Lectures on the Cosmic Microwave Background

Raphael M. Flauger
The University of Texas

Summer School on Cosmology, ICTP, June 2016

Lecture II

Anisotropies in the CMB

- The kinematic dipole
- Description of CMB Anisotropies
- General Relativity for Cosmologists - Part II
- Computing the Angular Power Spectra

The Kinematic Dipole

A bath of black body radiation provides a reference frame.

The Kinematic Dipole

A bath of black body radiation provides a reference frame.

Consider an observer moving relative to this frame

For black body at temperature T_0, the occupation number for each polarization state is

$$n(\vec{p}) = \frac{1}{e^{\frac{E(\vec{p})}{kT_0}} - 1}$$

A boost leaves occupation numbers invariant, only changes momenta

$$n(\vec{p}) = n_\Lambda(\vec{p}_\Lambda)$$

The Kinematic Dipole

Equivalently

$$n_\Lambda(\vec{p}) = n(\vec{p}_{\Lambda^{-1}}) = \frac{1}{e^{\frac{\gamma E(\vec{p})(1+\vec{\beta}\cdot\hat{p})}{kT_0}} - 1}$$

The temperature in the frame of the moving observer is

$$T_\beta(\hat{n}) = \frac{T_0}{\gamma(1-\vec{\beta}\cdot\hat{n})} \approx T_0 + T_0\vec{\beta}\cdot\hat{n} + \ldots$$

(using $\hat{n} = -\hat{p}$.)

(Peebles, Wilkinson 1968)

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(using $\hat{n} = -\hat{p}$.) **(Peebles, Wilkinson 1968)**

We observe intensity rather than temperature

$$I_\nu(\hat{n},\vec{\beta}) = \bar{I}_\nu + \left.\frac{d\bar{I}_\nu}{dT}\right|_{T_0} T_0\vec{\beta}\cdot\hat{n} + \ldots$$

The Kinematic Dipole

CMB experimentalists attempted to measure this effect soon after the discovery of the CMB and placed upper limits

First measurement of right ascension

Velocity of the Earth with Respect to the Cosmic Background Radiation

E. K. Conklin

Radio Astronomy Institute,
Stanford University,
Stanford, California.

Received March 17, 1969.

The Kinematic Dipole

First measurement of right ascension, declination, amplitude

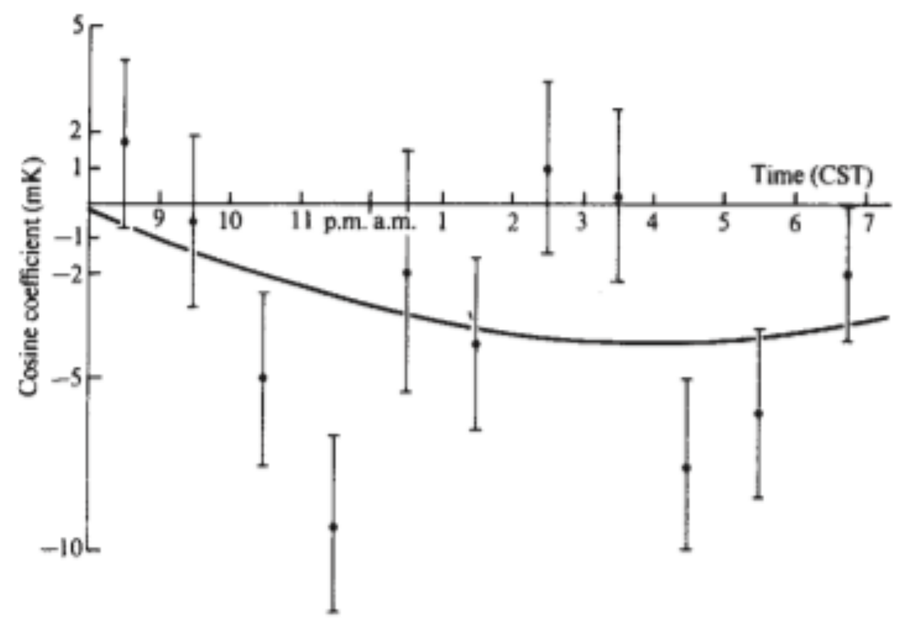
Isotropy of the 3 K Background

3.2 ± 0.8 mK

PAUL S. HENRY*

Joseph Henry Laboratories,
Department of Physics,
Princeton University,
Princeton, New Jersey 08540

Received May 17, 1971.



The Kinematic Dipole

and at higher significance

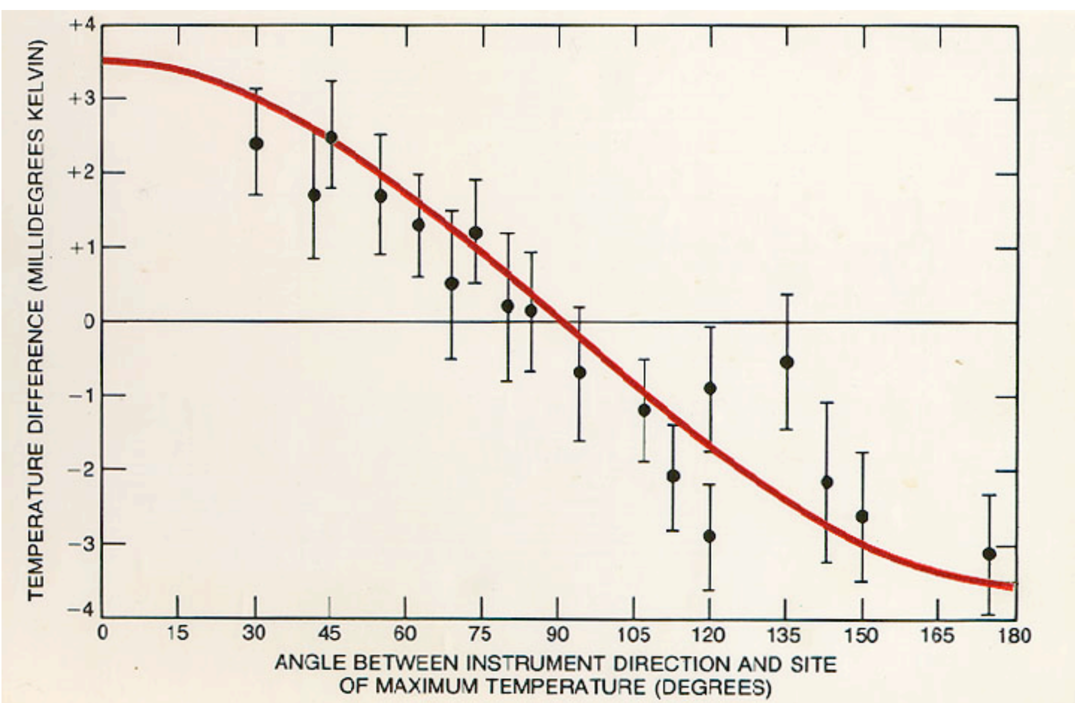
29.03.10 A Measurement of the Cosmic Microwave Background Anisotropy at 19 GHz. B.E. COREY and D.T. WILKINSON, PRINC. U. - A balloon-borne experiment on May 10-11, 1975 measured the large-scale anisotropy of the cosmic microwave background at 19 GHz. A Dicke-switched radiometer was used to measure the difference in radiation flux received by two horn antennas pointing 45° down from the zenith and 180° apart in azimuth. The apparatus was rotated about the vertical at 1 rpm to facilitate the removal of slow drifts in switch offset. Small corrections for galactic emission were applied to two of the nine hours of data. The data were fit to a model for the anisotropy given by $\Delta T = T_o(v/c) \cos\theta$, where θ is the angle between the line of sight and the earth's peculiar velocity $\vec{v}$. The best-fit parameters for $\vec{v}$ are $v = 270 \pm 70$ km/sec, $\alpha = 13^h \pm 2^h$, and $\delta = -25^{\circ} \pm 20^{\circ}$ where the errors are formal fitting errors. Possible systematic errors will be discussed. The results are only weakly dependent on the size of the galactic correction. This research was supported in part by the National Science Foundation.

