Supertranslations and superrotations

Geoffrey Compère

Université Libre de Bruxelles (ULB)

Workshop on Topics in Three Dimensional Gravity,
Trieste, March 22, 2016

Credits

- “Vacua of the gravitational field”,
  G.C., J. Long, arXiv :1601.04958
- “Classical static final state of collapse with supertranslation memory,
  G.C., J. Long, arXiv :1602.05197

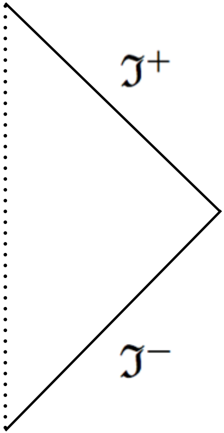
with inspiration from

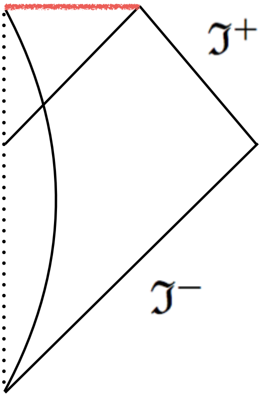
- “Aspects of the BMS/CFT correspondence”,
  G. Barnich, C. Troessaert, arXiv :1001.1541
- “Gravitational Memory, BMS Supertranslations and Soft Theorems, A. Strominger, A. Zhiboedov, arXiv :1411.5745

On this talk :

- Fascinating properties and algebra of symmetries of asymptotically flat spacetimes
- $4d \Rightarrow 3d \Rightarrow 4d$. Many lessons can be drawn from $3d$ to help understand $4d$ physics.
- Interplay between various concepts : asymptotic symmetries, gravitational memory, holography, black holes
- Tackle classical problems : gravitational collapse, cosmic censorship, black hole information paradox

Asymptotically flat spacetimes





No black hole in $3d$ Einstein-positive matter theory. [Ida, 2000]

Preambule : the BMS_3 and BMS_4 groups

The space of solutions to Einstein gravity with "reasonable" asymptotically flat boundary conditions can be expanded close to null infinity in a fixed gauge.

$$\begin{aligned} ds^2 &= -du^2 - 2dudr + r^2 d^2\Omega + \dots \\ &= -dv^2 + 2dvdr + r^2 d^2\Omega_{antipodal} + \dots \end{aligned}$$

The group of diffeomorphisms which

- preserve the form of the asymptotic metric, mapping one metric to another but preserving the gauge,
- are associated with finite and non-trivial canonical charges

is the asymptotic symmetry group.

Using "reasonable" boundary conditions, the asymptotic symmetry group was found to be the BMS_4 group in $4d$ [Bondi, van der Burg, Metzner, 1962] [Sachs, 1962] and the BMS_3 group in $3d$ [Ashtekar, Bicak, Schmidt, 1996]

What reasonable boundary conditions may mean ?

4d

- Admit Kerr, gravitational waves and electromagnetic fields
- Positive energy
- Allow to describe memory effects [Zeldovich, Polnarev, 1974] [Christodoulou, 1991]
- Allow to describe a semi-classical S-matrix which obeys all known theorems [Weinberg, 1965] [Cachazo, Strominger, 2014]
- Allow for small perturbations to decay (non-linear stability) [Christodoulou, Klainerman, 1993]

3d

- Admit “appropriate” matter fields
- Positive energy
- Flat region can be embedded in AdS_3

A translation in Minkowski spacetime

- (t, x, y, z)

$$\partial_z$$

- (t, r, θ, ϕ)

$$\cos\theta\partial_r - \frac{1}{r}\sin\theta\partial_\theta$$

- (u, r, θ, ϕ), retarded time $u = t - r$

$$-\cos\theta\partial_u + \cos\theta\partial_r - \frac{1}{r}\sin\theta\partial_\theta$$

The bms_4 algebra

$$bms_4 \simeq so(3,1) \oplus \text{Supertranslations}$$

Supertranslations are either translations or pure supertranslations. Pure supertranslations are (abelian) "higher harmonic angle-dependent translations"

$$T(\theta,\phi)\partial_u + \frac{1}{2}\nabla^2 T \partial_r - \frac{1}{r}(\partial_\theta T \partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi T \partial_\phi) + \dots$$

The solutions to $\nabla^2(\nabla^2+2)T = 0$ are the translations. Those are the $\ell = 0$ and $\ell = 1$ spherical harmonics, $T = 1$, $T = \cos\theta$, $T = \sin\theta\cos\phi$, $T = \sin\theta\sin\phi$.

