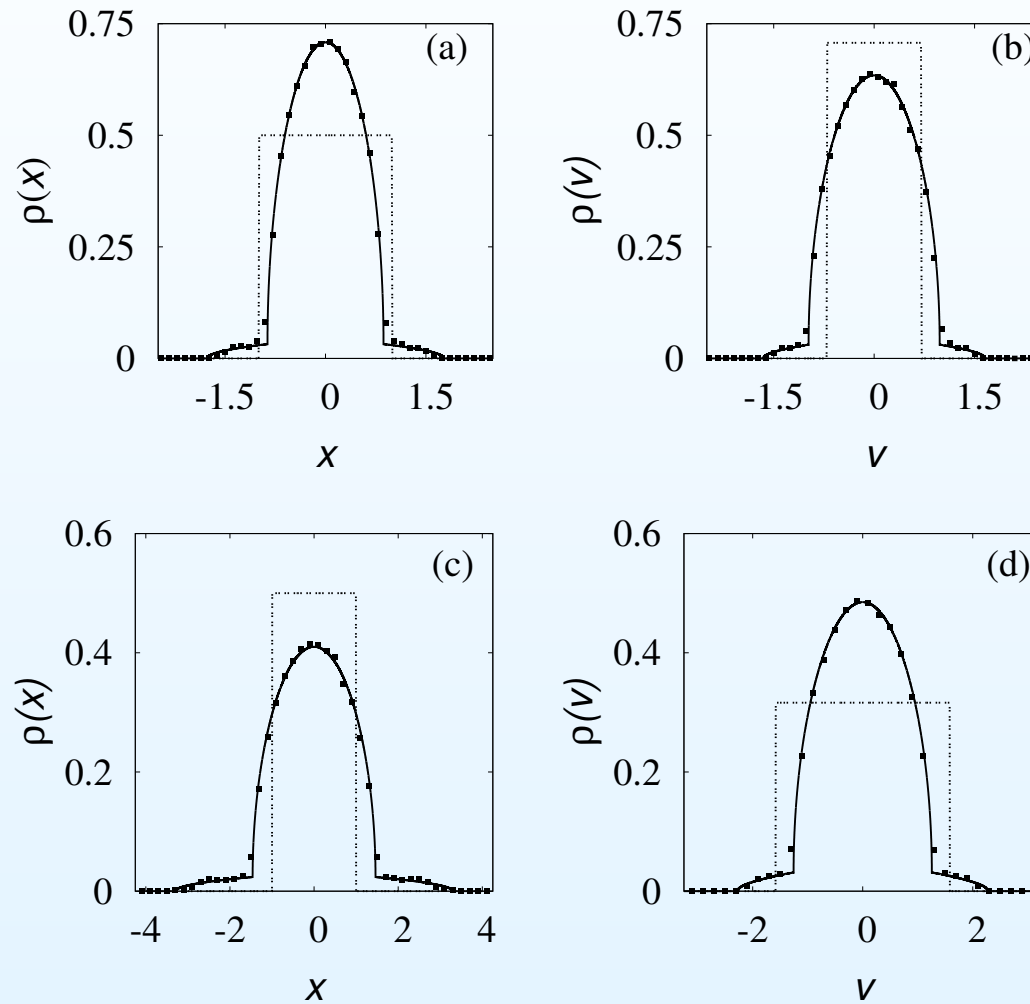
$$f(\vec{r}, \vec{v}) \leq \eta$$

1d Gravity



*Teles et al. MNRAS 417, L21 (2011)

Density and Velocity Distributions in 1d



Relaxation of 2d Gravitational Systems

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \frac{\nabla \psi}{m} \cdot \nabla_{\mathbf{v}} f = 0.$$

