HIGHER GENUS PARTITION FUNCTIONS FROM THREE DIMENSIONAL GRAVITY

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MOTIVATION

ENTANGLEMENT AND GEOMETRY

Hints from holography: emergence of geometry is closely related to entanglement structure of CFT.

- Entropy and area: $S = \frac{A}{4G_N}$ *[Bekenstein-Hawking'80s][Ryu-Takayanagi '06]*
- Entanglement wedge hypothesis: CFT subregion encodes gravitational EFT in region up to minimal surface
- Consistency of entanglement restricts geometry and gravitational dynamics

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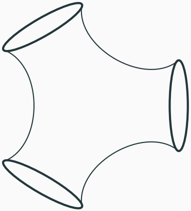
None of these address the qualitative structure of entanglement shared between many parties, e.g.

$$|W\rangle \propto |100\rangle + |010\rangle + |001\rangle \quad vs \quad |GHZ\rangle \propto |000\rangle + |111\rangle$$

Topologically nontrivial solutions to pure 3D gravity: multiboundary black holes *[Brill '95]*



Dual to entangled state on several copies of the CFT

$$|\Sigma\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$$

naturally defined in any theory by the path integral on a bordered Riemann surface Σ. *[Skenderis-van Rees '11]*

AdS dual is connected geometry only for some moduli.

E.g. thermofield double, Hawking-Page phase transition.

EDGES OF MODULI SPACE

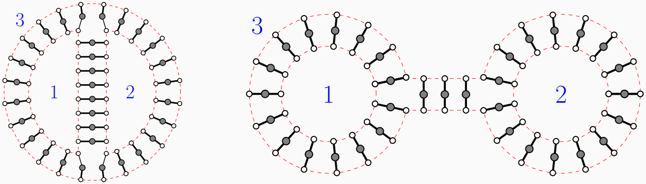
'Cold' limit *[Balasubramanian,Hayden,Maloney,Marolf,Ross '14]*

$$|\psi\rangle = \sum_{ijk} C_{ijk} e^{-\beta_1 H_1/2} e^{-\beta_2 H_2/2} e^{-\beta_3 H_3/2} |i\rangle_1 |j\rangle_2 |k\rangle_3$$

Dual: disconnected copies of AdS, entanglement is $O(c^0)$.

'Hot' limit *[Marolf,HM,Peach,Ross '15]*

Each region in local TFD, purified by some other region



Entanglement is local and bipartite. Dual: $\ell_{\text{horizons}} \gg \ell_{AdS}$

Wavefunction evaluated on field configuration ϕ computed by

$$\langle\phi|\Sigma\rangle = \int_{\Phi(\partial\Sigma)=\phi} \mathcal{D}\Phi e^{-I_\Sigma[\Phi]}$$

Norm $\langle\Sigma|\Sigma\rangle$ computed by inserting complete set of field configurations: path integral on Σ and a reflected copy, sewn along boundaries.

Calculates the partition function on ‘Schottky double’ X of Σ, so $\langle\Sigma|\Sigma\rangle = \mathcal{Z}(X)$ (generalise by inserting operators).

Phases come from dominance of different saddle point geometries in dual gravitational path integral for $\mathcal{Z}(X)$.

MOTIVATIONS

- Phase structure of geometric states
- Symmetry breaking and non-handlebodies *[Yin '07]*
- Computation of Rényi entropies *[Faulkner '13]*
- Universal (vacuum module) part of any CFT
- Mathematical: Kähler potential for Weil-Petersson metric on Teichmüller space *[Takhtajan-Zograf '88]*

PROBLEM AND SOLUTION

Partition function: may do the path integral on any geometry

$$\mathcal{Z}(X) = \int \mathcal{D}\Phi \, e^{-I_X[\Phi]}$$

Example: for $X = \text{space} \times S^1_\beta$, get $\mathcal{Z} = \sum_E e^{-\beta E}$

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For CFT, interesting dependence is on the conformal structure of X. In 2 dimensions, equivalent to complex structure, so X is naturally a Riemann surface.