What are supertranslations in the bulk ?

The extended bms_4 algebra

[Barnich, Troessaert, 2010]

$$bms_4 \simeq \text{Superrotations}^* \oplus \text{Supertranslations}^*$$

where

$$\begin{aligned} \text{Superrotations}^* &\simeq \text{Vir}^* \oplus \text{Vir}^*, \\ \text{Supertranslations}^* &\simeq \text{Regular supert.} \oplus \text{Meromorphic supert.} \end{aligned}$$

The Lorentz subalgebra

$$so(3,1) \simeq sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R}) \subset \text{Vir}^* \oplus \text{Vir}^*$$

is generated by global conformal transformations on the sphere. The rest of the algebra has generators which contain meromorphic functions, with poles on S^2.

The extended bms_4 algebra : comments

The algebra is not realized as asymptotic symmetry algebra, at least in the standard sense :

- The Kerr black hole has infinite meromorphic supertranslation momenta. [Barnich, Troessaert, 2010]
- Minkowski acted upon with a finite superrotation diffeomorphism has negative energy. [G.C., Long, 2016]

The superrotations still have a role to play :

- Superrotation charges are finite and can be non-trivial [Barnich, Troessaert, 2011] [Flanagan, Nichols, 2015] [G.C., Long, 2016]
- The subleading soft graviton theorem has been related to the Ward identity of the superrotation algebra [Kapec, Lysov, Pasterski, Strominger, 2014] [Campiglia, Laddha, 2015]

The bms_3 algebra

In $3d$: Poincaré $\simeq so(2,1) \oplus \mathbb{R}^3$. The Poincaré algebra is

$$\begin{aligned} i[R_m, R_n] &= (m-n)R_{m+n}, \\ i[R_m, T_n] &= (m-n)T_{m+n}, \\ i[T_m, T_n] &= 0, \qquad m, n = -1, 0, 1. \end{aligned}$$

1+2 Translations $T_0 = \partial_t$; $T_1 + T_{-1} = \partial_x$, $i(T_1 - T_{-1}) = \partial_y$

1+2 Lorentz transformations $R_0 = \partial_\phi$; $R_1 + R_{-1}$, $i(R_1 - R_{-1})$

The algebra can be promoted as an asymptotic symmetry algebra of asymptotically flat spacetimes, for $n \in \mathbb{Z}$:

$$\begin{aligned} bms_3 &\simeq Superrotations\,(R_n) \oplus Supertranslations\,(T_n) \\ &\simeq \quad \text{Virasoro} \quad \oplus \quad \widehat{u(1)} \end{aligned}$$

[Ashtekar, Bicak, Schmidt, 1996] [Barnich, G.C., 2007]

The BMS_3 group is $\mathrm{Diff}(S^1) \ltimes \mathrm{Vect}(S^1)$ [Barnich, Oblak, 2014].

The bms_3 algebra : comments

Limit from Brown-Henneaux In large $\ell \to \infty$ limit, $AdS_3 \to Mink_3$. The exact symmetries are contracted as $so(2,2) \to iso(2,1)$. The asymptotic symmetries with Brown-Henneaux/Dirichlet boundary conditions are contracted as

$$\mathrm{Vir} \bigoplus \mathrm{Vir} \to Superrotations \bigoplus Supertranslations$$

[Barnich, G.C., 2007]

Isomorphism The bms_3 algebra is also isomorphic to the infinite-dimensional extension of the $2d$ Galilean conformal algebra. [Bagchi, Gopakumar, 2009]

4d supertranslations and memories

After the passage of either gravitational waves or null matter between two detectors placed in the asymptotic null region, the detectors generically acquire a finite relative displacement and a finite time shift.

This is the *memory effect*. Historically, it is refered to as the linear memory effect for null matter [Zeldovich, Polnarev, 1974] and the non-linear memory or Christodoulou effect for gravitational waves [Christodoulou, 1991].

Memory effects follow from the existence of the supertranslation field $C(\theta, \phi)$ which is effectively shifted by a supertranslation after the passage of radiation as [Geroch, Winicour, 1981]

$$\delta_T C(\theta, \phi) = T(\theta, \phi).$$

Memory effects are a 2.5PN General Relativity effect. [Damour, Blanchet, 1988]

Memory effects cannot be detected by LIGO.