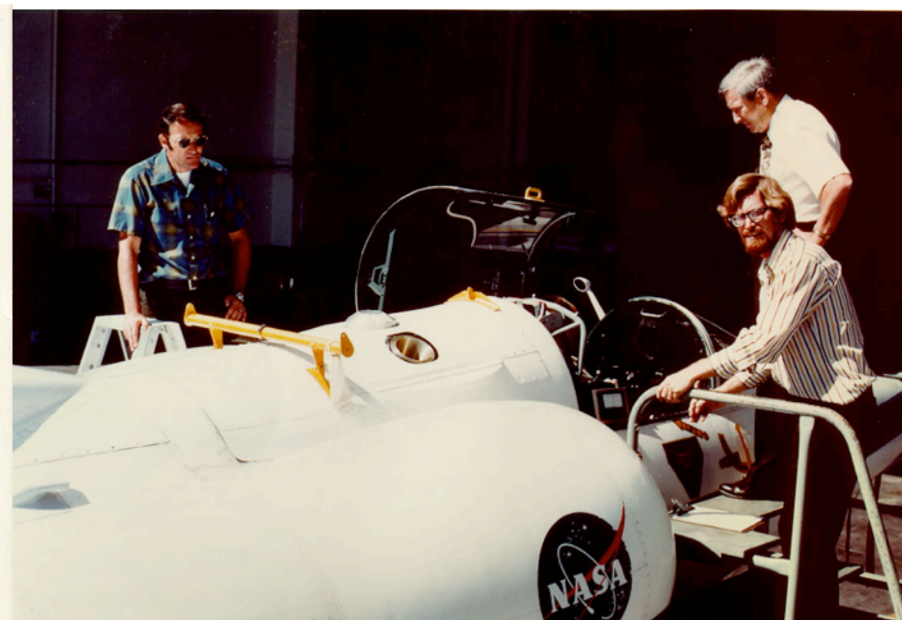
The Kinematic Dipole

Detection of Anisotropy in the Cosmic Blackbody Radiation

G. F. Smoot, M. V. Gorenstein, and R. A. Muller
Lawrence Berkeley Laboratory and Space Sciences Laboratory, University of California, Berkeley, California 94720
(Received 6 July 1977)

We have detected anisotropy in the cosmic blackbody radiation with a 33-GHz (0.9 cm) twin-antenna Dicke radiometer flown to an altitude of 20 km aboard a U-2 aircraft. In data distributed over two-thirds of the northern hemisphere, we observe an anisotropy which is well fitted by a first-order spherical harmonic with an amplitude of $(3.5 \pm 0.6) \times 10^{-3}$ °K, and direction [11.0 ± 0.6 h right ascension (R.A.) and $6° \pm 10°$ declination (dec)]. This observation is readily interpreted as due to motion of the earth relative to the radiation with a velocity of 390 ± 60 km/sec.





The Kinematic Dipole

Motion relative to the CMB beyond the dipole

At second order*

$$I_\nu(\hat{n},\vec{\beta}) = \bar{I}_\nu + \left.\frac{d\bar{I}_\nu}{dT}\right|_{T_0} T_0 \beta P_1(\hat{\beta}\cdot\hat{n}) + \frac{1}{3}\beta^2 P_2(\hat{\beta}\cdot\hat{n}) \left(\frac{d}{dT} T^2 \left.\frac{d\bar{I}_\nu}{dT}\right|_{T_0} \right) + \ldots$$

(Knox, Kamionkowski 2002)

* Ignoring frequency dependent monopole contribution and higher order terms

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kinematic quadrupole

(Knox, Kamionkowski 2002)

Frequency dependence would allow to distinguish between kinematic and primordial quadrupole.

Difficult to detect because of foreground contamination, but maps are corrected for this effect

* Ignoring frequency dependent monopole contribution and higher order terms

The Kinematic Dipole

Motion relative to the CMB beyond the dipole

Fluctuations are also affected

For $$n(\vec{p}) = \frac{1}{e^{\frac{E(\vec{p})}{kT(\hat{n})}} - 1}$$

with $T(\hat{n}) = T_0 + \Delta T(\hat{n})$, $\hat{n} = -\hat{p}$

$$\Delta I_\nu(\hat{n}, \vec{\beta}) = \left.\frac{d\bar{I}_\nu}{dT}\right|_{T_0} \left[T_0 \vec{\beta}\cdot\hat{n} + \Delta T\left(\hat{n} - \vec{\beta}_\perp)\right)(1 + f_\nu \vec{\beta}\cdot\hat{n})\right] + \ldots$$

relativistic aberration

modulation

where $\vec{\beta}_\perp = \vec{\beta} - \hat{n}(\vec{\beta}\cdot\hat{n})$ and $f_\nu = \frac{h\nu}{kT}\coth\left(\frac{h\nu}{kT}\right) - \frac{1}{2}$

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This has been measured by Planck! **(Planck 2013, 1303.5087)**

Primary CMB Anisotropies

The temperature and polarization anisotropies may be defined after subtraction of kinematic dipole and quadrupole as

Stokes parameters

$$\Delta I_\nu(\hat{n}, \psi(\hat{n})) = \left.\frac{d\bar{I}_\nu}{dT}\right|_{T_0} \left[\Delta T\,(\hat{n}) + Q(\hat{n})\cos\left(2\psi(\hat{n})\right) + U(\hat{n})\sin\left(2\psi(\hat{n})\right)\right]$$

polarization angle

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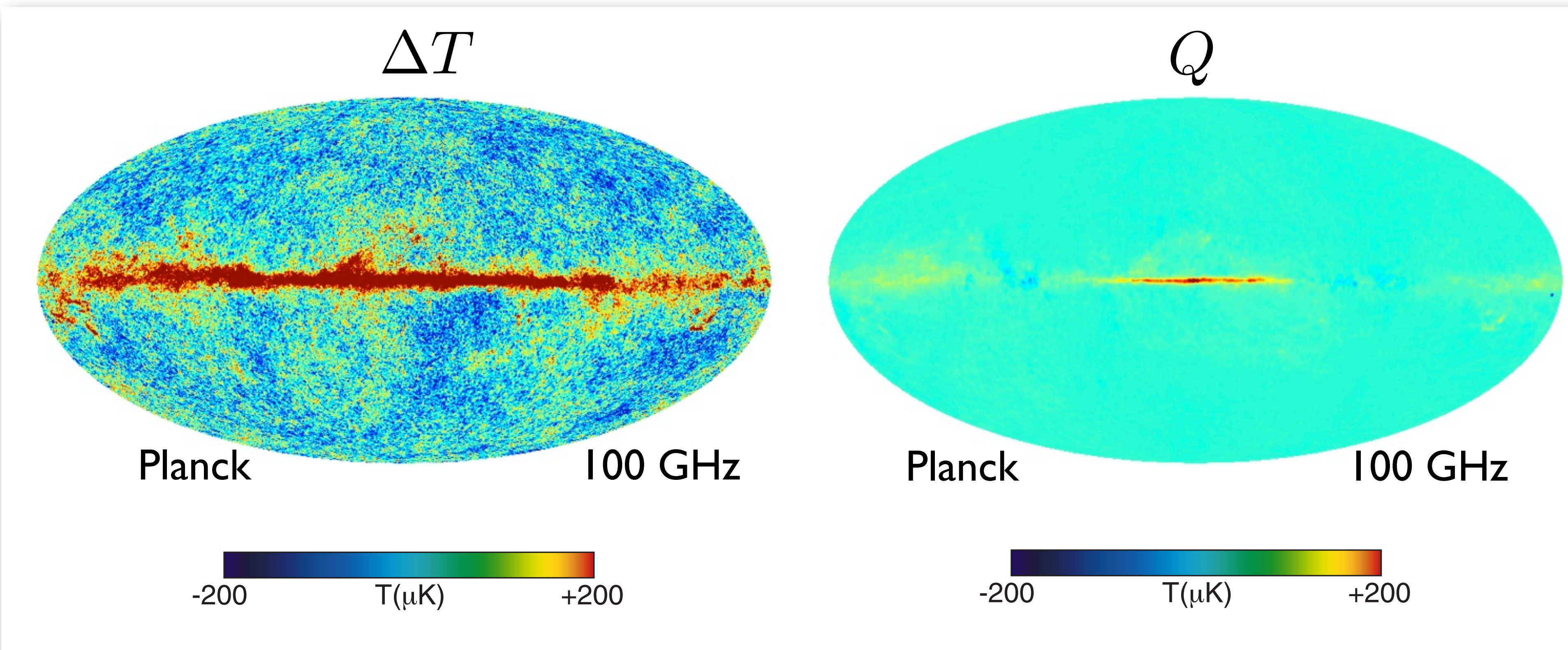
polarization angle

$$= \Delta I_\nu(\hat{n})^{ij} e_i(\psi(\hat{n})) e_j(\psi(\hat{n}))$$

Primary CMB Anisotropies

Temperature and Stokes parameters are usually shown as color-coded maps



Primary CMB Anisotropies

Only the correlation functions can be predicted by theory

$$\begin{aligned}
&\langle \Delta T(\hat{n}) \Delta T(\hat{n}') \rangle \quad , \\
&\langle \Delta T(\hat{n}) \left[Q(\hat{n}') + iU(\hat{n}') \right] \rangle \quad , \\
&\langle \left[Q(\hat{n}) + iU(\hat{n}) \right] \left[Q(\hat{n}') + iU(\hat{n}') \right] \rangle \quad , \\
&\langle \left[Q(\hat{n}) + iU(\hat{n}) \right] \left[Q(\hat{n}') - iU(\hat{n}') \right] \rangle \quad .
\end{aligned}$$

as well as higher n-point functions

Primary CMB Anisotropies

For data analysis and comparison with theory, it is more convenient to use multipole coefficients

$$a_{T,\ell m} = \int d^2\hat{n}\, Y_\ell^{m*}(\hat{n})\Delta T(\hat{n})$$

$$a_{P,\ell m} = \int d^2\hat{n}\, {}_2Y_\ell^{m*}(\hat{n})\left(Q(\hat{n}) + iU(\hat{n})\right)$$

