Today: Observational evidence for the “standard model” of cosmology
morning: the Hot Big Bang model
afternoon: the $\Lambda$CDM model

Tuesday: Special epochs of the universe (recombination, nucleosynthesis, inflation)

Wednesday: Structure formation
Danger: Astronomers at work!

Time: \[ 1 \text{ Gyr} = 10^9 \text{ yr} \approx 3.2 \times 10^{16} \text{ s} \approx 10^{60} t_{\text{planck}} \]

age of the Sun = 4.57 Gyr

time since Big Bang = 13.7 Gyr

Distance: \[ 1 \text{ Mpc} = 10^6 \text{ parsecs} \approx 3.1 \times 10^{22} \text{ m} \approx 10^{57} d_{\text{planck}} \]

distance to Andromeda Galaxy \( \approx 0.75 \text{ Mpc} \)

distance to Coma Cluster of galaxies \( \approx 100 \text{ Mpc} \)

Mass: \[ 1 \text{ M}_\odot \approx 2.0 \times 10^{30} \text{ kg} \approx 10^{38} m_{\text{planck}} \]

mass of Milky Way Galaxy \( \approx 10^{12} \text{ M}_\odot \)

mass of Coma Cluster \( \approx 10^{15} \text{ M}_\odot \)

Olbers’ Paradox: The night sky is dark.

\[ \Sigma_{\text{sky}} \approx 5 \times 10^{-17} \text{ watts m}^{-2} \text{ arcsec}^{-2} \]

\[ \Sigma_\odot \approx 5 \times 10^{-3} \text{ watts m}^{-2} \text{ arcsec}^{-2} \]

Stars attached to celestial sphere: no paradox.

Infinite universe filled with stars: PARADOX!

Cunningham, *The Cosmological Glass*, 1559

Digges, *A Perfect Description of the Celestial Orbs*, 1576
Stars are opaque spheres, with typical radius 
\[ R_\ast \sim R_\odot \sim 7 \times 10^8 \text{ m} \sim 2 \times 10^{-14} \text{ Mpc}. \]

The number density of stars is
\[ n_\ast \sim 10^9 \text{ Mpc}^{-3}. \]

How far can you see, on average, before your line of sight intercepts a star?

\[
\lambda = \frac{1}{n_\ast (\pi R_\ast^2)} \sim \frac{1}{(10^9 \text{ Mpc}^{-3})(10^{-27} \text{ Mpc}^{-2})} \sim 10^{18} \text{ Mpc}
\]

In an infinite universe (or one reaching to \( r > 10^{18} \text{ Mpc} \)), the sky is paved with stars, with surface brightness
\[ \Sigma \sim \Sigma_\odot \sim 5 \times 10^{-3} \text{ W m}^{-2} \text{ arcsec}^{-2}. \]

The night sky in our universe has a surface brightness smaller by
\[ \textbf{14 orders of magnitude}. \]

Which of my assumptions was wrong?
Possible resolutions of Olbers’ Paradox:

1) Distant stars are hidden by opaque material. (This doesn’t work in the long run.)

2) The universe has finite size: $r << 10^{18}$ Mpc. (Or stars occupy only a finite volume.)

3) The universe has finite age: $ck << 10^{18}$ Mpc. (Or stars have existed for a finite time.)

4) Distant stars have low surface brightness.

Hubble’s Law: Galaxies show a redshift proportional to their distance.

An emission (or absorption) line has wavelength $\lambda_e$ in the light source’s frame of reference, and wavelength $\lambda_0$ in the observer’s frame of reference.

$$z = \frac{\lambda_0 - \lambda_e}{\lambda_e}$$

$z > 0 \Rightarrow$ redshift

$z < 0 \Rightarrow$ blueshift
1923: Arthur Eddington compiles a list of 41 galaxy wavelength shifts (mostly measured by Vesto Slipher).  

36 redshifts, 5 blueshifts. Assuming classical Doppler shift, the mean radial velocity is  

$$v_r = c z = +540 \text{ km s}^{-1}$$

1923: Edwin Hubble estimates galaxy distances using Cepheid variable stars.

Hubble: ‘The corresponding distance [to the Andromeda Galaxy] is about 285,000 parsecs.’

actually 0.75 Mpc
1929: Hubble shows that galaxies have a measured redshift proportional to estimated distance.