More precisely, supertranslation memories follow from an angle-dependent energy conservation law deduced from Einstein's equations integrated over a finite retarded time interval of $\mathfrak{I}^+$: [Strominger, Zhiboedov, 2014]

$$-\frac{1}{4}\nabla^2(\nabla^2+2)(C|_{u_2}-C|_{u_1}) = m|_{u_2} - m|_{u_1} + \int_{u_2}^{u_1} du T_{uu},$$

$$T_{uu} \equiv \frac{1}{4} N_{zz}N^{zz} + 4\pi G \lim_{r\to\infty}[r^2 T^{matter}_{uu}].$$

The supertranslation shift can be constructed from the radiation flux history. It allows to compute the shift of the geodesic deviation vector s^A, $A=\theta,\phi$

$$s^A|_{u_2} - s^A|_{u_1} \sim \frac{1}{r}\partial^A\partial_B(C|_{u_2}-C|_{u_1})s^B$$

This is a classical effect of Einstein gravity, $O(\hbar^0)$.

What is the supertranslation field in the bulk ?

In 3d, part of the answer is the phase space of analytic solutions to vacuum Einstein gravity with Dirichlet boundary conditions : [Barnich, Troessaert, 2010]

$$ds^2 = \Theta(\phi)du^2 - 2dudr + 2\left(\Xi(\phi) + \frac{u}{2}\partial_\phi\Theta(\phi)\right)dud\phi + r^2 d\phi^2.$$

The transformation laws of $\Theta(\phi)$ under bms_3 is

$$\delta_{T,R}\Theta = R\partial_\phi\Theta + 2\partial_\phi R\Theta - 2\partial_\phi^3 R$$

This is the coadjoint representation of the Virasoro algebra.

We deduce that $\Theta(\phi)$ is the superrotation field itself plus a zero mode. The zero mode is the mass (a conical defect). In order to concentrate on the supertranslation field, we set

$$\Theta = -1 \quad \text{(no conical defect)}.$$

This sets to the supertranslation charge to 0 (rest frame).

The transformation law of $\Xi(\phi)$ under a supertranslation is then

$$\delta_T \Xi = -\partial_\phi T - \partial_\phi^3 T.$$

We deduce that $\Xi(\phi)$ is a composite field in terms of the supertranslation field $C(\phi)$ plus a zero mode

$$\Xi(\phi) = 4GJ - \partial_\phi(1+\partial_\phi^2)C, \qquad \delta_T C = T.$$

The zero mode is attributed to the spin of a massless particle. It creates a dislocation responsible for closed timelike curves. So we set $J = 0$. The metric becomes

$$ds^2 = -du^2 - 2dud(r + C(\phi) + \partial_\phi^2 C(\phi)) + r^2 d\phi^2.$$

$$ds^2 = -du^2 - 2dud(r + C(\phi) + \partial_\phi^2 C(\phi)) + r^2 d\phi^2.$$

We switch to static coordinates $\rho = r + \partial_\phi^2 C(\phi) + C(\phi) - C_{(0)}$, $t = u + \rho$.

The shift of C by its zero mode ensures that the space coordinate ρ is not affected by time shifts.

The metric becomes [G.C., Long, 2016]

$$ds^2 = -dt^2 + d\rho^2 + (\rho - \rho_{SH}(\phi))^2 d\phi^2.$$

In the rest frame, supertranslations only act spatially, except the zero mode which is a time translation.
Coordinates break down at the *supertranslation horizon*

$$\rho = \rho_{SH}(C) \equiv \partial_\phi^2 C(\phi) + C(\phi) - C_{(0)}.$$

The metric describes Poincaré vacua

The BMS_3 conserved charges are

$$\begin{aligned}
\mathcal{Q}_T &= 0 \qquad \text{(No momenta)} \\
\mathcal{Q}_R &= \int_0^{2\pi} d\phi R(\phi)\partial_\phi \rho_{SH} \\
&= \int_0^{2\pi} d\phi R(\phi)\partial_\phi \left(\partial_\phi^2 C(\phi) + C(\phi) - C_{(0)}\right) \\
&= -\int_0^{2\pi} d\phi C(\phi)(\partial_\phi^2 + 1)\partial_\phi R(\phi) \quad \text{(No Lorentz charges).}
\end{aligned}$$

$\Rightarrow$ All Poincaré charges are zero.