Primary CMB Anisotropies

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$$a_{E,\ell\, m} \equiv -(a_{P,\ell\, m} + a^*_{P,\ell\, -m})/2$$

$$a_{B,\ell\, m} \equiv i(a_{P,\ell\, m} - a^*_{P,\ell\, -m})/2$$

under parity

$$a_{E,\ell\, m} \to (-1)^\ell a_{E,\ell\, m} \qquad \text{“gradient”}$$

$$a_{B,\ell\, m} \to -(-1)^\ell a_{B,\ell\, m} \qquad \text{“curl”}$$

Primary CMB Anisotropies

The correlations are then encoded in the angular power spectra

$$\left\langle a_{T,\ell\, m} a^*_{T,\ell'\, m'} \right\rangle = C_{TT,\ell} \delta_{\ell\ell'} \delta_{mm'} \ ,$$

$$\left\langle a_{T,\ell\, m} a^*_{E,\ell'\, m'} \right\rangle = C_{TE,\ell} \delta_{\ell\ell'} \delta_{mm'} \ ,$$

$$\left\langle a_{E,\ell\, m} a^*_{E,\ell'\, m'} \right\rangle = C_{EE,\ell} \delta_{\ell\ell'} \delta_{mm'} \ ,$$

$$\left\langle a_{B,\ell\, m} a^*_{B,\ell'\, m'} \right\rangle = C_{BB,\ell} \delta_{\ell\ell'} \delta_{mm'} \ ,$$

For Gaussian fluctuations these contain all the information, for non-Gaussian fluctuations we would need higher n-point functions

Primary CMB Anisotropies

These angular power spectra can be calculated for a given model, and they can be estimated from the sky maps by

$$a_{T,\ell m}^{\rm obs} = \int d^2\hat{n}\, Y_\ell^{m\,*}(\hat{n})\Delta T(\hat{n})$$

$$C_{TT,\ell}^{\rm obs} \equiv \frac{1}{2\ell+1}\sum_m \left|a_{T,\ell\, m}^{\rm obs}\right|^2$$

Primary CMB Anisotropies

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$$a^{\rm obs}_{T,\ell m} = \int d^2\hat{n}\, Y_\ell^{m*}(\hat{n})\Delta T(\hat{n})$$

estimator

(assumes full sky)

$$C^{\rm obs}_{TT,\ell} \equiv \frac{1}{2\ell+1}\sum_m \left|a^{\rm obs}_{T,\ell\, m}\right|^2$$

This estimator is unbiased

$$\langle C^{\rm obs}_{TT,\ell}\rangle = C_{TT,\ell}$$

average over different realizations

Primary CMB Anisotropies

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estimator

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$$C^{\rm obs}_{TT,\ell} \equiv \frac{1}{2\ell+1}\sum_m \left|a^{\rm obs}_{T,\ell\, m}\right|^2$$

Cosmic variance

$$\langle (C^{\rm obs}_{TT,\ell} - C_{TT,\ell})^2\rangle = \frac{2}{2\ell+1} C^2_{TT,\ell}$$

Primary CMB Anisotropies

First claim of a detection in reanalysis of Relikt I data

Anisotropy of the microwave background radiation

I. A. Strukov, A. A. Bryukhanov, D. P. Skulachev, and M. V. Sazhin

Space Research Institute, Russian Academy of Sciences, Moscow
and P. K. Shternberg State Astronomical Institute, Moscow

(Submitted January 19, 1992)

Pis'ma Astron. Zh. **18**, 387–395 (May 1992)

New results from analysis of data on the anisotropy of the background radiation at 37 GHz (spaceborne experiment Relikt 1) are presented. The relative magnitude of the quadrupole component was estimated with 90% confidence for an inflationary perturbation spectrum: $6\cdot10^{-6} < \Delta T_2/T < 3.3\cdot10^{-5}$. An anomaly of the microwave radiation has been found, with 99% confidence, in a region with area $\simeq 1$ sr near the point with coordinates $\alpha \simeq 1^h30^m$ and $\delta \simeq -10°$ ($l = 150°$ and $b = -70°$). The magnitude of this anomaly is $\Delta T_b = -71 \pm 43$ μK with 90% confidence. We discuss possible sources of the anomaly.

Single frequency measurement at 37 GHz

Primary CMB Anisotropies

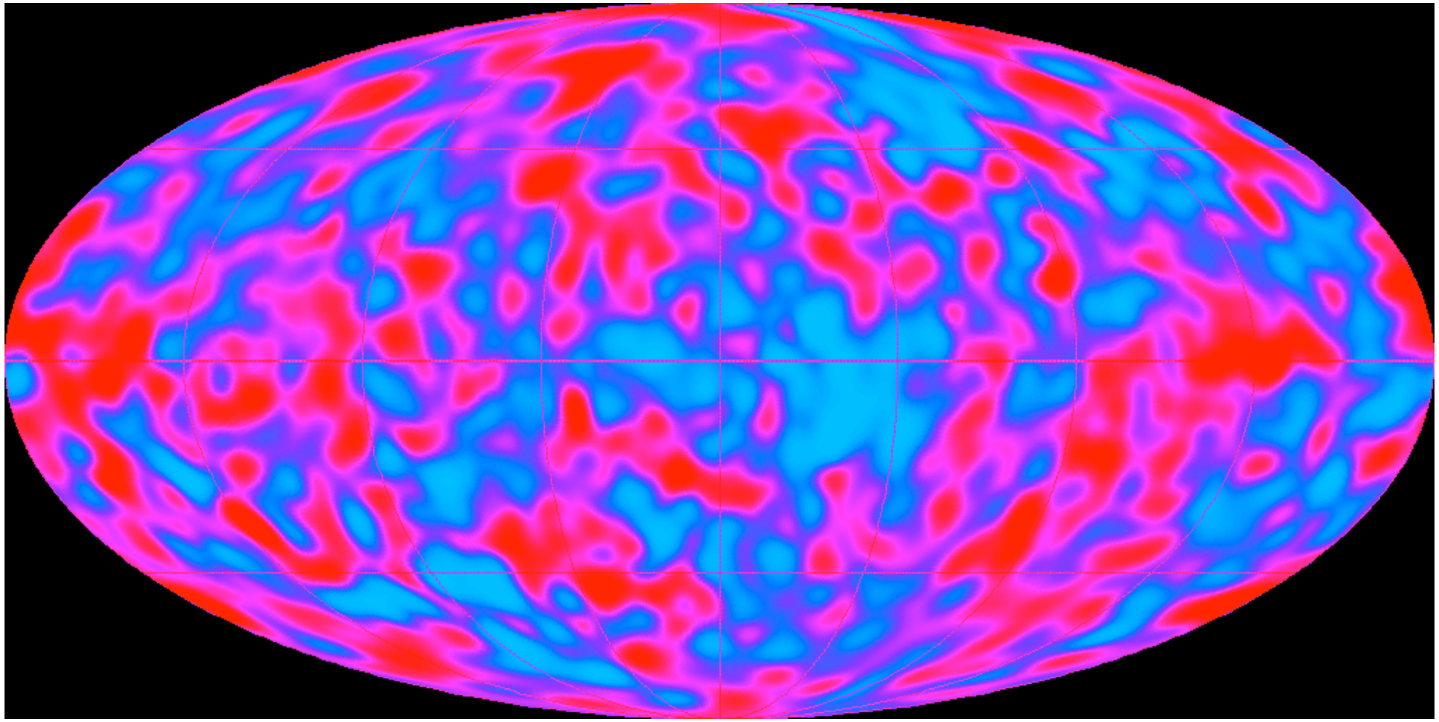
STRUCTURE IN THE *COBE*[1] DIFFERENTIAL MICROWAVE RADIOMETER FIRST-YEAR MAPS

G. F. SMOOT,[2] C. L. BENNETT,[3] A. KOGUT,[4] E. L. WRIGHT,[5] J. AYMON,[2] N. W. BOGGESS,[3] E. S. CHENG,[3]
G. DE AMICI,[2] S. GULKIS,[6] M. G. HAUSER,[3] G. HINSHAW,[4] P. D. JACKSON,[7] M. JANSSEN,[6]
E. KAITA,[7] T. KELSALL,[3] P. KEEGSTRA,[7] C. LINEWEAVER,[2] K. LOEWENSTEIN,[7] P. LUBIN,[8]
J. MATHER,[3] S. S. MEYER,[9] S. H. MOSELEY,[3] T. MURDOCK,[10] L. ROKKE,[7]
R. F. SILVERBERG,[3] L. TENORIO,[2] R. WEISS,[9] AND D. T. WILKINSON[11]