Each CFT gives a function on moduli space of Riemann surfaces.

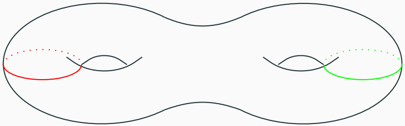
Holography: on-shell action of bulk $\mathcal{M}$ with boundary $\partial\mathcal{M} = X$.

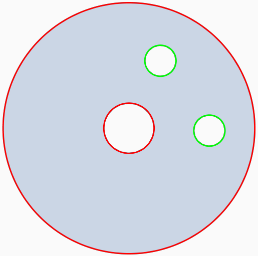
Possible to find solutions $\mathcal{M}$ with $\partial\mathcal{M} = X$ in 3D pure gravity because it's locally trivial: $\mathcal{M} = \mathbb{H}_3/\Gamma$ for $\Gamma \subseteq ISO(\mathbb{H}_3)$

- $ISO(\mathbb{H}_3) = SO(3,1) \equiv PSL(2,\mathbb{C})$
- Acts on boundary $\partial\mathbb{H}_3 = \mathbb{P}^1$ by Möbius maps $w \mapsto \frac{aw+b}{cw+d}$
- Need $X \approx \mathbb{P}^1/\Gamma$ as quotient of Riemann sphere

The appropriate construction is Schottky uniformisation

Cut $2g$ holes in the sphere and glue them in pairs with some Möbius maps $L_1, \ldots, L_g$. This makes a genus g surface:





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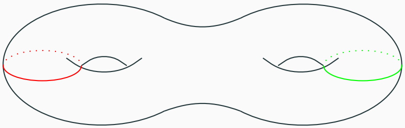

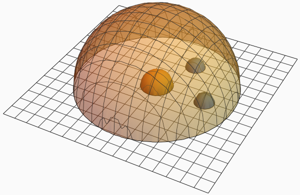

The action of L_i extends into $\mathbb{H}_3$. Fundamental region of bulk bounded by hemispheres, identified in pairs. (*Handlebodies*)

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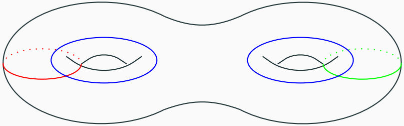

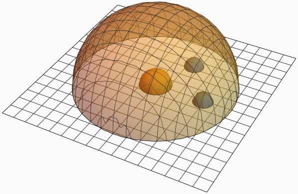

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Multiple solutions for any given Riemann surface boundary X: choice of g independent cycles to fill

Now evaluate action:

$$I = \frac{-1}{16\pi G_N}\left[\int_{\mathcal{M}} d^3x\sqrt{g}(R+2) + 2\int_{\partial\mathcal{M}} d^2x\sqrt{\gamma}(\kappa-1) + \text{constant}\right]$$

Divergent! Cutoff depends on choice of boundary metric

$$ds^2 = e^{2\phi(w,\bar{w})}dwd\bar{w} \implies \text{cutoff at } z = \epsilon\, e^{-\phi} + \cdots$$

Dependence on choice of metric gives the conformal anomaly:

$$\log\mathcal{Z}[e^{2\omega}\gamma] = \log\mathcal{Z}[\gamma] + \frac{c}{24\pi}\int d^2x\sqrt{\gamma}\left((\nabla\omega)^2 + \mathcal{R}\omega\right)$$

ACTION!

Canonical choice of metric: constant curvature $\mathcal{R} = -2$.