Superrotation charges are non-zero and characterize the supertranslation field 1-to-1.

$\Rightarrow$ Obstruction at shrinking circle. Existence of a defect.

Finite supertranslation diffeomorphism

The solution with supertranslation field is diffeomorphic to Minkowski spacetime.

$$ds^2 = -dt^2 + dx_s^2 + dy_s^2 = -dt^2 + d\rho^2 + (\rho - \rho_{SH}(\phi))^2 d\phi^2$$

The finite diffeomorphism is

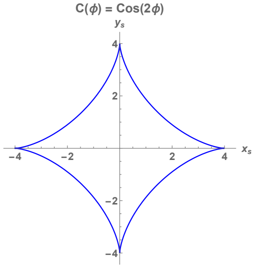
$$\begin{aligned} x_s &= \rho \cos\phi - C(\phi)\cos\phi + C'(\phi)\sin\phi, \\ y_s &= \rho \sin\phi - C(\phi)\sin\phi - C'(\phi)\cos\phi. \end{aligned}$$

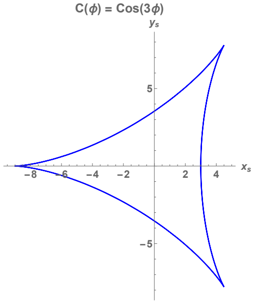
It is invertible outside of the supertranslation horizon

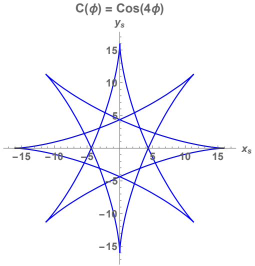
$$\rho > \rho_{SH}(\phi) = C''(\phi) + C(\phi)$$

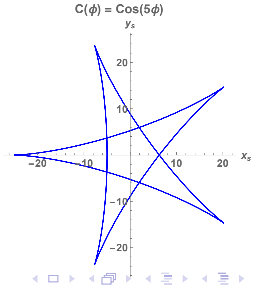
It generates superrotation charges $Q_R = \int_0^{2\pi} d\phi R'(\phi)\rho_{SH}(\phi)$.

Supertranslation horizon

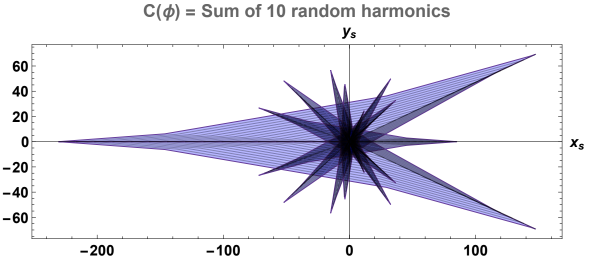








Supertranslation horizon



The static gauge for the vacua breaks down at the supertranslation horizon.

The defect which sources superrotation charges lies in the interior region.

Finite supertranslation diffeomorphism

$$ds^2 = -dt^2 + d\rho_s^2 + \rho_s^2 d\phi^2 = -dt^2 + d\rho^2 + (\rho - \rho_{SH}(\phi))^2 d\phi^2$$

The finite diffeomorphism is

$$\begin{aligned} \rho_s^2 &= (\rho - C)^2 + (C')^2, \\ \tan\phi_s &= \frac{(\rho - C)\sin\phi - C'\cos\phi}{(\rho - C)\cos\phi + C'\sin\phi}. \end{aligned}$$

For $C = a_x \cos\phi + b_x \sin\phi$, it is exactly the coordinate change from polar coordinates around the origin to polar coordinates around a translated origin by (a_x, b_x). The metric is preserved ($\rho_{SH} = 0$).

Supertranslation diffeomorphisms are generalizations of "changing the origin of coordinates".

Limit from AdS_3

The general metric of Einstein gravity with Brown-Henneaux boundary conditions is

$$ds^2 = \ell^2 \frac{dr^2}{r^2} - \left(r dx^+ - \ell^2 \frac{L_-(x^-)dx^-}{r}\right)\left(r dx^- - \ell^2 \frac{L_+(x^+)dx^+}{r}\right)$$

[Bañados, 1998] It represents $AdS_3/BTZ/\dots$ with holographic gravitons generated by the holographic stress-tensor $T_{++} = L_+(x^+)$, $T_{+-} = 0$, $T_{--} = L_-(x^-)$ of a dual CFT_2.

