Spectrum of the 2+1 black hole

Workshop of 3D Gravity
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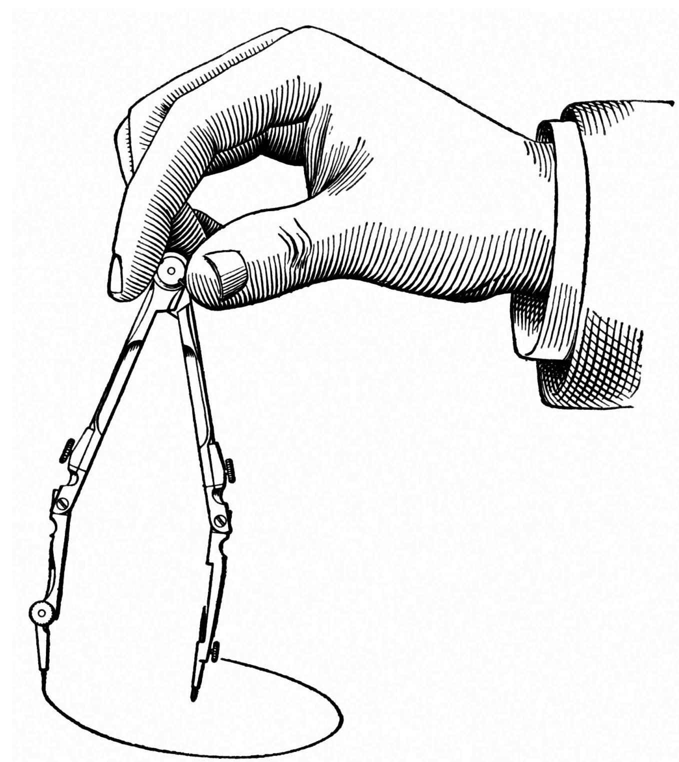
Jorge Zanelli (z@cecs.cl)
CECs –Valdivia (Chile)

I. What is Gravity?

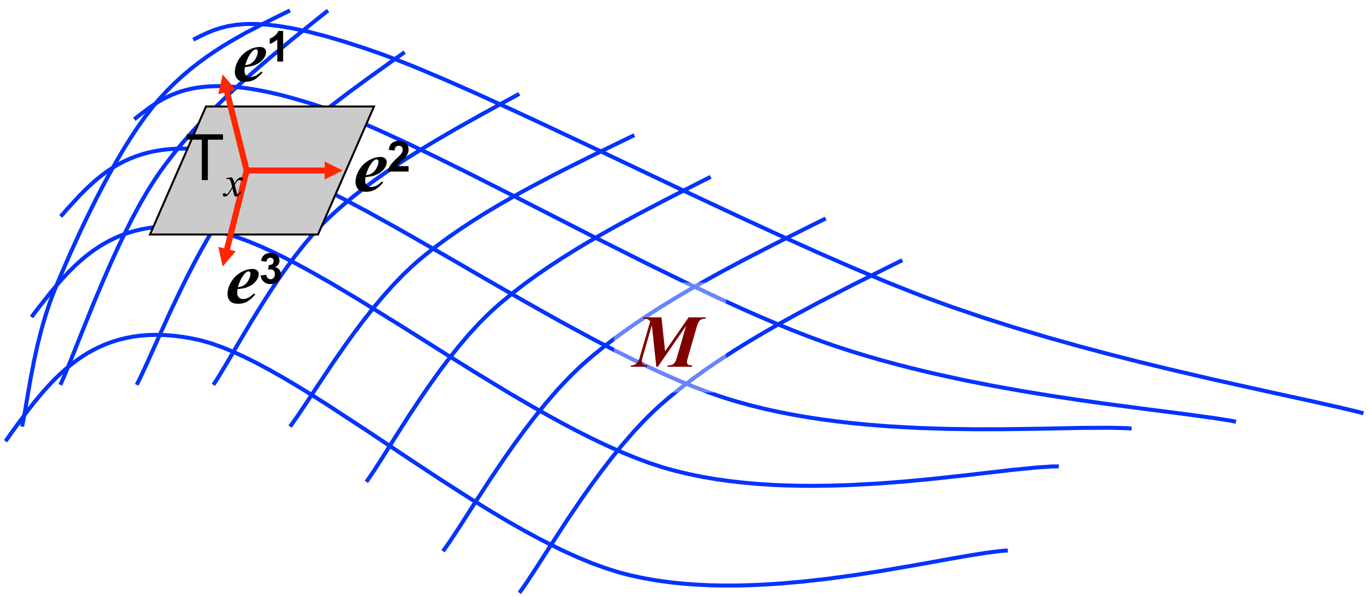
Geometry has two ingredients:

- Metric structure (length/area/volume, scale) $\longrightarrow e^a$
- Affine structure (parallel transport, congruence) $\longrightarrow \omega^a{}_b$



- Spacetime *M* is a differentiable manifold (up to isolated singularities in sets of zero-measure).

- *M* admits a tangent space T_x at each point.



- An open set around any point x is *diffeomorphic* to an open set in the tangent.

Metric Structure

- Each tangent space T_x is a copy of Minkowski space.
- Since Minkowski space is endowed with the Lorentzian metric, the diffeomorphism induces a metric structure on *M*:

$$dz^a = e^a_\mu(x)dx^\mu \equiv e^a$$

(e^a: vielbein, soldering form, local orthonormal frame).

$$\begin{aligned} ds^2 &= \eta_{ab}dz^a dz^b \\ &= \eta_{ab}e^a_\mu e^b_\nu dx^\mu dx^\nu \equiv g_{\mu\nu}dx^\mu dx^\nu \end{aligned}$$

Metric on *M*:

$$g_{\mu\nu}(x) = \eta_{ab}e^a_\mu(x)e^b_\nu(x)$$

- The vielbein can be viewed as the Jacobian matrix that relates differential forms in T_x and *M*.

$$dz^a = e^a_\mu(x)dx^\mu$$

- This is a mapping between differentials in two distinct spaces.

- It can also be viewed as the operator that relate tensors in each space, for example, the metric:

$$g_{\mu\nu}(x) = \eta_{ab}e^a_\mu(x)e^b_\nu(x)$$

M $\qquad$ T

- The metric is a *composite*
- e^a_μ is the square root of the metric, $e^a_\mu \sim \sqrt{g_{\mu\nu}}$

Equivalence Principle

A sufficiently small vicinity of any point of *M* can be accurately approximated by T_x .