Received 1992 April 21; accepted 1992 June 12

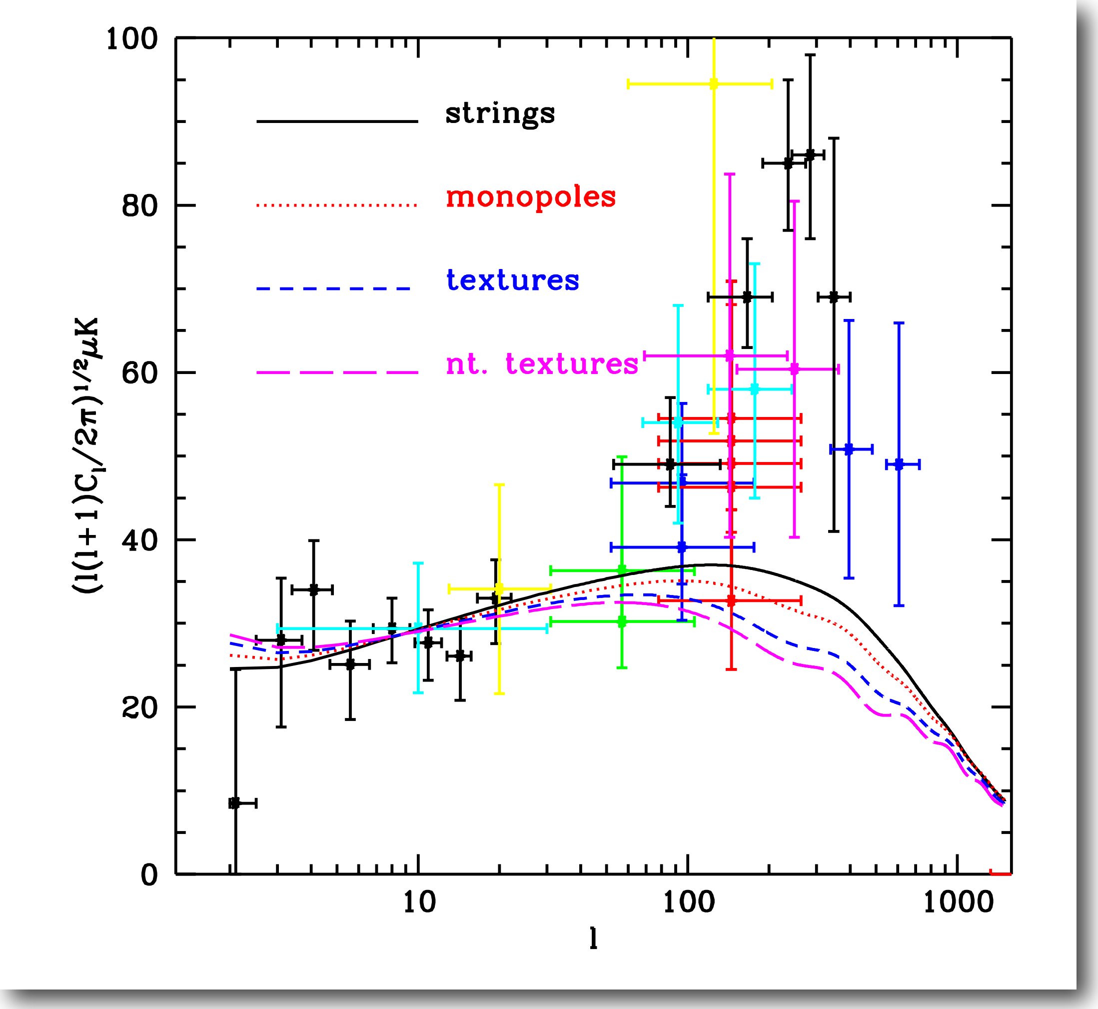
ABSTRACT

The first year of data from the Differential Microwave Radiometers (DMR) on the *Cosmic Backgoround Explorer* (*COBE*) show statistically significant ($>7\ \sigma$) structure that is well described as scale-invariant fluctuations with a Gaussian distribution. The major portion of the observed structure cannot be attributed to known systematic errors in the instrument, artifacts generated in the data processing, or known Galactic emission. The structure is consistent with a thermal spectrum at 31, 53, and 90 GHz as expected for cosmic microwave background anisotropy.



Primary CMB Anisotropies

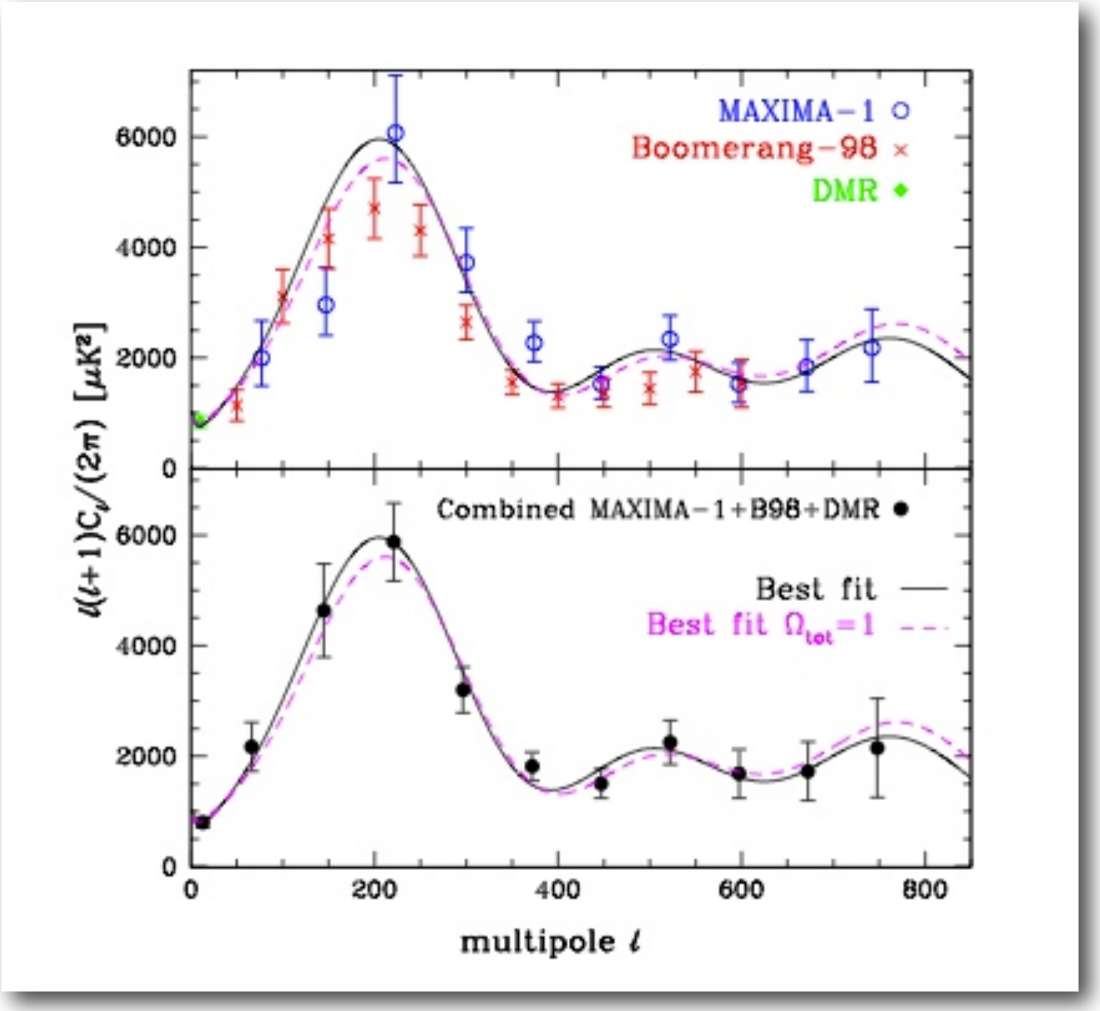
1997



(Pen, Seljak, Turok 1997)