The flat limit $\ell \to \infty$ is well-defined in Null Gaussian coordinates [Barnich, Gomberoff, Gonzalez, 2012] . After canceling the superrotation field and angular momentum ($L_+ = L_-$) and taking $\ell \to \infty$, $L_+(\phi) \simeq \partial_\phi \rho_{SH}(\phi)$ and we find the vacua

$$ds^2 = -dt^2 + d\rho^2 + (\rho - \rho_{SH}(\phi))^2 d\phi^2$$

with zero Poincaré charges as a limiting solution of AdS_3.

The finite $4d$ vacuum supertranslation

We can generalize to $4d$. [G.C., Long, 2016]

After a long analysis, the finite BMS supertranslation diffeomorphism of Minkowski spacetime is found to be

$$
\begin{aligned}
t_s &= t + C_{(0,0)}, \\
x_s &= (\rho - C + C_{(0,0)}) \sin\theta\cos\phi + \csc\theta\sin\phi\partial_\phi C - \cos\theta\cos\phi\partial_\theta C, \\
y_s &= (\rho - C + C_{(0,0)}) \sin\theta\sin\phi - \csc\theta\cos\phi\partial_\phi C - \cos\theta\sin\phi\partial_\theta C, \\
z_s &= (\rho - C + C_{(0,0)}) \cos\theta + \sin\theta\partial_\theta C.
\end{aligned}
$$

At past or future null infinity, the infinitesimal version matches with BMS supertranslations after using the mapping rule

$$
\xi_T^{(BMS_\pm)} = \xi_T^{(stat)} - \delta_T x^\mu_{(BMS_\pm)} \frac{\partial}{\partial x^\mu_{(BMS_\pm)}}
$$

The Poincaré vacua of Einstein gravity

The resulting metric is

$$ds^2 = -dt^2 + dx_s^2 + dy_s^2 + dz_s^2 = -dt^2 + d\rho^2 + g_{AB} d\theta^A d\theta^B,$$

where $\theta^A = \theta, \phi$ and

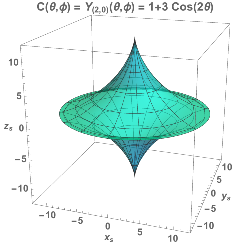
$$\begin{aligned} g_{AB} &= (\rho - C)^2 \gamma_{AB} - 2(\rho - C) D_A D_B C + D_A D_E C D_B D^E C, \\ &= (\rho \gamma_{AC} - D_A D_C C - \gamma_{AC} C) \gamma^{CD} (\rho \gamma_{DB} - D_D D_B C - \gamma_{DB} C) \end{aligned}$$

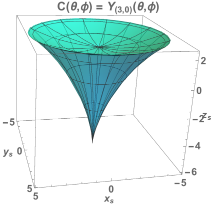
We checked that the 10 Poincaré charges are zero. The superrotation charges are finite and non-trivial.

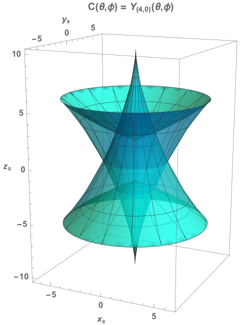
The metric models the degenerate Poincaré vacuum which encodes memory effects in Einstein gravity.

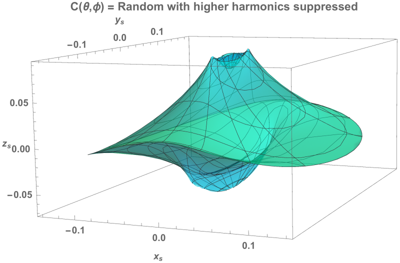
Maybe our universe is patched with such vacua, originating from a pregeometric phase.

Isometric embedding of the supertranslation horizon









Memories from 4*d* Gravitational Collapse

The final static ($J = 0$) metric after spherical gravitational collapse, if analytic, is diffeomorphic to the Schwarzschild metric. [No hair theorems]

[Carter, Hawking, Robinson, 1971-1975] [Chrusciel, Costa, 2008] [Alexakis, Ionescu, Klainerman, 2009]

But memory effects accumulate before and during collapse, so the final metric is in a different BMS vacuum that the global vacuum.