- In a sufficiently small region of spacetime a reference frame can always be found in which the laws of physics are those of *special relativity* (*free fall*).

- The laws of physics are invariant under *local Lorentz transformations*.

- General Reativity is a nonabelian gauge theory for the group $SO(3,1)$ in 4*D*. (40 *years before Yang & Mills!*)

Affine Structure

- The differential operations appropriate for a dynamical theory compatible with GR should be *Lorentz covariant,*

$$D_\mu u^a = \partial_\mu u^a + \omega^a{}_{b\mu} u^b$$

where u^a is a Lorentz vector and $\omega^a{}_{b\mu}$ is the connection for the Lorentz group.

- This means that under Lorentz transformations, u^a and $D_\mu u^a$ transform in the same manner,

$$u^a \to u'^a = \Lambda^a{}_b u^b,\ \Lambda^a{}_b \in SO(D-1,1)$$

$$Du^a \to (Du^a)' = \Lambda^a{}_b (Du^b)$$

- The covariant derivative and the curvature 2-form are defined uniquely by the Lorentz connection 1-form

$$\omega^a = \omega^a{}_{b\mu} dx^\mu$$

- *Covariant derivative:*

$$Du^a = du^a + \omega^a{}_b u^a$$

- *Curvature:*

$$DDu^a = R^a{}_b u^b, \qquad R^a{}_b = d\,\omega^a{}_b + \omega^a{}_c \omega^c{}_b$$

- *Bianchi identity:*

$$D(R^a_b u^b) = R^a_b D u^b$$

or $$DR^a_b = dR^a_b + \omega^a_c R^c_b - \omega^a_c R^c_b \equiv 0$$

- Thus, the fundamental ingrdients for a dynamical theory of the spacetime geometry are:

e^a	*vielbein* (1-form)
$\omega^a{}_b$	*connection* (1-form)
$R^a{}_b = d\omega^a{}_b + \omega^a{}_c\omega^c{}_b$	*curvature* (2-form)
$T^a = de^a + \omega^a{}_c e^c$	*torsion* (2-form)
$\varepsilon_{a_1 a_2 \cdots a_D}, \quad \eta_{ab}$	*invariant tensors* (0-forms)

- *By taking successive derivatives no new functionally independent building blocks are produced:*

$$DR^a_b \equiv 0, \quad DT^a = R^a_b e^b, \quad (dd = 0).$$

- The most general D-dimensional Lorentz invariant Lagrangian built with these ingredients is a polynomial in e^a, $\omega^a{}_b$, and their exterior derivatives.

- We assume nothing about the invertibility of the vielbein, or the metric, and *do not include the Hodge **.

- The use of exterior derivatives guarantees that all field equations will be first order in e^a $\omega^a{}_b$.

- If torsion is set to zero, the equations will be at most second-order for the metric.

If no further ingredients or assumptions are added, there is a finite family of Lagrangians that can be constructed with these elements in each dimension.

II. What is 3D Gravity?

The most general Lagrangian for 3D gravity is a 3-form made out of e^a, $\omega^a{}_b$ and their exterior derivatives, and *quasi*-invariant under local Lorentz transformations:

$$I[e,\omega] = \int (\varepsilon_{abc}[R^{ab} + \tfrac{\alpha}{3} e^a e^b]e^c + \lambda e^a T_a + \gamma[\omega^a{}_b d\omega^b{}_a + \tfrac{2}{3}\omega^a{}_b \omega^b{}_c \omega^c{}_a])$$

The field equations read

$$R^{ab} + \Lambda e^a e^b = 0, \qquad T^a + \tau \varepsilon^{abc} e_b e_c = 0$$

Spacetimes of *constant* (Lorentz) *curvature* and *constant torsion*.

Splitting the connection:

$$\omega^a{}_b = \bar{\omega}^a{}_b + \kappa^a{}_b$$

Torsion-free part ($\bar{\omega}^a{}_b$)

Contorsion ($\kappa^a{}_b$)

where $D_{\bar{\omega}} e^a = de^a + \bar{\omega}^a{}_b e^b \equiv 0.$

Then:

$$T^a = \kappa^a{}_b e^b$$

$$R^a{}_b = \bar{R}^a{}_b + D_{\bar{\omega}} \kappa^a{}_b + \kappa^a{}_c \kappa^c{}_b$$

Lorentz curvature ($R^a{}_b$)

Riemann curvature ($\bar{R}^a{}_b$)

This is true for any dimension.

What does it mean for (2+1)-dimensional geometries?

In 2+1 dimensions,

$$R^{ab} + \Lambda_0 e^a e^b = 0, \qquad T^a + \tau \varepsilon^{abc} e_b e_c = 0$$

$$T^a = \kappa^a{}_b e^b \Rightarrow \kappa^a{}_b = \tau \varepsilon^a{}_{bc} e^c$$

$$\Rightarrow R^a{}_b = \bar{R}^a{}_b + \cancel{D_{\bar{\omega}} \kappa^a{}_b} + \kappa^a{}_c \kappa^c{}_b = \bar{R}^a{}_b + \tau^2 e^a e_b$$

It turns out that in a (2+1)-dimensional space of constant negative curvature *it is always possible to choose* $\omega^a{}_b = 0$ *in an open region*. (Lorentzian version of Adams theorem.)

$$\Rightarrow \Lambda_0 = 0, \quad R^a{}_b = 0, \quad \bar{R}^a{}_b = -\tau^2 e^a e_b$$

Locally AdS_3 spacetimes are Lorentz flat.

III. What is a 3D BH?

The 2+1 black holes are spherically symmetric, static (or stationary) spacetimes whose metrics satisfy

$$\bar{R}^{ab} - l^{-2} e^a e^b = 0$$

- They are labeled by two constants of integration: *mass* (*M*) and *angular momentum* (*J*).

- These spaces have the *same constant negative curvature* ($-l^{-2}$) for all values of *M* and *J*.