Primary CMB Anisotropies

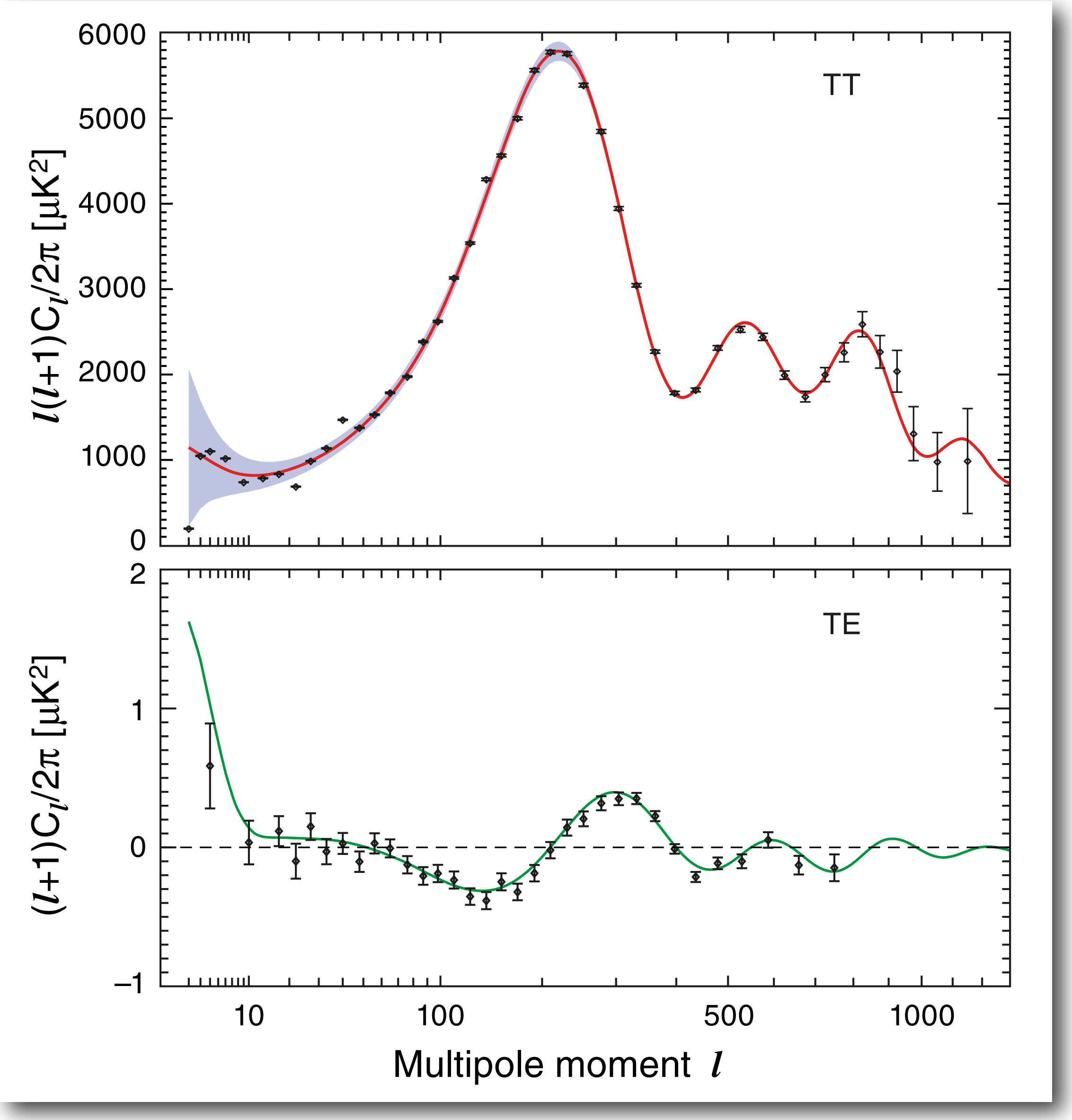
2001



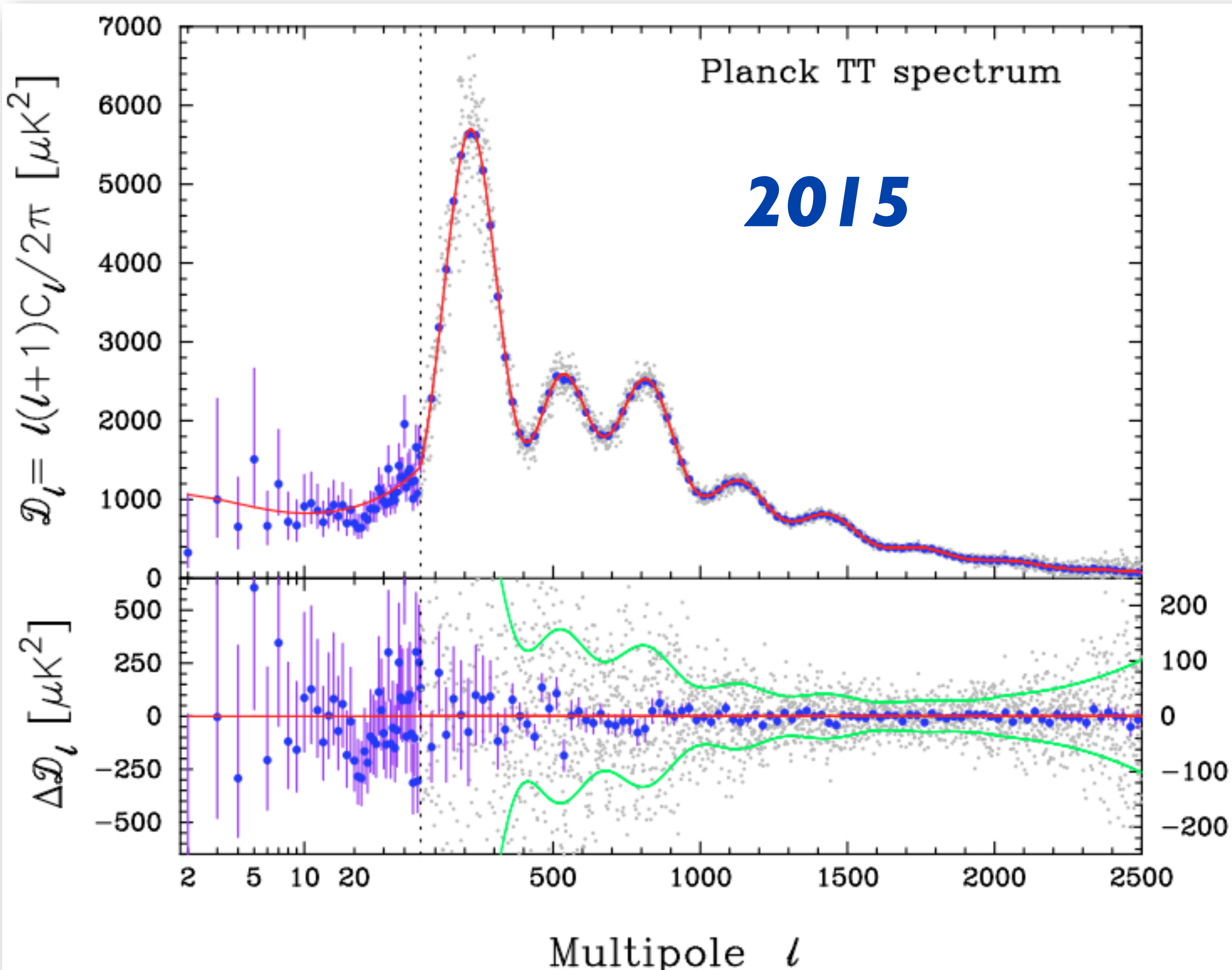
(Jaffe, et al. 2001)

Primary CMB Anisotropies

2010



Primary CMB Anisotropies



General Relativity - Part II

We saw that the composition and evolution of the FLRW universe are described by

$$ds^2 = -dt^2 + a^2 d\vec{x}^2$$

$$T = \bar{\rho} dt^2 + a^2 \bar{p} d\vec{x}^2$$

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To describe the anisotropies, we must consider small perturbations around the FLRW background

$$ds^2 = (-1 + h_{00})dt^2 + 2h_{0i} dt dx^i + (a^2 \delta_{ij} + h_{ij}) dx^i dx^j$$

$$T = (\bar{\rho} + \delta T_{00}) dt^2 + 2\delta T_{0i} dt dx^i + (a^2 \bar{p} \delta_{ij} + \delta T_{ij}) dx^i dx^j$$

General Relativity - Part II

Under an infinitesimal coordinate transformation

$$x^\mu \to x^\mu + \epsilon^\mu(x)$$

the perturbations transform

$$\Delta h_{00} = -2\frac{\partial \epsilon_0}{\partial t}$$
$$\Delta h_{0i} = -\frac{\partial \epsilon_i}{\partial t} - \frac{\partial \epsilon_0}{\partial x^i} + 2\frac{\dot{a}}{a}\epsilon_i$$
$$\Delta h_{ij} = -\frac{\partial \epsilon_i}{\partial x^j} - \frac{\partial \epsilon_j}{\partial x^i} + 2a\dot{a}\epsilon_0$$

We can use and choice of coordinates (or gauge) that is convenient

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We can use and choice of coordinates (or gauge) that is convenient, e.g. synchronous gauge

$$h_{00} = 0 \qquad\qquad h_{0i} = 0$$

General Relativity - Part II

In synchronous gauge

$$ds^2 = -dt^2 + (a^2\delta_{ij} + h_{ij})dx^i dx^j$$

$$\delta T = \delta\rho\, dt^2 - 2(\bar{\rho} + \bar{p})\delta u_i dt dx^i + \left(a^2(\delta p\, \delta_{ij} + \pi_{ij}) + \bar{p} h_{ij}\right) dx^i dx^j$$

We can decompose the perturbations into scalar, vector, and tensor perturbations.

$$\delta u_i = \partial_i \delta u + \delta u_i^V$$

$$h_{ij} = a^2(A\delta_{ij} + \partial_i\partial_j B + \partial_i C_j^V + \partial_j C_i^V + h_{ij}^T)$$

$$\pi_{ij} = \partial_i\partial_j \pi^S + \partial_i \pi_j^V + \partial_j \pi_i^V + \pi_{ij}^T$$

General Relativity - Part II

In synchronous gauge

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$$h_{ij} = a^2(A\delta_{ij} + \partial_i\partial_j B + \partial_i C_j^V + \partial_j C_i^V + h_{ij}^T)$$

$$\pi_{ij} = \partial_i\partial_j \pi^S + \partial_i \pi_j^V + \partial_j \pi_i^V + \pi_{ij}^T$$

Rotational invariance of the background imply that these do not mix and we can study one at a time.