and

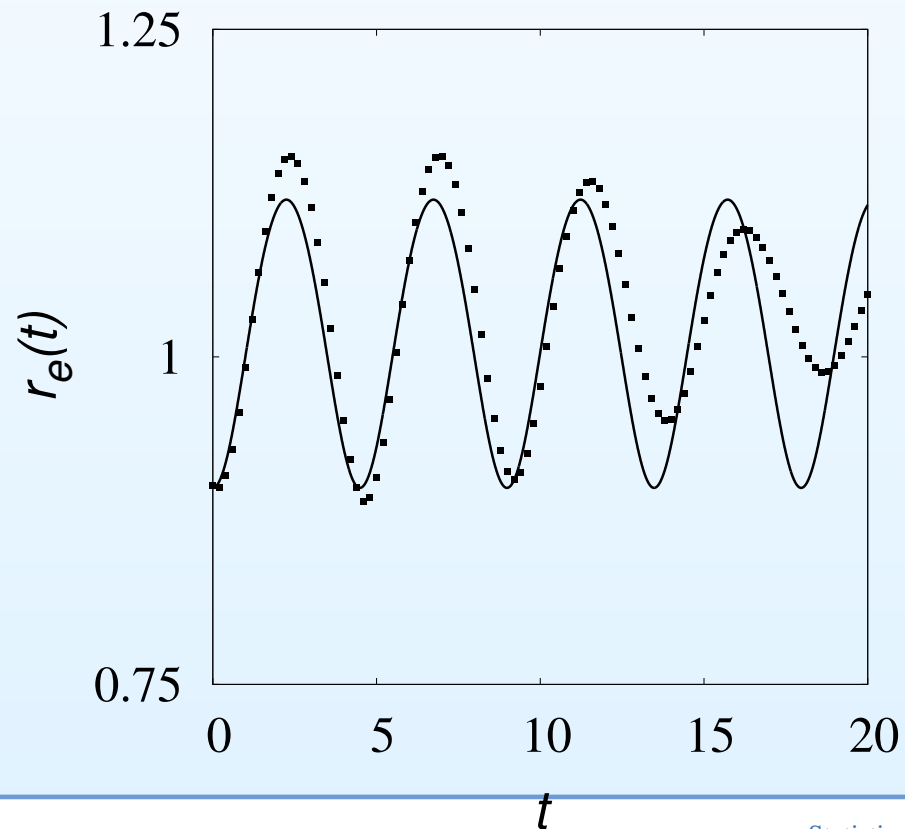
$$\nabla^2 \psi = 4\pi G m n(\mathbf{r})$$

where

$$n(\mathbf{r}) = N \int f(\mathbf{r}, \mathbf{v}) d^2 \mathbf{v}$$

Envelope Oscillations

$$\ddot{r}_e(t) + \frac{2GM}{r_e(t)} = \frac{v_m^2 r_m^2}{r_e^3(t)} .$$



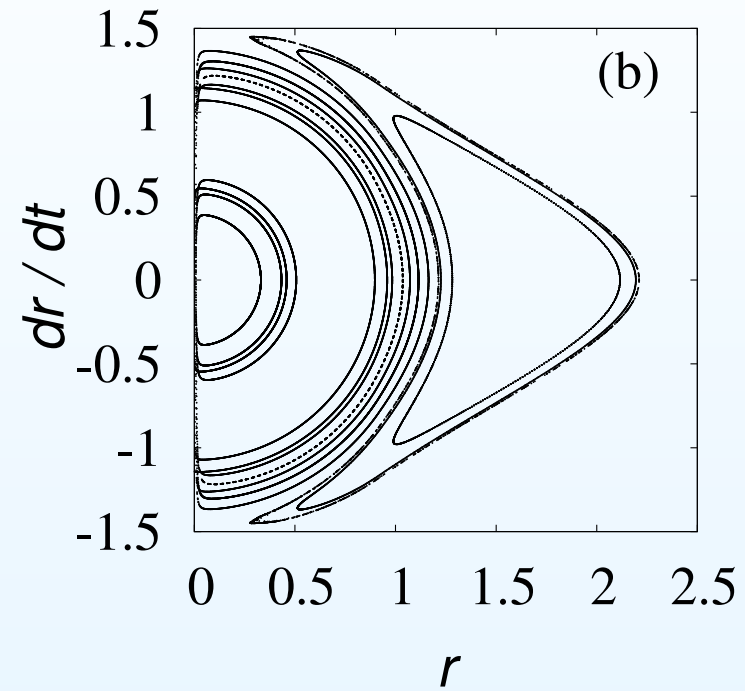
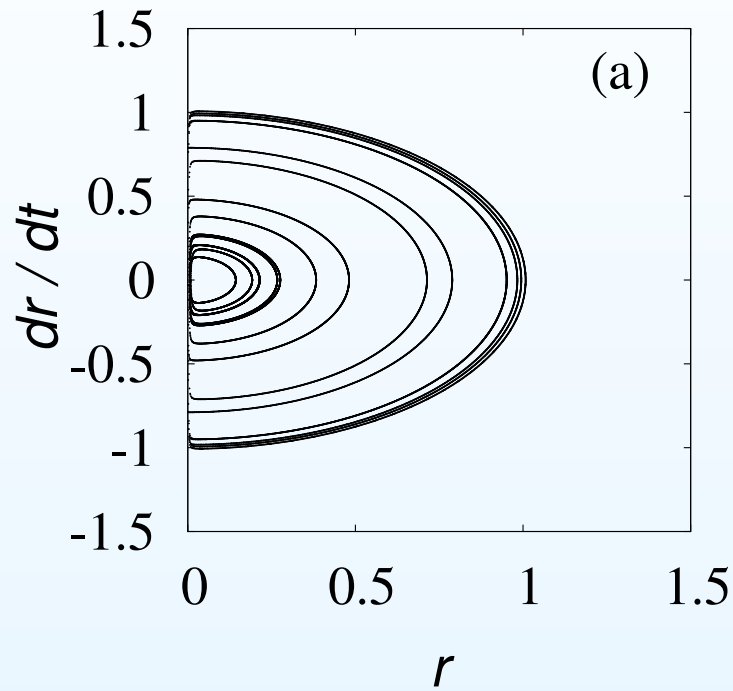
Test Particle