(N.B.: From now on we take $l=1$)

The 2+1 black hole metric

$(l=1)$

Static case

$J=0$

$$ds^2 = -(r^2 - M)dt^2 + \frac{dr^2}{(r^2 - M)} + r^2 d\phi^2$$

$$r_+^2 = M$$

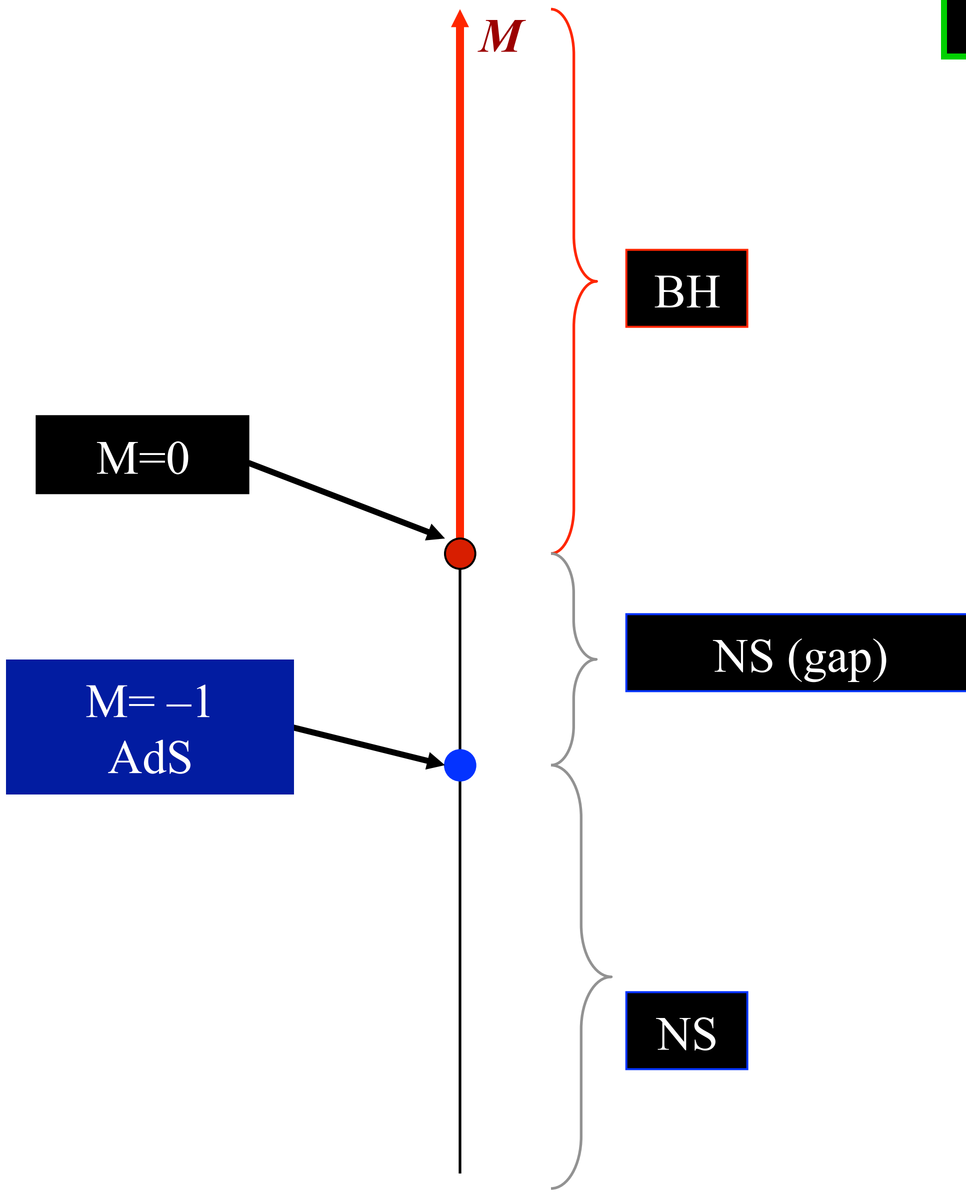
$M \geq 0$ → Black hole; horizon $r_+ = M^{1/2}$

$M = -1$ → AdS spacetime ($\Lambda = -1$)

$M<0$ → No horizon: *Naked Singularities*

2+1 BH spectrum

(J=0)



Spinning 2+1 black hole

$J \neq 0$ case

$$ds^2 = -f^2(r)dt^2 + \frac{dr^2}{f^2(r)} + r^2(Ndt + d\phi)^2$$

$$f^2 = -M + r^2 + \frac{J^2}{4r^2}$$

$$N = -\frac{J}{2r^2}$$

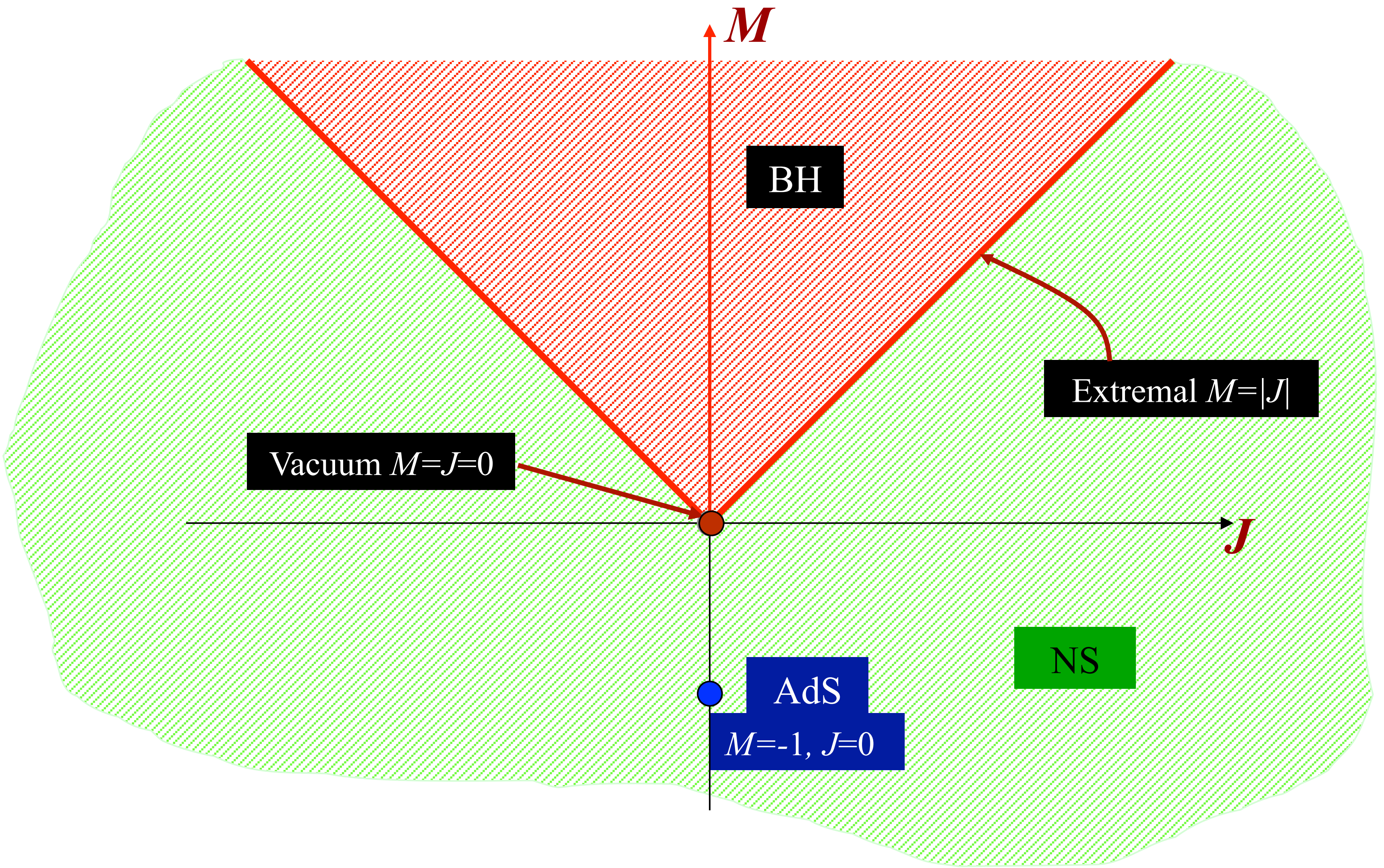
$$r_\pm^2 = \frac{M}{2}\left(1 \pm \sqrt{1 - \frac{J^2}{M^2}}\right) \in \mathrm{R}^+ \Leftrightarrow M \geq |J|$$