Two questions :

- What is the final state of collapse $g_{\mu\nu}(M, C(\theta,\phi))$?
- How does the supertranslation field $C(\theta,\phi)$ compares to the final mass M ?

The Schwarzschild metric

It admits Weyl conformally flat sections. This is manifest in isotropic coordinates $(t, \rho_s, \theta_s, \phi_s)$:

$$ds^2 = -\frac{\left(1 - \frac{M}{2\rho_s}\right)^2}{\left(1 + \frac{M}{2\rho_s}\right)^2} dt^2 + \left(1 + \frac{M}{2\rho_s}\right)^4 \left(d\rho_s^2 + \gamma_{AB} d\theta^A d\theta^B\right)$$

where

$$\begin{aligned} \gamma_{AB} d\theta^A d\theta^B &= d\theta_s^2 + \sin^2\theta_s d\phi_s^2, \\ \rho_s &= \infty \text{ at spatial infinity} \\ \rho_s &= \frac{M}{2} \text{ at the event horizon} \end{aligned}$$

The Schwarzschild metric embedded in the BMS supertranslation vacuum

$$ds^2 = -\frac{\left(1-\frac{M}{2\rho_s}\right)^2}{\left(1+\frac{M}{2\rho_s}\right)^2}dt^2 + \left(1+\frac{M}{2\rho_s}\right)^4\left(d\rho^2 + g_{AB}d\theta^A d\theta^B\right)$$

where

$$\begin{aligned} g_{AB} &= (\rho\gamma_{AC} - D_A D_C C - \gamma_{AC} C)\gamma^{CD}(\rho\gamma_{DB} - D_D D_B C - \gamma_{DB} C) \\ \rho_s^2 &= (\rho - C)^2 + D_A C D^A C \end{aligned}$$

Remarks :

- When $C = 0$, this is Schwarzschild
- Obtained by finite supertranslation diffeomorphism
- The non-trivial Poincaré charges are just the energy M
- There are superrotation charges quadratic in C

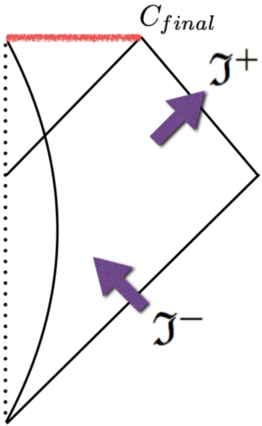
The Schwarzschild metric with BMS hair

In comparison with ["Soft hair on black holes", Hawking, Perry, Strominger, 2016]

- Agree : The hair is soft (zero energy). Supermomenta commute with the Hamiltonian.
- $O(\hbar^0)$, not $O(\hbar^1)$. The classical nature of the BMS hair is rooted in the classical memory effect. The metric are angles/distances which are classically observable (on the contrary electromagnetic hair is encoded in phases measurable only by a quantum apparatus). $O(\hbar^0)$ correction is compatible with quantum theory arguments allowing for a resolution of Hawking's paradox [Mathur, 2009]
- I don't see how linear/small diffeomorphisms could capture the hair. A linearized diffeomorphism would give only the linearized metric, valid close to $\mathfrak{I}^+$ or $\mathfrak{I}^-$. But non-linear effects in the bulk follow from Einstein's equations.

How much supertranslation hair ?

What is the final value of $C(\theta, \phi)$?



It depends upon the fluxes and Bondi mass at $\mathfrak{I}^+$ and $\mathfrak{I}^-$.

How much supertranslation hair ?

Assuming junction conditions joining $\mathscr{I}^+_-$ and $\mathscr{I}^-_+$ [Strominger, 2013] and boundary conditions on radiation [Christodoulou, Klainerman, 1993], Einstein's equations give

$$-\frac{1}{4}\nabla^2(\nabla^2+2)(C|_{final}(\theta,\phi) - C|_{in}(\pi-\theta,\phi+\pi))$$
$$= m|_{final} - m|_{in} + \int_{-\infty}^{+\infty} du T_{uu}(\theta,\phi) - \int_{-\infty}^{+\infty} dv T_{vv}(\pi-\theta,\phi+\pi)$$

This is the global angle-dependent energy conservation law for asymptotically flat spacetimes. [Geroch, Winicour, 1980] [Strominger, Zhiboedov, 2014] [G.C., Long, 2016]

Spherically symmetric collapse of a null shell
$\Rightarrow C|_{final} = 0$ (metric described by Vaidya metric).