$$\ddot{r}_i(t) - \frac{L_i^2}{r_i^3(t)} = \begin{cases} -\frac{r_i(t)}{r_e^2(t)} & \text{for } r_i(t) \leq r_e(t) \\ -\frac{1}{r_i(t)} & \text{for } r_i(t) \geq r_e(t) \end{cases}$$

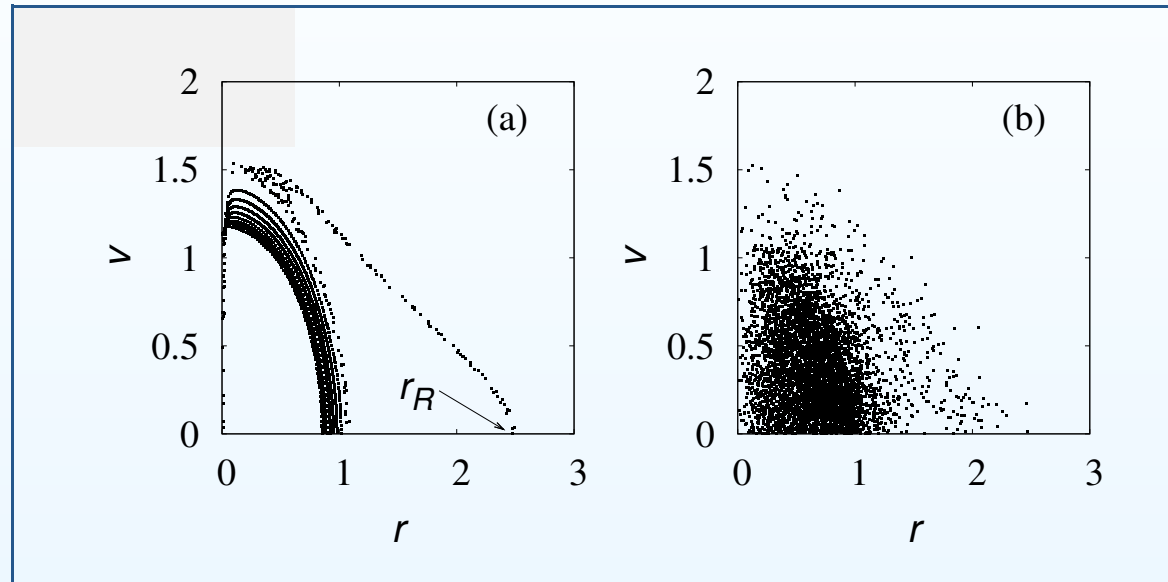
where

$$L_i = |\mathbf{r}_i \times \mathbf{v}_i|$$

Poincaré Section



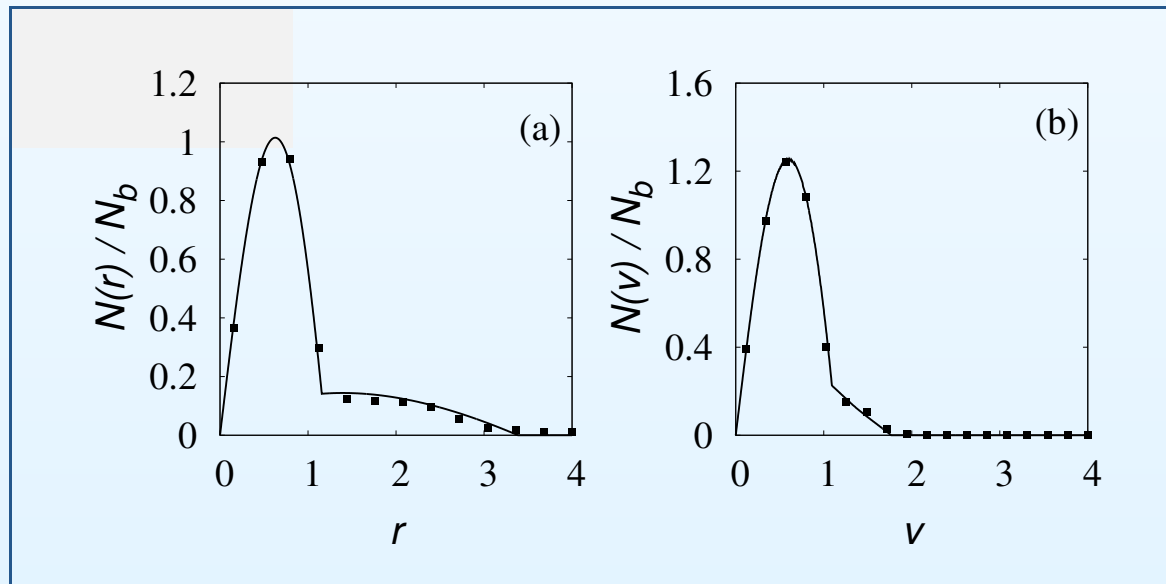
Resonance and Simulations



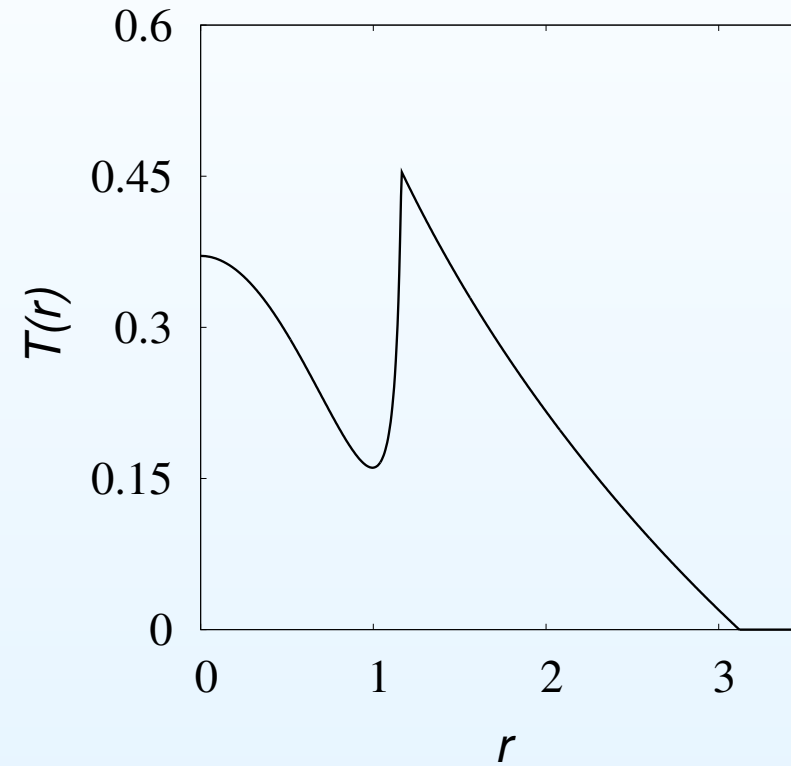
$$\begin{aligned} f(\mathbf{r}, \mathbf{v}) &= \eta \Theta(\epsilon_F - \epsilon(\mathbf{r}, \mathbf{v})) \\ &+ \chi \Theta(\epsilon(\mathbf{r}, \mathbf{v}) - \epsilon_F) \Theta(\epsilon_R - \epsilon(\mathbf{r}, \mathbf{v})) \end{aligned} \quad (1)$$

Density and Velocity Distributions

$$\int d^2\mathbf{r} d^2\mathbf{v} \left[\frac{\mathbf{v}^2}{2} + \frac{\psi(r)}{2} \right] f(\mathbf{r}, \mathbf{v}) = \epsilon_0$$
$$\int d^2\mathbf{r} d^2\mathbf{v} f(\mathbf{r}, \mathbf{v}) = 1, \quad (2)$$