$M \geq |J| > 0$ → BH; horizons $r_\pm \geq 0$

$M = -1$, $J = 0$ → AdS

$M < |J|$, $\neq -1$ → Naked singularites

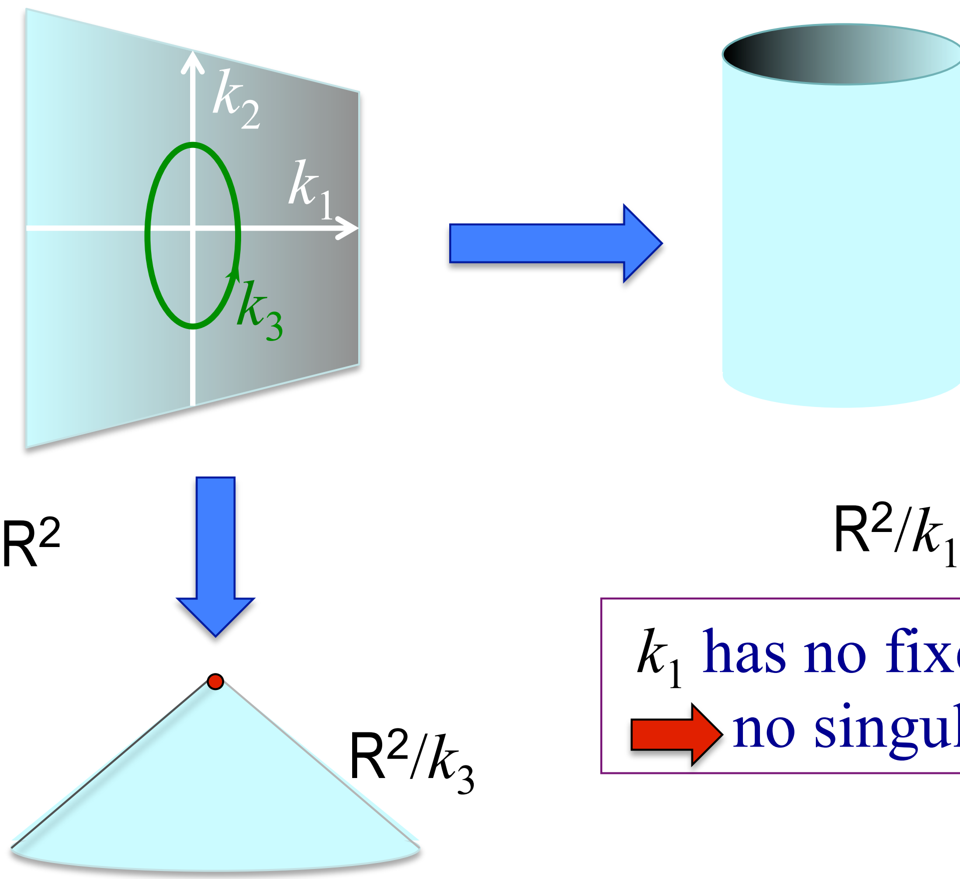
Spinning 2+1 black hole states



IV. How is the 3D BH possible?

Identifications by Killing vectors respect *local geometry*:

k_1, k_2, k_3: isometries



k_1 has no fixed points
➡ no singularities

$r=0$ fixed point of k_3
➡ conical singularity

- The BH geometries can be obtained by identifications in the covering AdS_{2+1}, defined by the pseudosphere

$$-(x^0)^2 - (x^1)^2 + (x^2)^2 + (x^3)^2 = -1$$

which has 6 Killing vectors:

$$k = \tfrac{1}{2} k^{ab} (x_a \partial_b - x_b \partial_a) = \tfrac{1}{2} k^{ab} J_{ab} \in so(2,2)$$

Therefore, the BH geometry is *locally* AdS but not globally so.

- The isometries of the resulting manifold (spatial rotations & time translations) correspond to the symmetry group of of the Killing vector used in the quotient.

- What Killing vectors are used in the identification?

Black hole identifications

AdS_{2+1}: $$-(x^0)^2-(x^1)^2+(x^2)^2+(x^3)^2=-1$$

- $k_{+-}=r_+J_{12}-r_-J_{03}$ Generic (spinning) bh, $r_+>r_-\geq 0$.
- $k_{Ext}=r_+(J_{01}-J_{23})+\frac{1}{2}(J_{12}+J_{03}+J_{02}-J_{13})$
  Extremal bh, $r_+=r_->0$, $M=J\neq 0$.
- $k_0=\frac{1}{2}(J_{12}+J_{03}+J_{02}-J_{13})$
  Zero mass bh, $r_+=r_-=0$, or $M=J=0$.