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Metric invariant under quotient group: for $L \in \Gamma$,

$$e^{2\phi(Lw)}d(Lw)d\overline{(Lw)} = e^{2\phi(w)}dwd\bar{w} \implies \phi(Lw) = \phi(w) - \frac{1}{2}\log\left|L'(w)\right|^2$$

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Multiple solutions for given X: ϕ helps to match moduli

1. Solve $\nabla^2\phi = e^{2\phi}$ on a fundamental region D for Γ
2. With boundary conditions $\phi(Lw) = \phi(w) - \frac{1}{2}\log|L'(w)|^2$
3. Match moduli by geodesic lengths in canonical metric
4. Evaluate on-shell action

$$I = -\frac{c}{24\pi}\int_D d^2w\,(\nabla\phi)^2 + \text{(boundary and constant terms)}$$

Action of [Takhtajan,Zograf '88], holography by [Krasnov '00]

ANALYTIC EXAMPLE: THE TORUS

A genus 1 Schottky group is generated by a single Möbius map, which we may choose to be $w \mapsto qw$, for $0 < |q| < 1$.

Canonical metric flat: ϕ harmonic, with $\phi(qw) = \phi(w) - \log|q|$

Solution: $\phi = -\log(2\pi|w|)$

$$ds^2 = e^{2\phi} dw d\bar{w} = \frac{dw d\bar{w}}{(2\pi)^2 w\bar{w}} = dz d\bar{z}$$

where $w = \exp(2\pi i z)$. Now z is identified as $z \sim z + 1 \sim z + \tau$, with $q = \exp(2\pi i \tau)$.

Evaluating action is straightforward: get $I = \frac{c}{12} \log|q|$

PHASES FOR THE TORUS

Different τ related by $PSL(2,\mathbb{Z})$ give the same complex structure, but different solutions.

As the moduli change smoothly, the dominant solution may change. First-order phase transitions at large c.

$$\log \mathcal{Z}(\tau) = 2\pi \frac{c}{12} \max \Im \left(\frac{a\tau + b}{c\tau + d} \right)$$

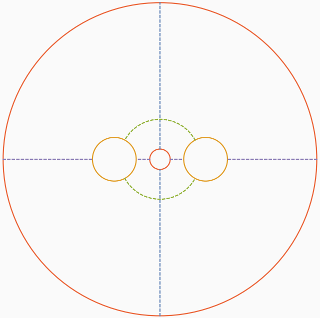
When $\tau = \frac{i\beta}{2\pi}$ is pure imaginary:

$$\log \mathcal{Z} = \frac{c}{12} \begin{cases} \beta & \beta \geq 2\pi \quad \textit{vacuum} \\ \frac{(2\pi)^2}{\beta} & \beta \leq 2\pi \quad \textit{Cardy} \end{cases}$$

This is the familiar Hawking-Page phase transition.

NUMERICAL SOLUTION

We need to solve Liouville's equation on this domain:



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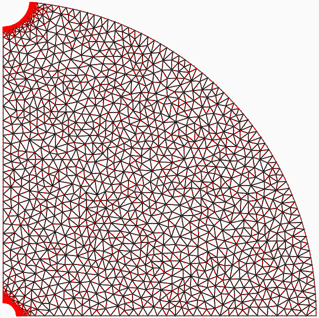
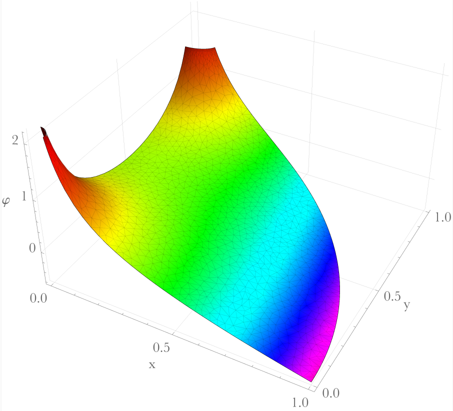
Nasty shaped region! Use finite element methods



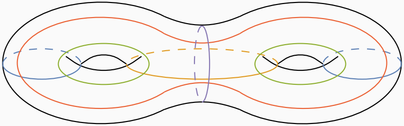
Approximate domain by triangles. Discretise the equation on these elements, and solve by Newton's method.

Solution for ϕ:



GENUS 2

Solve explicitly for a two-dimensional subspace of genus 2 moduli.