Kinetic Temperature Distributions



*Teles et al. J. Stat. Mech. P05007 (2010)

3d Gravity with $\mathcal{R}_0 = 1$

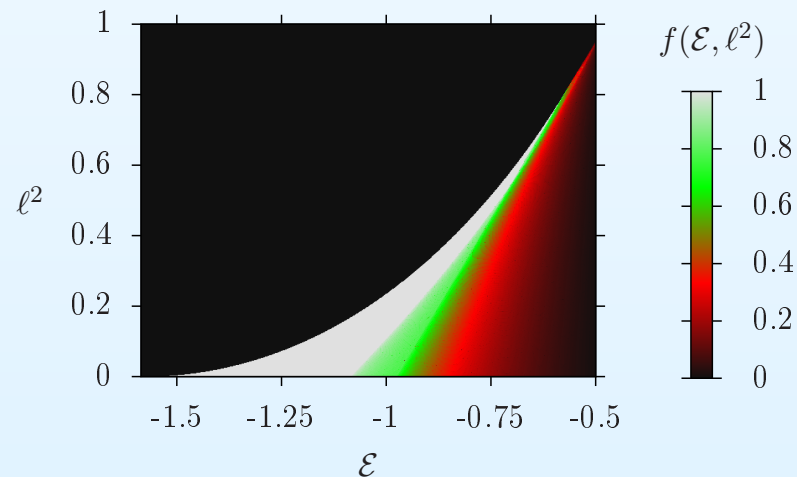
- Small envelope oscillations.
- Quasi-static mean-field potential $\psi(r)$.
- Coarse-grained distribution function:

$$f(\mathcal{E}, \ell^2) = \frac{n(\mathcal{E}, \ell^2)}{g(\mathcal{E}, \ell^2)}$$

Distribution Function

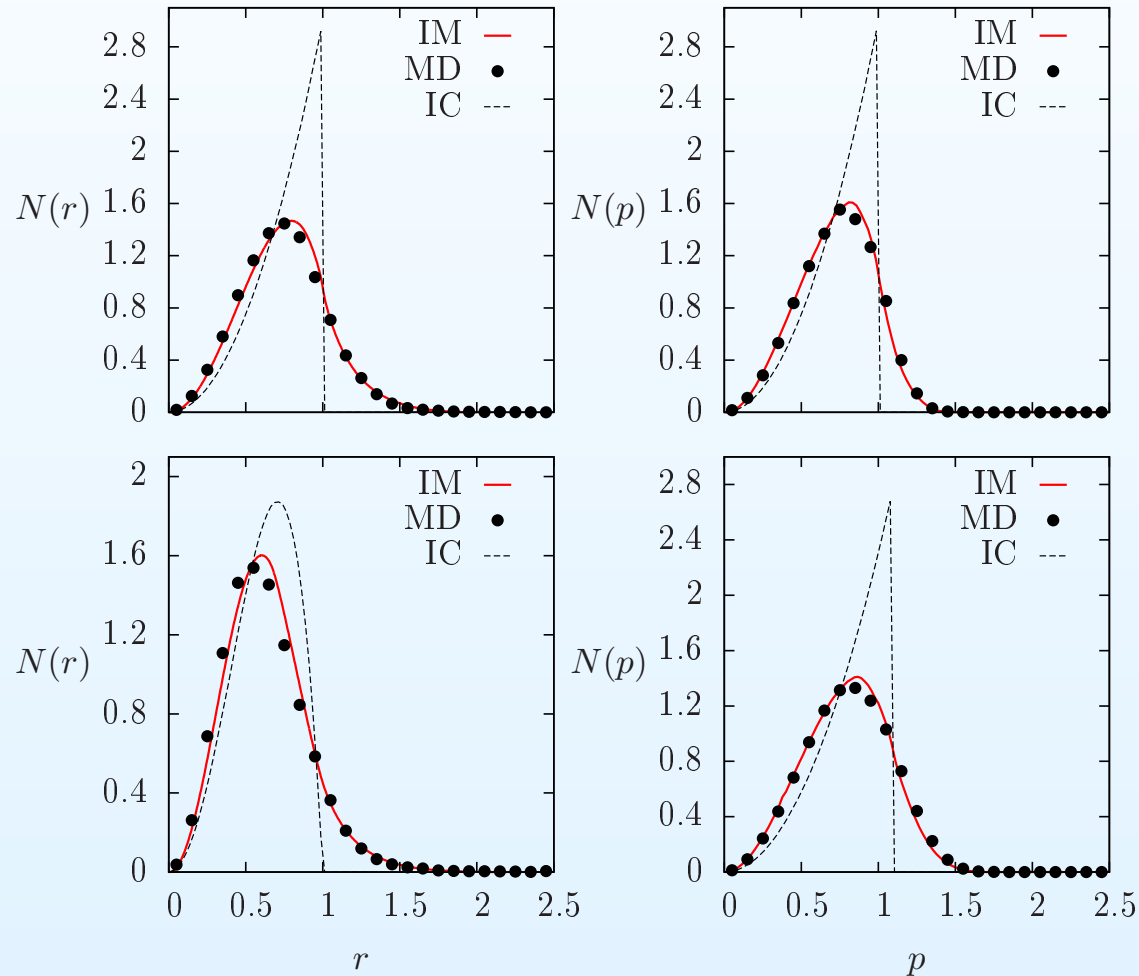
- Coarse-grained distribution function:

$$f(\mathcal{E}, \ell^2) = \frac{\int dr f_0 \left[r, \sqrt{2(\mathcal{E} - \psi(r))} \right] \frac{\Theta \left[\mathcal{E} - \frac{\ell^2}{2r^2} - \psi(r) \right]}{\sqrt{\mathcal{E} - \frac{\ell^2}{2r^2} - \psi(r)}}}{\int dr \frac{\Theta \left[\mathcal{E} - \frac{\ell^2}{2r^2} - \psi(r) \right]}{\sqrt{\mathcal{E} - \frac{\ell^2}{2r^2} - \psi(r)}}}$$



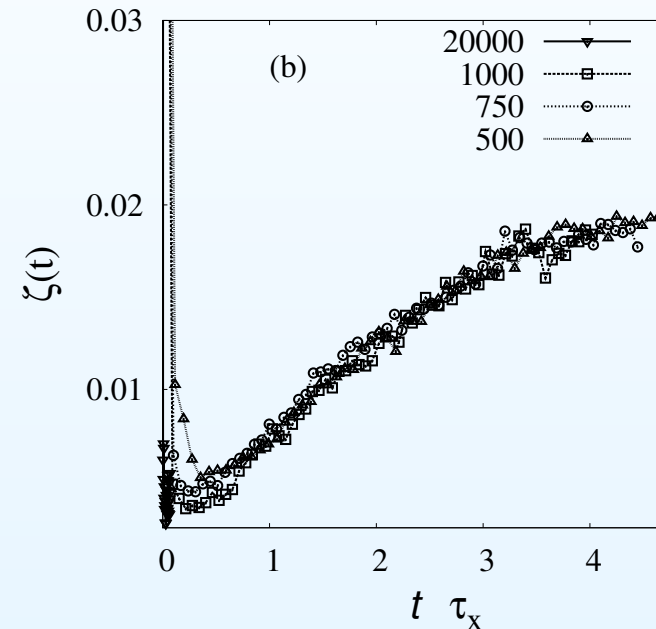
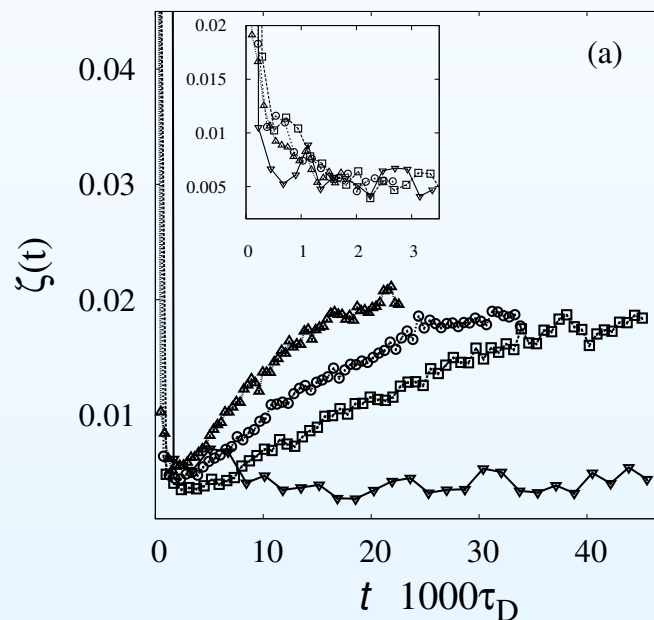
Marginal distributions