All of these are non-compact elements of SO(2,2)

- $k\cdot k>0$ → *No closed timelike curves*
- No fixed points → *No conical singularities*

Boosting black holes

The freedom to make Lorentz transformations in AdS_{2+1} can be exploited to change the identifying Killing vectors. The resulting black holes have different M and J .

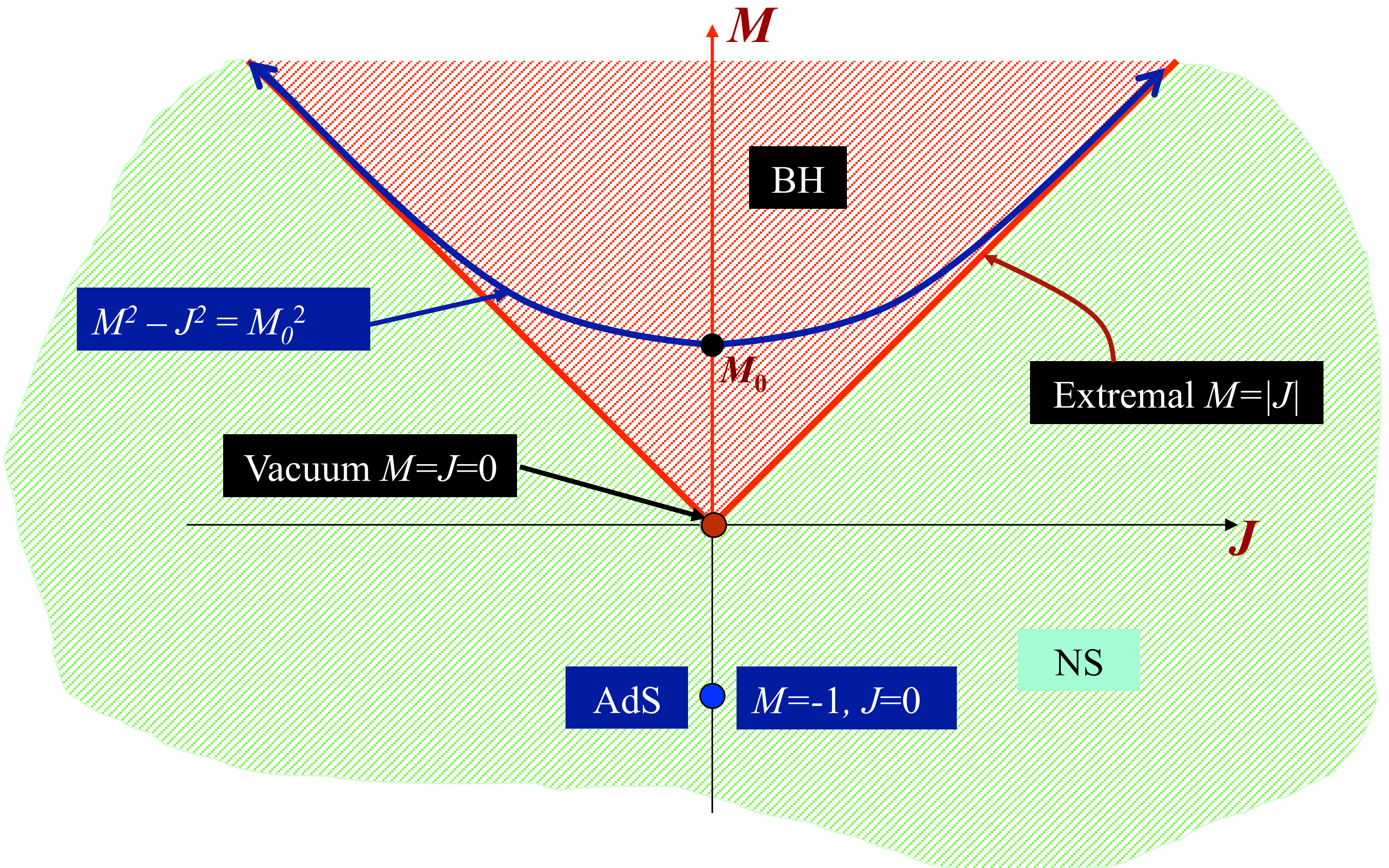
$$\frac{AdS_{2+1}}{k} \longrightarrow \frac{AdS_{2+1}}{k'}$$

In particular, a static BH ($M_0 \neq 0$, $J_0 = 0$) can turn into a spinning one ($M \neq 0$, $J \neq 0$) by a global Lorentz boost:

$$t = \frac{t_0 + \Omega\phi_0}{\sqrt{1-\Omega^2}}, \quad \phi = \frac{\phi_0 + \Omega t_0}{\sqrt{1-\Omega^2}}, \quad r^2 = r_0^2 + \frac{\Omega^2}{1-\Omega^2} M_0; \quad \Omega < 1$$

$$M = \frac{1+\Omega^2}{1-\Omega^2} M_0, \quad J = \frac{2\Omega}{1-\Omega^2} M_0, \quad \boxed{M^2 - J^2 = M_0^2}$$