General Relativity - Part II

Note that we can only generate a gradient field from scalar perturbations, not a curl

scalar modes	$C_{TT,\ell}, C_{TE,\ell}, C_{EE,\ell}$
vector modes	$C_{TT,\ell}, C_{TE,\ell}, C_{EE,\ell}, C_{BB,\ell}$
tensor modes	$C_{TT,\ell}, C_{TE,\ell}, C_{EE,\ell}, C_{BB,\ell}$

If vector modes are not sourced, they rapidly decay

So B-mode detection is an indirect measurement of gravitational waves

Equations of motion

To work out the equations, we must first know what epoch we should begin our calculation.

Early enough for rapid thermalization, not so early that other degrees of freedom appear in the plasma

$6 \times 10^6 K < T < 10^9 K$ is convenient

In ΛCDM

e^- p He

γ dark matter

ν cosmological constant

Equations of motion

How do we describe the various components?

Electrons and protons elastically scatter very efficiently. They can be described as one "baryon" fluid.

For cold dark matter a hydrodynamic description is also sufficient because it is extremely non-relativistic, i.e. "dust".

Neutrinos free-stream, leading to anisotropic stress. They are usually described by a Boltzmann hierarchy.

If we are interested in the polarization of photons we have to keep track of it and describe them by a Boltzmann hierarchy.

Equations of motion

Toy example:

Perturbations in a thermal gas of massless particles

Instead of keeping track of the trajectories of all particles, we will describe it by the phase space density

$$n(\vec{x}, \vec{p}, t) \equiv \sum_r \delta(\vec{x} - \vec{x}_r(t))\delta(\vec{p} - \vec{p}_r(t))$$

Since

$$\frac{d\vec{x}_r}{dt} = \hat{p}_r \quad \text{and} \quad \frac{d\vec{p}_r}{dt} = 0$$

it satisfies a collisionless Boltzmann equation

$$\frac{\partial n}{\partial t} = -\hat{p} \cdot \nabla n$$

Equations of motion

Toy example:

Like for photons, temperature perturbations are related to intensity perturbations by

$$\Delta I_\nu(\hat{n}) = \left.\frac{d\bar{I}_\nu}{dT}\right|_{T_0} \Delta T(\hat{n})$$

A differential measurement sensitive to all frequencies probes

$$\int_0^\infty d\nu \Delta I_\nu(\hat{n}) = \frac{4\Delta T(\hat{n})}{T_0} \int_0^\infty d\nu \bar{I}_\nu$$

This makes it natural to define the "temperature" anisotropy

$$\Delta_T(\vec{x}, \hat{p}) = \frac{1}{\bar{\bar{I}}} \int \frac{p^3 dp}{(2\pi)^3} \delta n(\vec{x}, p\,\hat{p})$$

Equations of motion

Toy example:

It satisfies

$$\frac{\partial \Delta_T(\vec{x}, \hat{p}, t)}{\partial t} + \hat{p} \cdot \nabla \Delta_T(\vec{x}, \hat{p}, t) = 0$$

Translational invariance suggests to look for solutions

$$\Delta_T(\vec{x}, \hat{p}, t) = \int \frac{d^3 q}{(2\pi)^3} \alpha(\vec{q}) \Delta_T(q, \mu, t) e^{i\vec{q}\cdot\vec{x}}$$

$\hat{q} \cdot \hat{p}$

$$\frac{\partial \Delta_T(q, \mu, t)}{\partial t} + iq\mu \Delta_T(q, \mu, t) = 0$$

(Of course, the solution to this equation is trivial, but let's keep going)

Equations of motion

Toy example:

The temperature anisotropies at the origin at some time t_0 are

$$\frac{\Delta T(\hat{n})}{T_0} = \frac{1}{4}\Delta_T(\vec{x}=0, -\hat{n}, t_0)$$

and

$$a_{T,\ell\, m} = \pi i^\ell \int \frac{d^3q}{(2\pi)^3}\alpha(\vec{q})Y^*_{\ell\, m}(\hat{q})\Delta_{T,\ell}(q, t_0)$$

with $\Delta_{T,\ell}(q, t_0)$ defined by

$$\Delta_T(q, \mu, t_0) = \sum_\ell (-i)^\ell (2\ell+1)P_\ell(\mu)\Delta_{T,\ell}(q, t_0)$$

Equations of motion

Toy example:

This suggests to derive equations directly for

$$\Delta_{T,\ell}(q, t_0)$$

These equations are called the Boltzmann hierarchy

In our toy example

$$\dot{\Delta}_{T,\ell}(q,t) + \frac{q}{2\ell+1}\left[(\ell+1)\Delta_{T,\ell+1}(q,t) - \ell\Delta_{T,\ell-1}(q,t)\right] = 0$$

Analogous equations can be derived for the polarization anisotropy.

Equations of motion

Beyond the toy example

For interacting particles one finds

$$\frac{\partial \Delta_T(q,\mu,t)}{\partial t} + iq\mu\Delta_T(q,\mu,t) = -\omega\Delta_T(q,\mu,t) + \omega F\left[\Delta_{T,0}(q,t), \Delta_{T,2}(q,t), t\right]$$

with formal solution

$$\Delta_T(q,\mu,t) = \Delta_T(q,\mu,t_i)e^{-iq\mu(t-t_i)}e^{-\omega(t-t_i)}$$
$$+\omega\int_{t_i}^{t} dt e^{-iq\mu(t-t')}e^{-\omega(t-t')}F[\Delta_{T,0}(q,t), \Delta_{T,2}(q,t), t]$$

Since only low multipoles appear in the collision terms, one can solve a truncation of the hierarchy and obtain the higher multipoles through this "line-of-sight integration"

Equations of motion

Beyond the toy example

The same derivation generalizes to a general spacetime

In this case define the phase space density

$$n(x^i, p_i, t) \equiv \sum_r \delta(x^i - x_r^i(t))\delta(p_i - p_{i\,r}(t))$$

The definition of momentum and the geodesic equation imply

$$\frac{dx^i}{dt} = \frac{p^i}{p^0} \qquad\qquad \frac{dp_i}{dt} = \frac{p^k p^l}{2p^0}\frac{\partial g_{kl}}{\partial x^i}$$

and
$$\frac{\partial n}{\partial t} + \frac{p^k}{p^0}\frac{\partial n}{\partial x^k} + \frac{1}{2}\frac{p^k p^l}{p^0}\frac{\partial g^{kl}}{\partial x^m}\frac{\partial n}{\partial p_m} = C$$

Derivation of the Boltzmann hierarchy as before but more tedious.

Equations of motion

Photons

$$\dot{\Delta}^{(S)}_{T,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\Delta^{(S)}_{T,\ell+1}(q,t) - \ell\Delta^{(S)}_{T,\ell-1}(q,t)\right]$$
$$= -\omega_c(t)\Delta^{(S)}_{T,\ell}(q,t) - 2\dot{A}_q\delta_{\ell,0} + 2q^2\dot{B}_q\left(\frac{1}{3}\delta_{\ell,0} - \frac{2}{15}\delta_{\ell,2}\right)$$
$$+\omega_c\Delta^{(S)}_{T,0}\delta_{\ell,0} + \frac{1}{10}\omega_c\Pi\delta_{\ell,2} - \frac{4}{3}\frac{q}{a}\omega_c\delta u_{b\,q}\delta_{\ell,1}$$

$$\dot{\Delta}^{(S)}_{P,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\Delta^{(S)}_{P,\ell+1}(q,t) - \ell\Delta^{(S)}_{P,\ell-1}(q,t)\right]$$
$$= -\omega_c(t)\Delta^{(S)}_{P,\ell}(q,t) + \frac{1}{2}\omega_c(t)\Pi(q,t)\left(\delta_{\ell,0} + \frac{1}{5}\delta_{\ell,2}\right)$$

with source function

$$\Pi = \Delta^{(S)}_{P,0} + \Delta^{(S)}_{T,2} + \Delta^{(S)}_{P,2}$$

Equations of motion

Photons

$$\dot{\Delta}^{(S)}_{T,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\Delta^{(S)}_{T,\ell+1}(q,t) - \ell\Delta^{(S)}_{T,\ell-1}(q,t)\right]$$
$$= -\omega_c(t)\Delta^{(S)}_{T,\ell}(q,t) - 2\dot{A}_q\delta_{\ell,0} + 2q^2\dot{B}_q\left(\frac{1}{3}\delta_{\ell,0} - \frac{2}{15}\delta_{\ell,2}\right)$$
$$+\omega_c\Delta^{(S)}_{T,0}\delta_{\ell,0} + \frac{1}{10}\omega_c\Pi\delta_{\ell,2} - \frac{4}{3}\frac{q}{a}\omega_c\delta u_{b\,q}\delta_{\ell,1}$$