Hubble 1929

Freedman et al. 2001

Hubble’s Law:

\[ cz = H_0 r \]

\( H_0 = \) ‘Hubble constant’ = 68 \( \pm \) 2 km s\(^{-1}\) Mpc\(^{-1}\)

\( 1/H_0 = \) ‘Hubble time’ = 14.4 \( \pm \) 0.4 Gyr

\( c/H_0 = \) ‘Hubble distance’ = 4400 \( \pm \) 100 Mpc

\( v_r = H_0 r \)

\( t = r / v_r = 1 / H_0 \)

independent of \( r \)
Hubble’s Law: result of homogeneous, isotropic expansion.

\[ r_{12}(t) = a(t)r_{12}(t_0) \]
\[ r_{23}(t) = a(t)r_{23}(t_0) \]
\[ r_{31}(t) = a(t)r_{31}(t_0) \]

\[ a(t) = \text{‘scale factor’} \]
homogeneous: \( a \) is function of \( t \), but not of \( \vec{r} \)

isotropic: \( a \) is scalar, not tensor

normalization: \( a(t) = 1 \) at \( t = t_0 = \text{now.} \)

\[ H(t) = \text{‘Hubble parameter’}, \]
\[ H_0 = H(t_0) = \text{‘Hubble constant’} \]
Hubble’s law is **consistent** with a Big Bang model, but does not **require** it.

| **Hot Big Bang** | **Steady State**  
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Cosmological principle: universe is spatially homogeneous &amp; isotropic (on large scales), but <strong>changes</strong> with time, becoming cooler &amp; less dense.</td>
<td><strong>Perfect</strong> cosmological principle: universe is spatially homogeneous &amp; isotropic (on large scales), and its global properties are <strong>constant with time</strong>.</td>
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### Steady state model:

Hubble constant $H_0$ is constant with time.

\[
\frac{dr}{dt} = H_0 r \quad \Rightarrow \quad r \propto e^{H_0 t}
\]

Mean density $\rho_0$ is constant with time.

\[
V \propto r^3 \propto e^{3H_0 t} \quad \Rightarrow \quad \dot{M} = \rho_0 \dot{V} = \rho_0 3H_0 V
\]

\[
\frac{\dot{M}}{V} = \rho_0 3H_0 \sim 6 \times 10^{-28} \text{ kg m}^{-3}\text{Gyr}^{-1}
\]
1963: “There are only 2½ facts in cosmology.”

1) The sky is dark at night.

2) The galaxies are receding from each other as expected in a uniform expansion.

2½) The contents of the universe have probably changed as the universe grows older.

Radio galaxy 3C295: $z = 0.464, \ z/H_0 = 6.7 \text{ Gyr}$

**Fact 3:** The universe contains a cosmic microwave background (CMB), discovered by Penzias & Wilson in 1965.

CMB is very well fitted by a blackbody spectrum (Planck function = Bose-Einstein distribution for massless bosons).

$$n(v)dv = \frac{8\pi}{c^3} \frac{v^2dv}{\exp(hv/kT) - 1}$$

$$T_0 = 2.7255 \pm 0.0006 \text{ K}$$
Blackbody spectra are produced by opaque objects: CMB tells us that the early universe was opaque.

Baryonic matter (protons, neutrons, & electrons) was ionized.

Rate at which photons scattered from free electrons was greater than the expansion rate of the universe ($\Gamma > H$).

Equivalently: mean free path for photons was shorter than the Hubble distance ($c/\Gamma < c/H$).

Then: opaque

Now: transparent

Violation of the perfect cosmological principle

CMB temperature dipole (red = foreground synchrotron emission in our galaxy) [NASA/WMAP]
CMB dipole anisotropy: due mainly to a Doppler shift from our motion through space.

subtract WMAP’s orbital motion about the Sun (~30 km s\(^{-1}\))
Sun’s orbital motion about the center of our galaxy (~220 km s\(^{-1}\))
our galaxy’s motion relative to Andromeda (~80 km s\(^{-1}\))

Local Group of galaxies is moving toward Hydra,
with \(v \approx 630 \text{ km s}^{-1} \sim 0.002c\)
CMB small-scale anisotropy: due to inhomogeneity at the time when photons last scattered.

\[ \left( \frac{\delta T}{T} \right)_{\text{rms}} \approx 10^{-5} \quad \left[ 5^\circ < \theta < 180^\circ \right] \]

At the time of last scattering, density and potential fluctuations were low in amplitude (\(\delta \varphi / \varphi \sim 10^{-5}\)).
Olbers’ paradox+Hubble’s law+CMB ➔
A universe described by a Hot Big Bang model
(began in a hot, dense state a finite time ago).

The cosmological principle (homogeneous & isotropic)
applies only on large scales today (>100 Mpc). In the past,
the universe was more nearly homogeneous & isotropic.

Expansion of a homogeneous & isotropic universe
is described by the Robertson-Walker metric
and the Friedmann equation.

Expansion of the universe is regulated by gravity.

**Newtonian gravity**

\[
\nabla^2 \phi = 4\pi G \rho
\]

potential \hspace{1cm} mass density

\[
d^2 \vec{r} \over dt^2 = -\nabla \phi
\]

Non-zero acceleration in regions where \( \rho = 0 \).

**General Relativity**

\[
G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}
\]

Einstein tensor
(spacetime curvature)

stress-energy tensor
(energy density \( \varepsilon \), pressure \( P \), etc...)

Non-zero curvature in regions where \( \varepsilon = P = 0 \).