Spinning 2+1 black hole states



V. Extended spectrum of the 3D BH

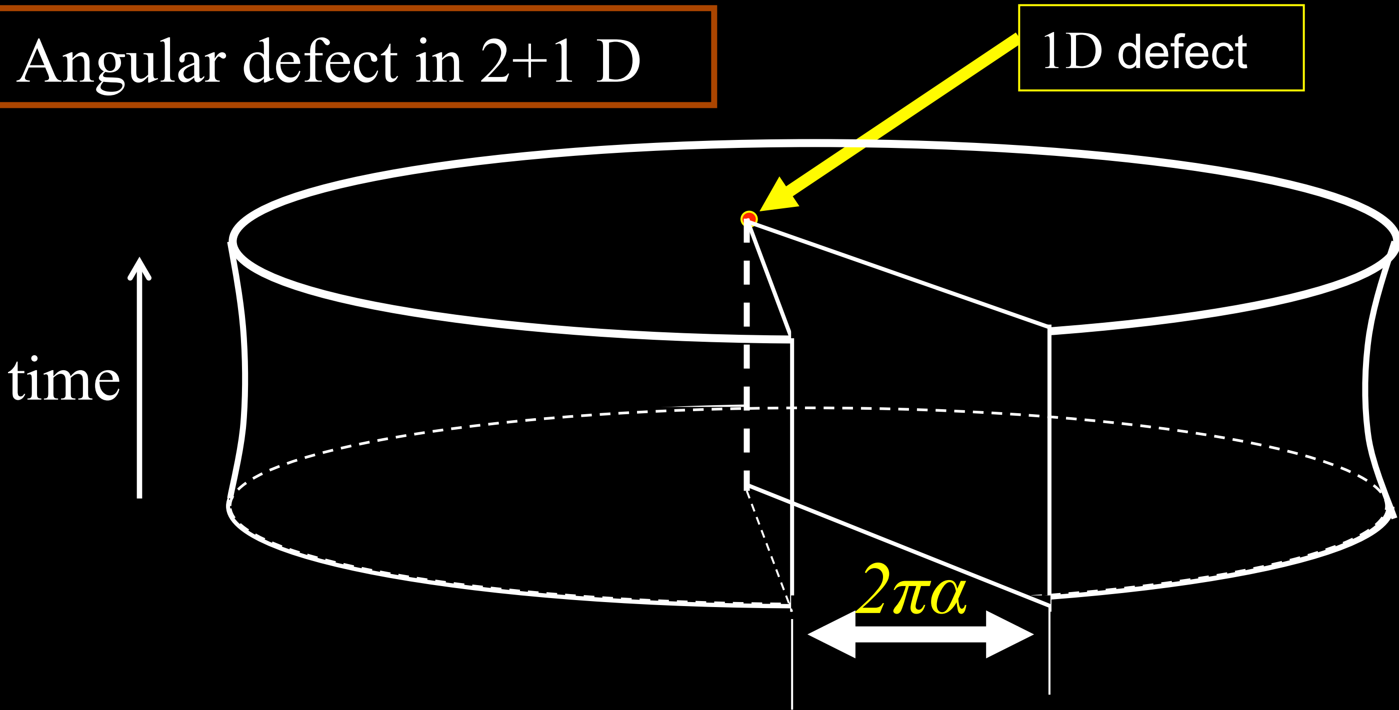
Topological defects are also spherically symmetric, static, localized, asymptotically *AdS* geometries.

They can also be produced by identifications with a spacelike Killing vector k is AdS_3.

If k belongs to the *compact* part of SO(2,2) (rotations) leaving *fixed points* ⇒ Naked conical singularity (conical defect)

The resulting quotient space (orbifold) AdS_3/k has less symmetry than AdS_3 due to the presence of the fixed points.

Angular defect in 2+1 D



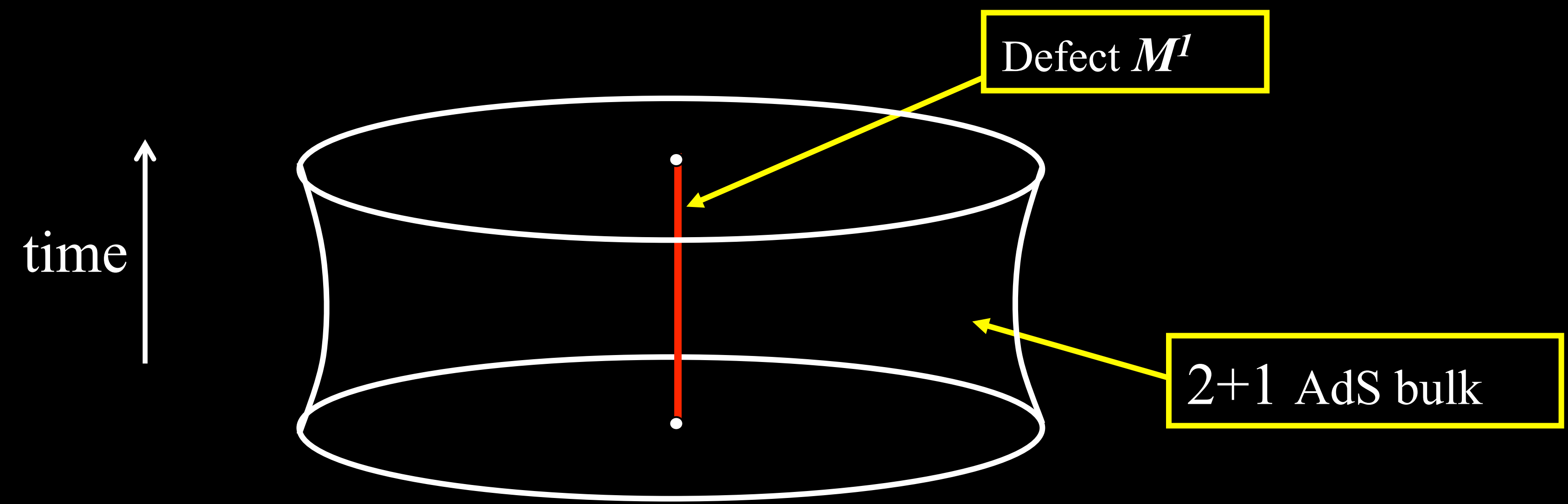
Identification in the x^1-x^2 plane generates a conical singularity at the set of fixed points

Killing vector: $k = -2\pi\alpha\,(x^1\partial^2 - x^2\,\partial^1)$

$$= -2\pi\alpha\,\partial_\varphi$$

Angular deficit ⟺ Curvature has δ-singularity

Conical singularity in 2+1 D



The defect is a conical singularity in AdS:

$$ds^2 = -\left(\rho^2+1\right)d\tau^2 + \left(\rho^2+1\right)^{-1} d\rho^2 + \rho^2 (1-\alpha)^2 d\varphi^2$$

and the curvature is constant *almost* everywhere

"Source"

$$\bar{R}^{ab} + e^a e^b = 2\pi\alpha\delta^{[ab]}_{[12]}\delta^{(2)}(x^1 x^2)dx^1 \wedge dx^2, \quad 0 \le \alpha \le 1$$

In appropriate coordinates, this looks like a black hole:

$$ds^2 = -\left(r^2 - M\right)d\tau^2 + \left(r^2 - M\right)^{-1}dr^2 + r^2 d\phi^2$$

But, with negative mass $M = -(1-\alpha)^2$, instead of an angular deficit $\Delta\phi = 2\pi\alpha$.