$$\dot{\Delta}^{(S)}_{P,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\Delta^{(S)}_{P,\ell+1}(q,t) - \ell\Delta^{(S)}_{P,\ell-1}(q,t)\right]$$
$$= -\omega_c(t)\Delta^{(S)}_{P,\ell}(q,t) + \frac{1}{2}\omega_c(t)\Pi(q,t)\left(\delta_{\ell,0} + \frac{1}{5}\delta_{\ell,2}\right)$$

with source function

$$\Pi = \Delta^{(S)}_{P,0} + \Delta^{(S)}_{T,2} + \Delta^{(S)}_{P,2}$$

Polarization sourced by temperature quadrupole

Equations of motion

Photons

The components of the stress tensor can be written as

$$
\begin{aligned}
\delta\rho_{\gamma\, q} &= \bar{\rho}_\gamma \Delta^{(S)}_{T,0}\,, \\
\delta p_{\gamma\, q} &= \frac{\bar{\rho}_\gamma}{3}\left(\Delta^{(S)}_{T,0} + \Delta^{(S)}_{T,2}\right)\,, \\
\delta u_{\gamma\, q} &= -\frac{3}{4}\frac{a}{q}\Delta^{(S)}_{T,1}\,, \\
q^2\pi^S_{\gamma\, q} &= \bar{\rho}_\gamma \Delta^{(S)}_{T,2}\,.
\end{aligned}
$$

At early times when Compton scattering is efficient

$$\Delta_{T,\ell} \to 0 \quad \text{for} \quad \ell \geq 2$$

$$\Delta_{P,\ell} \to 0$$

The Boltzmann hierarchy reduces to the equations of hydrodynamics

Equations of motion

Neutrinos

$$\dot{\Delta}^{(S)}_{\nu,\ell}(q,t) + \frac{q}{a(2\ell+1)}\left[(\ell+1)\Delta^{(S)}_{\nu,\ell+1}(q,t) - \ell\Delta^{(S)}_{\nu,\ell-1}(q,t)\right] = - 2\dot{A}_q\delta_{\ell,0} + 2q^2\dot{B}_q\left(\frac{1}{3}\delta_{\ell,0} - \frac{2}{15}\delta_{\ell,2}\right)$$

Baryons

Energy conservation

$$\delta\dot{\rho}_{b\,q} + \frac{3\dot{a}}{a}\delta\rho_{b\,q} - \frac{q^2}{a^2}\overline{\rho}_b\delta u_{b\,q} + \frac{1}{2}\overline{\rho}_b\left(3\dot{A}_q - q^2\dot{B}_q\right) = 0$$

Momentum conservation

$$\delta\dot{u}_{b\,q} + \frac{4}{3}\frac{\overline{\rho}_\gamma}{\overline{\rho}_b}\omega_c(t)\left(\delta u_{b\,q} + \frac{3}{4}\frac{a}{q}\Delta^{(S)}_{T,1}(q,t)\right) = 0$$

Equations of motion

Dark Matter

$$\delta\dot{\rho}_{c\,q} + \frac{3\dot{a}}{a}\delta\rho_{c\,q} + \frac{1}{2}\overline{\rho}_{c\,q}\left(3\dot{A}_q - q^2\dot{B}_q\right) = 0$$

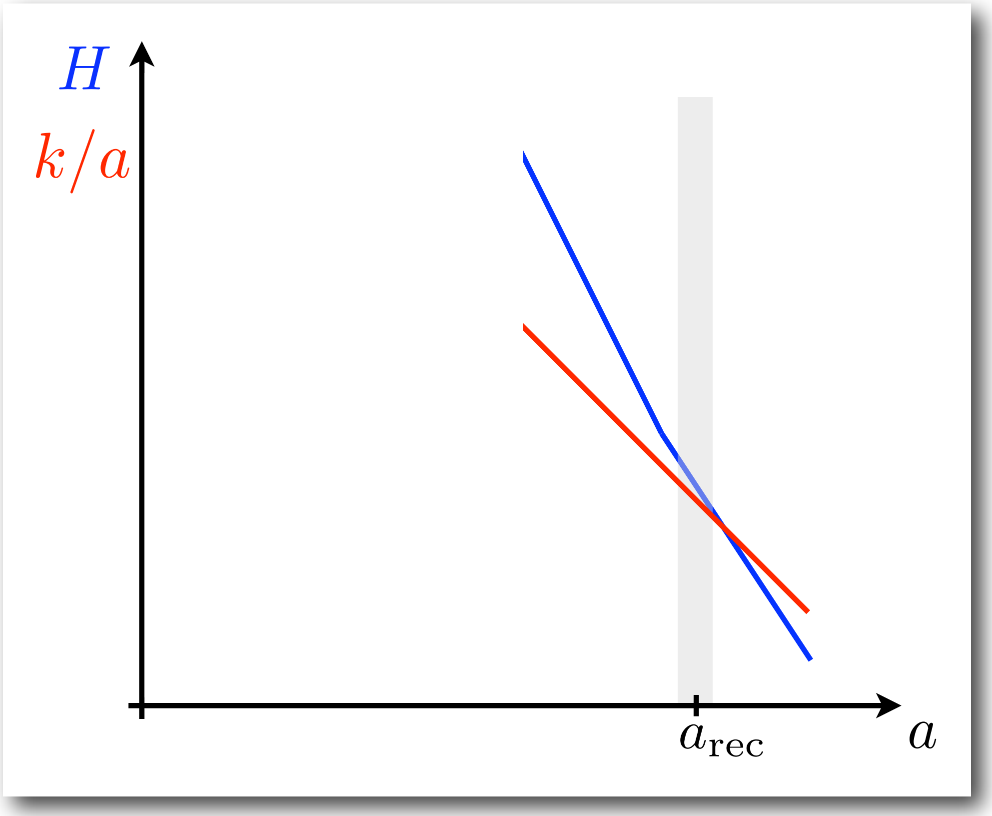
Scalar metric perturbations

$$\frac{q^2}{a^2}A_q + \frac{\dot{a}}{a}\left(3\dot{A}_q - q^2\dot{B}_q\right) = 8\pi G\left(\delta\rho_{q\,b} + \delta\rho_{q\,c} + \overline{\rho}_\gamma\Delta^{(S)}_{T,0} + \overline{\rho}_\nu\Delta^{(S)}_{\nu,0}\right)$$

$$\dot{A}_q = 8\pi G\left(\overline{\rho}_b\delta u_{b\,q} - \frac{a}{q}\overline{\rho}_\gamma\Delta^{(S)}_{T,1}(q,t) - \frac{a}{q}\overline{\rho}_\nu\Delta^{(S)}_{\nu,1}(q,t)\right)$$

Initial Conditions

What remains is the choice of initial conditions



All modes are "outside the horizon" at early times.

$$\frac{q}{a} \ll H$$

Initial Conditions

At early times the Boltzmann hierarchy for photons reduces to the equations of hydrodynamics

This suggests we can look for a solution of the form

$$\Delta_{T,0}^{(S)} = \Delta_{\nu,0}^{(S)} = \frac{4}{3}\frac{\delta\rho_c}{\overline{\rho}_c} = \frac{4}{3}\frac{\delta\rho_b}{\overline{\rho}_b} \equiv \Delta_0^{(S)}$$

$$\Delta_{\nu,1}^{(S)} \propto \Delta_{T,1}^{(S)} = -\frac{4}{3}\frac{q}{a}\delta u_{b\,q} \equiv \Delta_1^{(S)}$$

These are adiabatic initial conditions

Initial Conditions

In this limit $\mathcal{R}_q = \frac{A_q}{2} + H\delta u_q$ becomes a constant

and we can normalize our solution such that $\mathcal{R}_q \to \mathcal{R}_q^o$

Then during radiation domination

$$\Delta_0^{(S)}(q,t) = \frac{4}{3}\frac{q^2t^2}{a^2(t)}\mathcal{R}_q^o\,,$$

$$\Delta_1^{(S)}(q,t) = \frac{8}{27}\frac{q^3t^3}{a^3(t)}\mathcal{R}_q^o\,,$$

$$\Delta_{\nu,2}^{(S)}(q,t) = -\frac{16}{3(15+4f_\nu)}\frac{q^2t^2}{a^2(t)}\mathcal{R}_q^o\,,$$

$$A_q(t) = \left(2 - \frac{2}{3}\frac{5+4f_\nu}{15+4f_\nu}\frac{q^2t^2}{a^2(t)}\right)\mathcal{R}_q^o\,,$$

$$q^2\dot{B}_q(t) = \frac{20}{15+4f_\nu}\frac{q^2t}{a^2(t)}\mathcal{R}_q^o\,,$$

$$\Delta_{\nu,1}^{(S)}(q,t) = \frac{23+4f_\nu}{15+4f_\nu}\Delta_1^{(S)}(q,t)$$

Initial Conditions

These are the equations and initial conditions used by the Boltzmann codes such as CAMB or CLASS.