Propagating wave solutions.
We describe the local curvature of spacetime with a **metric**.

**Metric** = relation that gives the shortest distance between two neighboring points.

**Example:** 2-dimensional Euclidean (flat) space

\[
\begin{align*}
\text{(x,y) \rightarrow (x+dx,y+dy)} & \\
\text{ds}^2 = dx^2 + dy^2 & \quad \text{or, in polar coordinates,} \\
\text{ds}^2 = dr^2 + r^2 \, d\theta^2
\end{align*}
\]

**Example:** 3-d Euclidean (flat) space

\[
\begin{align*}
\text{(x,y,z) \rightarrow (x+dx,y+dy,z+dz)} & \\
\text{ds}^2 = dx^2 + dy^2 + dz^2 & \quad \text{or, in spherical coordinates,} \\
\text{ds}^2 = dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) & = dr^2 + r^2 \, d\Omega^2
\end{align*}
\]

**Example:** 4-d Minkowski spacetime [metric of special relativity]

\[
\begin{align*}
\text{(t,x,y,z) \rightarrow (t+dt,x+dx,y+dy,z+dz)} & \\
\text{ds}^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 & \quad \text{or, with spherical coordinates,} \\
\text{ds}^2 = -c^2 dt^2 + dr^2 + r^2 \, d\Omega^2
\end{align*}
\]
Spacetime curvature can be complicated.

However, if the curvature of 3-d space is homogeneous & isotropic, there are only 3 possibilities.

<table>
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<tr>
<th>Flat (Euclidean)</th>
<th>Positive curvature</th>
<th>Negative curvature</th>
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<td>$ds^2 = dr^2 + r^2 d\Omega^2$</td>
<td>$ds^2 = dr^2 + R^2 \sin^2(r/R) d\Omega^2$</td>
<td>$ds^2 = dr^2 + R^2 \sinh^2(r/R) d\Omega^2$</td>
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Combine homogeneous & isotropic expansion (or contraction) with homogeneous & isotropic curvature of space.

The result is the Robertson-Walker metric:

$$ds^2 = -c^2 dt^2 + a(t)^2 [dr^2 + S_\kappa (r)^2 d\Omega^2 ]$$

$$S_\kappa = \begin{cases} 
R_0 \sin(r / R_0) & [\kappa = +1] \\
r & [\kappa = 0] \\
R_0 \sinh(r / R_0) & [\kappa = -1]
\end{cases}$$

Also known as the Friedmann-Robertson-Walker (FRW) metric or the Friedmann-Lemaître-Robertson-Walker (FLRW) metric.
The assumption of homogeneity & isotropy is extremely powerful!

With this assumption, all you need to know about spacetime curvature is:

- curvature constant $\kappa = +1, 0, \text{ or } -1$
- radius of curvature $R_0$ (if $\kappa \neq 0$)
- scale factor $a(t)$

Robertson-Walker metric:

$$ds^2 = -c^2 dt^2 + a(t)^2 [dr^2 + S_\kappa (r)^2 d\Omega^2]$$

Time coordinate $t =$ cosmological proper time [or cosmic time]
(measured by an observer who sees isotropic expansion).

Radial coordinate $r =$ proper distance at time $t_0.$
(length of a spatial geodesic when $a(t_0) = 1$).
Proper distance increases as 
\[ d_p(t) = a(t) \quad d_p(t_0) = a(t) \quad r \]

Radius of curvature increases as 
\[ R(t) = a(t) \quad R_0 \]

Wavelength of light moving freely through space increases as 
\[ \lambda(t) = a(t) \quad \lambda_0 \]

Light was emitted with wavelength \( \lambda_e \) at time \( t_e \), and observed with wavelength \( \lambda_0 \) at time \( t_0 \). The redshift is...

\[ z = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{\lambda_0 - a(t_e)\lambda_0}{a(t_e)\lambda_0} = \frac{1}{a(t_e)} - 1 \]

\[ z = 0.464; \quad a(t_e) = 1/(1+z) = 0.68 \]

\[ z = 8.68; \quad a(t_e) = 1/(1+z) = 0.103 \]

Monotonically expanding Big Bang model: larger \( z \) \( \Rightarrow \) smaller \( a(t_e) \) \( \Rightarrow \) earlier \( t_e \) \( \Rightarrow \) greater \( r = d_p(t_0) \)

\( r, \theta, \phi \) are comoving coordinates
The curvature of spacetime is related to its energy content by Einstein’s field equation:

\[ G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \]

If space is homogeneous & isotropic, this reduces to the Friedmann equation:

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3c^2} \varepsilon(t) - \frac{\kappa c^2}{R_0^2} \frac{1}{a(t)^2}
\]

\[
H(t)^2 = \frac{8\pi G}{3c^2} \varepsilon(t) - \frac{\kappa c^2}{R_0^2} \frac{1}{a(t)^2}
\]