The exceptional cases are:

$\alpha = 0, M = -1$ → no deficit (AdS spacetime)

$\alpha = 1, \quad M = 0$ → maximum deficit (*Vacuum* bh)

For $-1 < M < 0$, these are ***naked*** but otherwise harmless conical singularities

… like any brane

- The identification produces a *0-brane* located at the fixed point of $k = -2\pi\alpha\,\partial_\varphi$

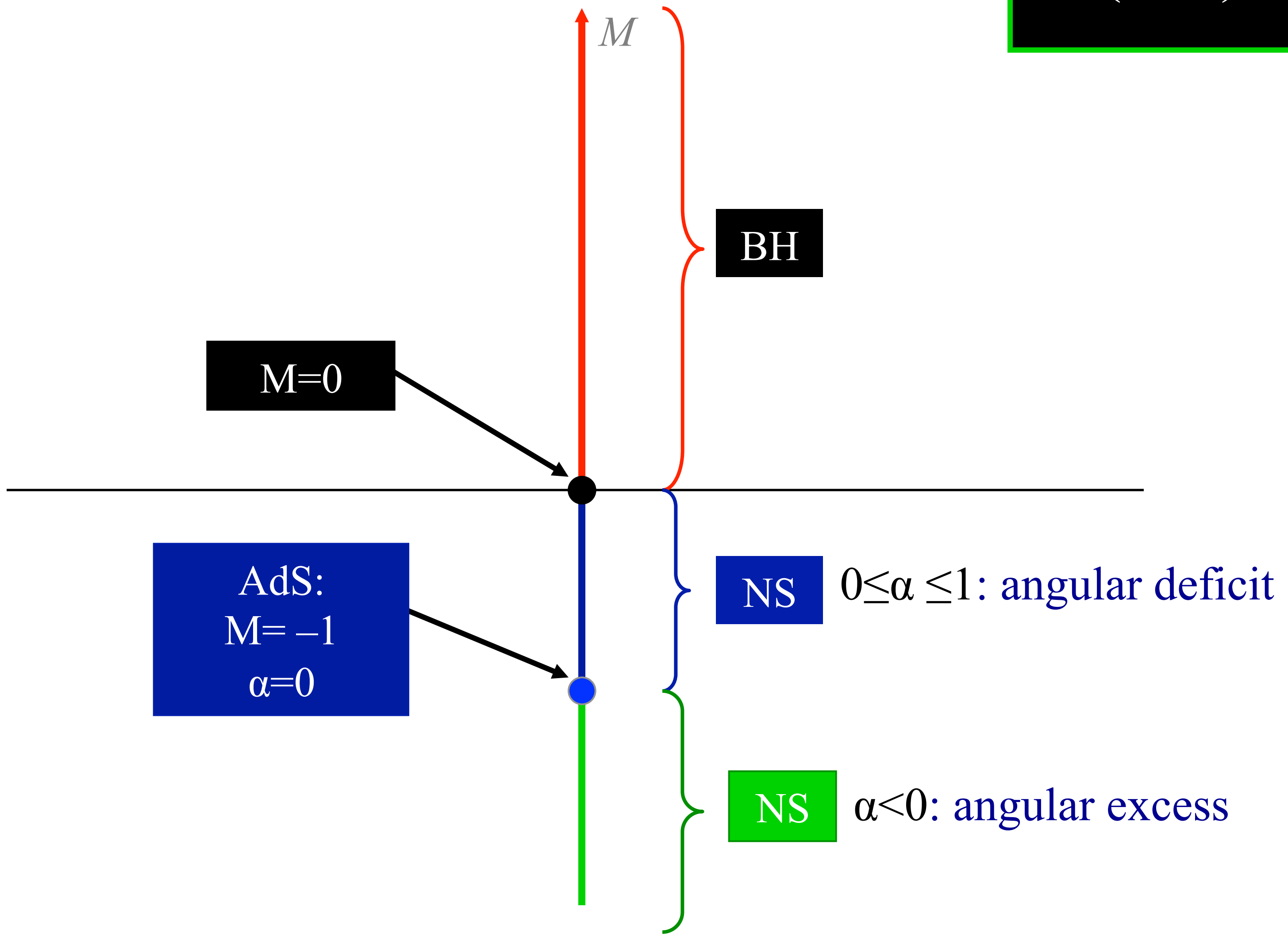
- There is a curvature singularity, proportional to the angular deficit $2\pi\alpha$,

$$\bar{R}^{ab} + e^a e^b = 2\pi\alpha\delta^{[ab]}_{[12]}\delta(\Sigma_{12})d\Omega^2_\Sigma$$

- This curvature singularity can be interpreted as a *point particle* of mass $m=(1-\alpha)^2$ (0-brane), or as the result of working in a spacetime of *nontrivial topology* from which some points have been removed.
- In both cases m measures the angular rotation of a parallel-transported vector along a loop around $r = 0$.

2+1 *BH-NS spectrum*

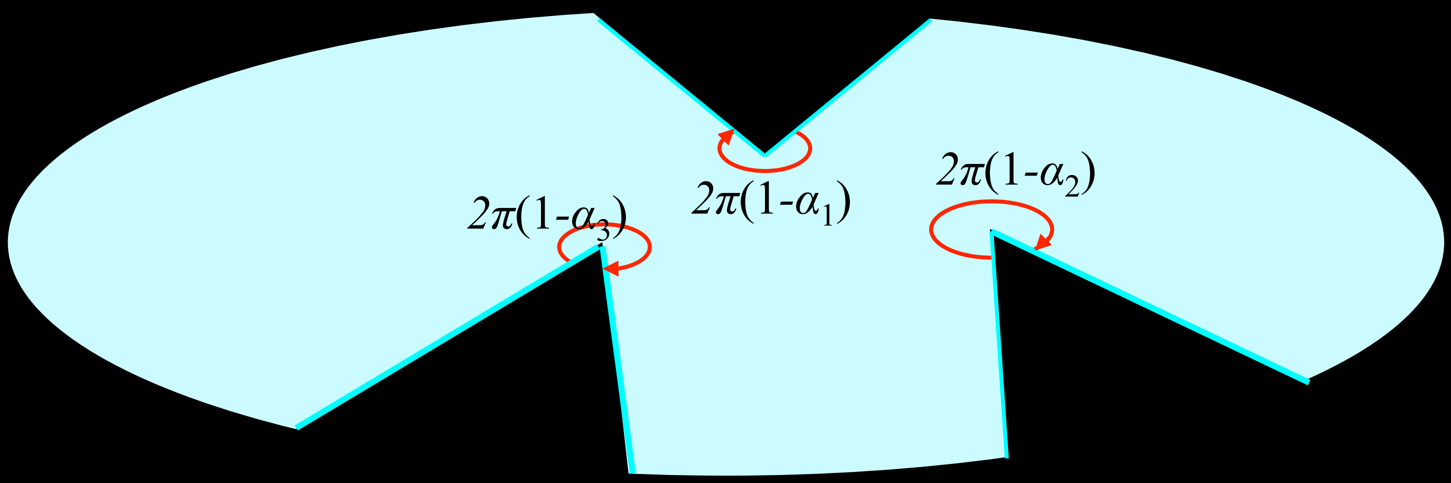
(*J*=0)



The angular deficits and excesses are additive: two conical singularities with deficits α_1 and α_2 can coalesce to form a new CS with deficit $\alpha_1 + \alpha_2$.