- Initial waterbag and parabolic initial distributions



Crossover to Boltzmann Distribution

- $$\chi(t) = \frac{1}{N^2} \int_0^\infty [N(r, t) - N_{ch}(r)]^2 d^2\mathbf{r}$$



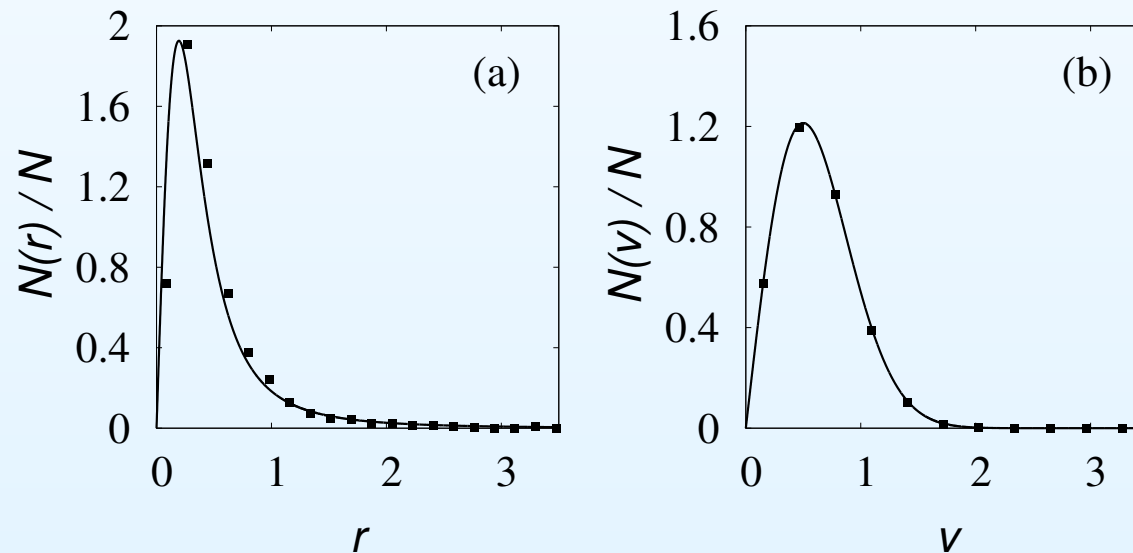
- collapse time $\tau_x \approx N^{1.35} \tau_D$, where $\tau_D = r_m / \sqrt{2GM}$

*Teles et al. J. Stat. Mech. P05007 (2010)