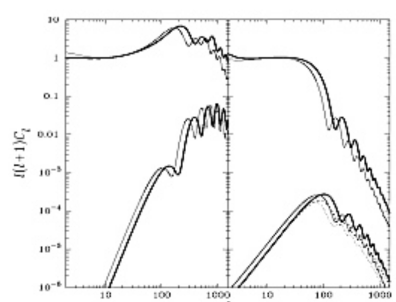
With the solution at hand, one computes

$$a_{T,\ell m}^{(S)} = \pi T_0 i^\ell \int d^3q \; \alpha(\mathbf{q}) Y_\ell^{m*}(\hat{q}) \Delta_{T,\ell}^{(S)}(q, t_0)$$

or directly

$$C_{TT,\ell}^{(S)} = \pi^2 T_0^2 \int q^2 dq \; \left| \Delta_{T,\ell}^{(S)}(q, t_0) \right|^2$$

similarly for polarization and tensor contribution



Code for Anisotropies in the Microwave Background

by Antony Lewis and Anthony Challinor

CLASS

the Cosmic Linear Anisotropy Solving System

From eV to Inflation

$$C_{XX,\ell}^{(S)} = 4\pi T_0^2 \int \frac{dk}{k} \Delta_{\mathcal{R}}^2(k) \left| \int_0^{\tau_0} d\tau S_X^{(S)}(k,\tau) j_\ell(k(\tau_0 - \tau)) \right|^2$$

From eV to Inflation

Initial Conditions

Late time evolution

$$C_{XX,\ell}^{(S)} = 4\pi T_0^2 \int \frac{dk}{k} \Delta_{\mathcal{R}}^2(k) \left| \int_0^{\tau_0} d\tau S_X^{(S)}(k,\tau) j_\ell(k(\tau_0 - \tau)) \right|^2$$

Physics of Recombination

Geometry

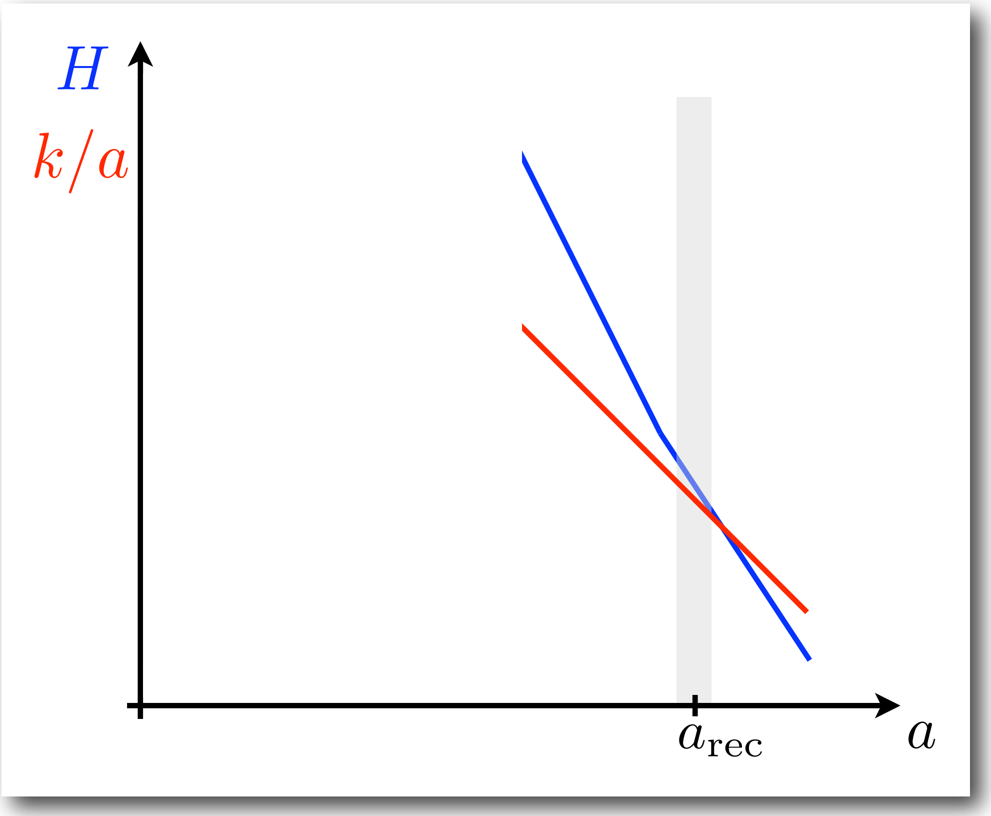
From eV to Inflation

So far, these are initial conditions for the system of equations that governs the evolution of the universe from around few keV to the present

In this limit, the system has 5 solutions that do not decay, one "adiabatic" solution and 4 "isocurvature" solutions. **(Bucher et al. 1999)**

Experimentally, only the adiabatic solution seems excited for which $\mathcal{R}$ is constant.

From eV to Inflation



We can extrapolate backwards very easily at least until the temperatures become high enough for new degrees of freedom to appear.

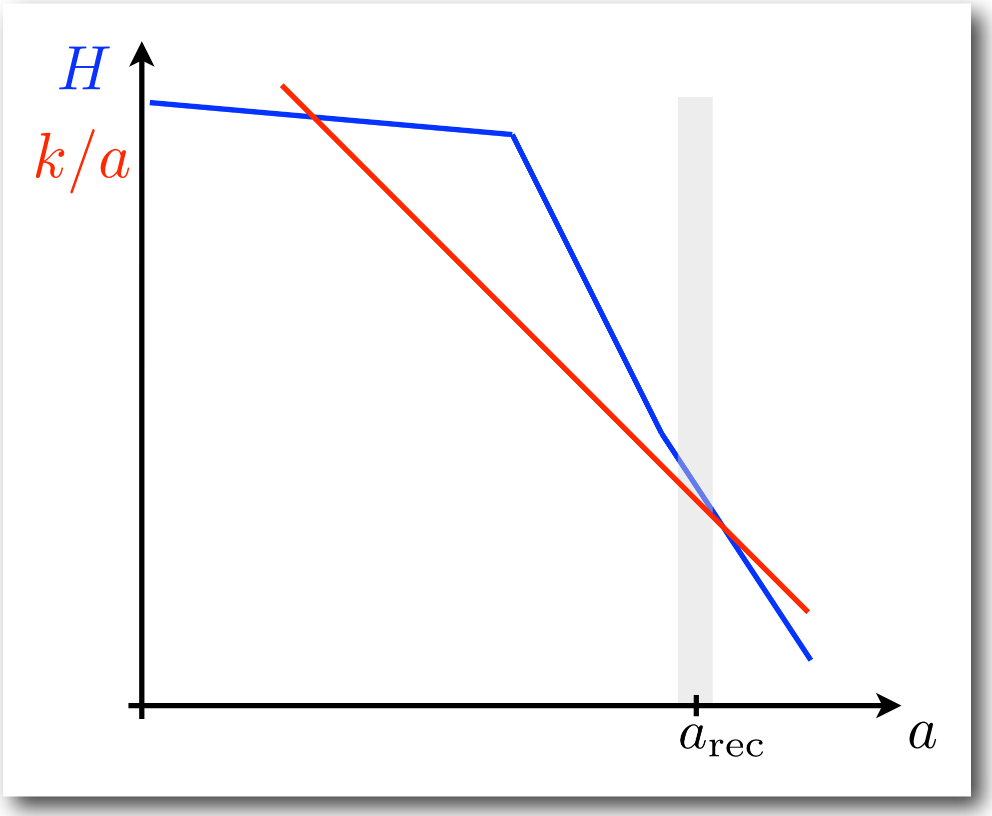
From eV to Inflation

Outside the horizon, this adiabatic solution with constant $\mathcal{R}$ exists not only for the matter content present below a few keV, but for a general matter content. **(Weinberg 2009)**

To generate the perturbations causally, they cannot have been outside the horizon very early on.
This requires a phase with

$$\frac{d}{dt}\left(\frac{q}{a|H|}\right) < 0 \qquad \text{(e.g. inflation or bounce)}$$

From eV to Inflation



The perturbations are generated as quantum fluctuations while inside the horizon, and then exit the horizon.

From eV to Inflation

There are two cases in which the solution with constant $\mathcal{R}$ is known to be an attractor:

- Single field inflation
- Phase of thermal equilibrium without conserved charges.

In single field inflation, the anisotropies in the CMB directly tell us about the inflationary dynamics!

From eV to Inflation

For standard single field slow-roll inflation, the primordial spectrum of scalar perturbations is

$$\Delta_{\mathcal{R}}^2(q) = \frac{H^2(t_q)}{8\pi^2\epsilon(t_q)} \approx \Delta_{\mathcal{R}}^2 \left(\frac{q}{q_*}\right)^{n_s-1}$$

with $$n_s = 1 - 4\epsilon_* - 2\delta_*$$

and $$\epsilon = -\frac{\dot{H}}{H^2} \qquad \delta = \frac{\ddot{H}}{2H\dot{H}}$$

and the 3-pt function too small to be observed.