$\alpha > 0$	particles
$\alpha = 0$	vacuum
$\alpha < 0$	antiparticles

- Several singularities can be put together provided $\Sigma\alpha_i \leq 1$



Spinning case

These static particle states can also be boosted:

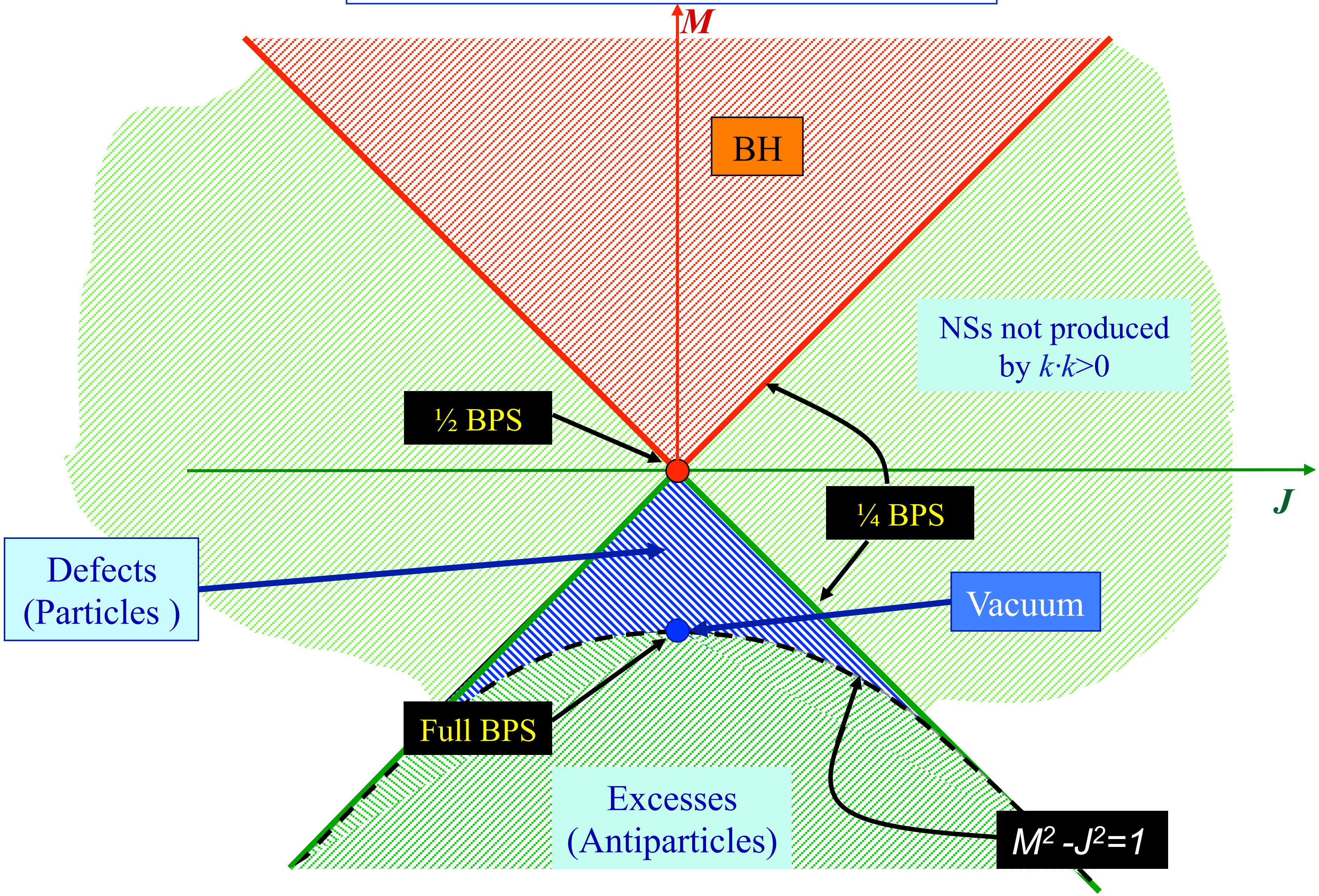
$$ds^2 = -f^2(r)dt^2 + \frac{dr^2}{f^2(r)} + r^2(Ndt + d\phi)^2$$

$$f^2 = -M + r^2 + \frac{J^2}{4r^2}, \quad N = -\frac{J^2}{2r^2}; \quad \boxed{M \leq -|J|}$$

- Spinning massive BHs and NSs admit globally defined Killing spinors for $M=J$.
- Those configurations correspond to bosonic supersymmetric, perturbatively stable, lowest energy states (BPS).

AdS vacuum:	$M = -1$	$\rightarrow$ full susy
0-mass BH/NS:	$M = J = 0$	$\rightarrow 1/2$ susy
Extremal BH/NS:	$\pm J = M < 0$	$\rightarrow 1/4$ susy

2+1 *BH-NS spectrum*



VI. Summary

- 3D black holes ($M \geq 0$) and conical singularities ($M < 0$) are quotients of AdS_3 by Killing vectors, AdS_3 / k where $$k = \tfrac{1}{2}\lambda^{ab} J_{ab}, \quad J_{ab} \in so(2,2), \;\; k \cdot k > 0$$
- The values of M and J are determined by the KV used in the identification.
- BHs & CSs are locally AdS_3 solutions of Einstein's equations.
- Any simply connected patch of their geometries is *Lorentz flat*, admitting global boosts. In particular, spinning BHs & CSs can be obtained boosting static states, with $M^2 - J^2 = M_0^2$.

- For $|M| = |J|$ BH & CS states are BPS [supersymmetric]

- Topological defects can also be endowed with abelian charges. If the electric charge and the angular deficit are balanced, these 0-branes are also BPS states.

- Similar $2p$-branes can also be constructed for $D \geq 2p+3$; They can also be charged and minimally coupled to nonabelian C-S forms.

- ....

References

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Thanks!