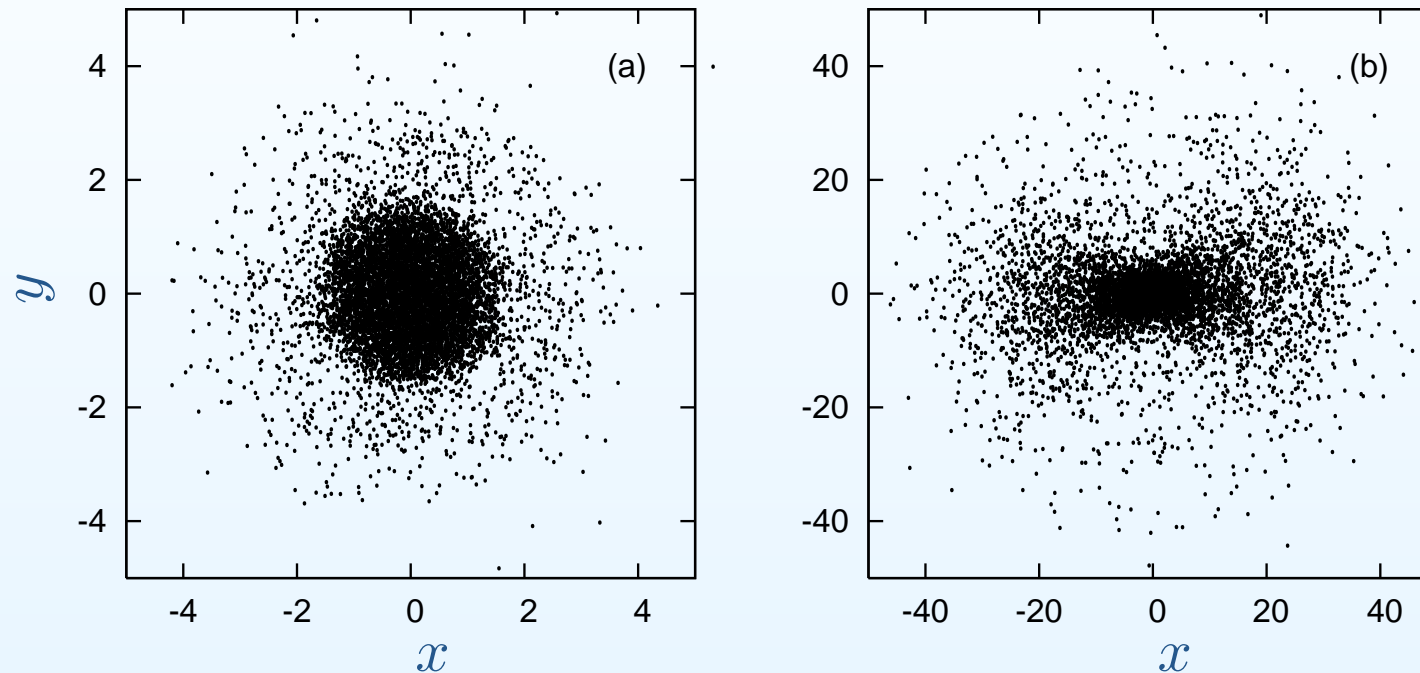
Finite N : Boltzmann Distribution

- $f_{MB} = C e^{-\beta(\mathbf{v}^2/2 + \psi(\mathbf{r}))}$
- Poisson-Boltzmann equation:

$$\nabla^2 \psi = \frac{4\pi^2 C}{\beta} e^{-\beta\psi}$$



Symmetry Breaking



2D system: Panel (a) $\mathcal{R}_0 = 2$; Panel (b) $\mathcal{R}_0 = 6.25$

Pakter, Marcos, and Y. Levin, Phys. Rev. Lett. **111**, 230603 (2013).

Symmetry Breaking in d-Dimensions

- Initial distribution:

$$f_0(\mathbf{r}, \mathbf{v}) = \frac{d^2}{C_d^2 r_m^d v_m^d} \Theta(r_m - r) \Theta(v_m - v)$$

- Envelope in position:

$$X_i(t) = \sqrt{(d+2)\langle x_i^2 \rangle}$$

- Envelope in velocity:

$$V_i(t) = \sqrt{(d+2)\langle v_i^2 \rangle}$$

Envelope Equation

$$\dot{X}_i^2 + X_i \ddot{X}_i = V_i^2 - (d + 2) \left\langle x_i \frac{\partial \psi}{\partial x_i} \right\rangle$$

and

$$V_i \dot{V}_i = -(d + 2) \left\langle \dot{x}_i \frac{\partial \psi}{\partial x_i} \right\rangle$$

where

$$\nabla^2 \psi = C_d n(\mathbf{r}, t)$$

Envelope Equation

$$\ddot{X}_i = \frac{\epsilon_i^2}{X_i^3} - \frac{d}{2} X_i g_i(X_1, \dots, X_d)$$

where emittance

$$\epsilon_i^2(t) = X_i^2 V_i^2 - \dot{X}_i^2 X_i^2 = \epsilon_i^2(0)$$

and

$$g_i(X_1, \dots, X_d) = \int_0^\infty \frac{ds}{(X_i^2 + s) \prod_{j=1}^d (X_j^2 + s)^{1/2}}$$

Symmetry Breaking

- Virial number $\mathcal{R}_0 \equiv \frac{2K}{(2-d)U} = v_m^2 = \epsilon^2(0)$
- Symmetric mode $\bar{X} = (\sum_i X_i)/d$ is unstable for $d > 3$
- Oscillations of \bar{X} result in the growth of antisymmetric mode.
- Instability in $d = 2$ for $\mathcal{R}_0 < 0.255893\dots$ and $\mathcal{R}_0 > 2.55819\dots$
- Instability in $d = 3$ for $\mathcal{R}_0 < 0.388666\dots$ and $\mathcal{R}_0 > 1.61133\dots$

Conclusions

- Systems with long-range interactions lack ergodicity and mixing.
- The proposed core-halo distribution describes plasmas, 1d and 2d self-gravitating systems, HMF, etc.
- There is a significant degree of universality.
- Non-symmetric QSS are possible for gravitational systems.
- 3d gravity is very complicated because of particle evaporation, however, progress is being made...
- **Review:** Levin, Pakter, Rizzato, Telles, Benetti, *Nonequilibrium statistical mechanics of systems with long-range interactions*, Phys. Rep. **535**, 1 (2014).