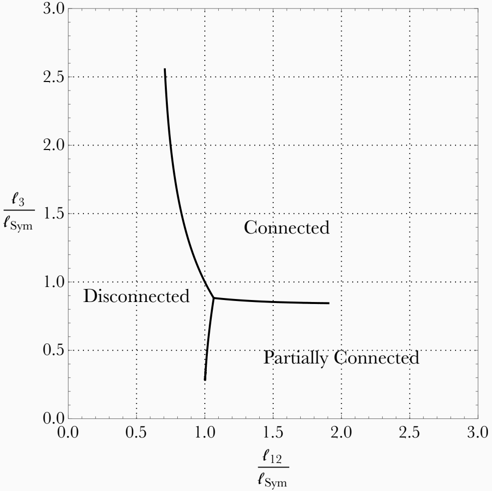


Corresponds to three-boundary wormhole with two equal horizon sizes $\lambda_1 = \lambda_2$. Use moduli ℓ_{12}, ℓ_3.

Conformal automorphisms $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Three phases: connected, disconnected (3×AdS), partially connected (AdS+BTZ)

[Same family of surfaces: single-exterior black hole with rectangular torus behind horizon; three different Rényi entropies]

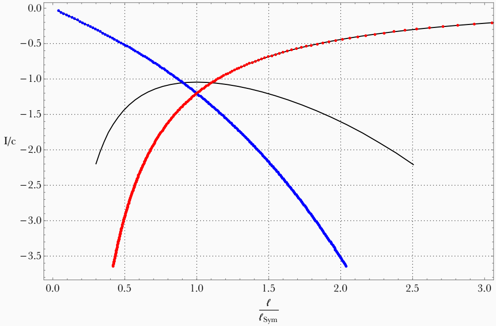


Enhanced symmetries: D_6 along line $\ell_1 = \ell_2 = \ell_3$, and D_4 at connected/disconnected phase boundary.

Modular transformation swaps connected and disconnected phases.

ACTION ON Z_3-SYMMETRIC LINE

Connected, disconnected, and symmetry breaking phases.



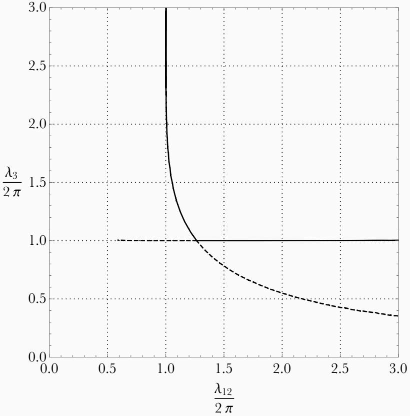
$I = 0$ corresponds to a non-handlebody *[Maldacena-Maoz '04]*

Transition point $\ell = \ell_{\text{Sym}} = 2\log(2+\sqrt{3})$ at surface of enhanced symmetry $y^2 = x^6 - 1$ (genus 2 analogue of $\tau = i$)

HORIZON SIZES

Horizon lengths λ_i easy to calculate from Schottky group data.

(In fact, all geodesic lengths/entanglement entropies *[HM '14]*)



At the edges of moduli space, $\lambda \to 2\pi$: pinching reduces to genus one case. Very good approximation! Useful perturbation expansion?

Torus wormhole: horizon length at least $\lambda \approx 22.3$, much larger than thermal state transition ($\lambda = 2\pi$).

WHAT WE'VE LEARNED

Can use genus 2 data to deduce certain genus 3 results in symmetric situations.

For example $\mathbb{Z}_4$-symmetric four boundary wormhole has transition for horizons $\lambda \approx 7.62$.

Can deduce that 'intrinsically 4-party entanglement' *[BHMMR'14]* must exist, if internal moduli unimportant.

But internal moduli become relevant for $\lambda \gtrsim 2.12$!

- Genus 2 phase diagram mapped out
- Some analytic results from symmetries
- No spontaneous (replica) symmetry breaking
- All handlebodies appear to dominate non-handlebodies
- Pinching limits checked
- Perturbation around pinching may be very useful
- Phase transitions put strong constraints on geometries and possible structure of entanglement