How much supertranslation hair?

Non-spherically symmetric collapse of a null shell is constrained by the null energy condition

$$T_{vv}(\theta, \phi) \geqslant 0.$$

Assuming all matter arrives at $v = 0$,

$$T_{vv} = \left(\frac{M + M \sum P_{l,m} Y_{l,m}(\theta, \phi)}{4\pi r^2} + O(r^{-3}) \right) \delta(v)$$

we get the complicated constraint

$$\sum P_{l,m} Y_{l,m}(\theta, \phi) \geqslant -1.$$

How much supertranslation hair?

In the ideal case (no outgoing radiation, no initial mass, only ingoing collapsing radiation), the solution to the global energy conservation law is

$$C(\theta,\phi) = M \sum_{\ell \geqslant 2,m} \frac{4(-1)^\ell}{(\ell-1)\ell(\ell+1)(\ell+2)} P_{l,m} Y_{l,m}(\theta,\phi)$$

with the constraint

$$\sum P_{l,m} Y_{l,m}(\theta,\phi) \geqslant -1.$$

which bounds C from above and below (from compactness). So, for a general non-spherically symmetric collapse we expect (think binary black hole merger or accretion)

$$|C(\theta,\phi)| \simeq M \qquad \text{(leading order classical effect)}$$

Competition between supertranslation horizon and infinite redshift surface

$$ds^2 = -\frac{\left(1-\frac{M}{2\rho_s}\right)^2}{\left(1+\frac{M}{2\rho_s}\right)^2}dt^2 + \left(1+\frac{M}{2\rho_s}\right)^4 \left(d\rho^2 + g_{AB}d\theta^A d\theta^B\right)$$

where $\rho_s^2 = (\rho - C)^2 + D_A C D^A C$. The infinite redshift surface is located at $\rho = \rho_H(\theta, \phi)$ solution to

$$\frac{M^2}{4} = (\rho_H - C)^2 + D_A C D^A C.$$

- When $C \ll M$, this is a black hole with event horizon
- When $D_A C D^A C > \frac{M^2}{4}$, there is no infinite redshift surface.
  $\Rightarrow$ Probable violation of the weak cosmic censorship
- But it turns out that for all cases studied, $D_A C D^A C \leqslant \frac{M^2}{4}$ from the weak energy condition bound !
  $\Rightarrow$ New test of the weak cosmic censorship

Supertranslation and Killing horizons

Simplest axisymmetric $\ell = 2$ case :

$$-\frac{M}{12}(3\cos^2\theta - 1) \leqslant C(\theta,\phi) \leqslant \frac{M}{6}(3\cos^2\theta - 1)$$

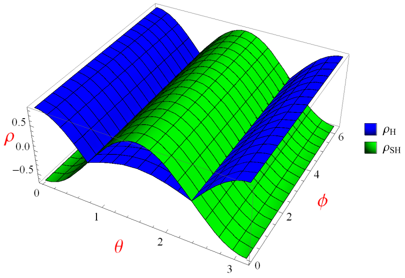


Figure: Upper bound : $C(\theta,\phi) = \frac{M}{6}(3\cos^2\theta - 1)$.

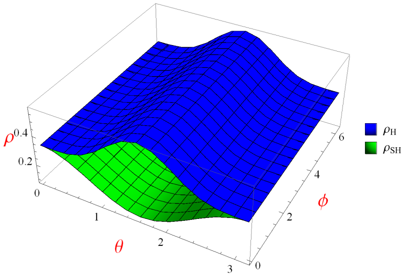


Figure: Lower bound : $C(\theta,\phi) = -\frac{M}{12}(3\cos^2\theta - 1)$.

The Killing horizon ρ_H can be partly hidden behind the supertranslation horizon ρ_{SH}.

On the 4*d* Superrotation field

What about the vacua with supertranslation and superrotation fields ?

We need to apply a finite combined supertranslation and superrotation diffeomorphism to Minkowski :

$$g_{\mu\nu}(\gamma_{z\bar{z}}, C(z,\bar{z}), G(z), u, r) = \frac{\partial x_s^\alpha}{\partial x^\mu} \eta_{\alpha\beta}(\gamma_{z\bar{z}}, r) \frac{\partial x_s^\beta}{\partial x^\nu}.$$

with

$$\begin{aligned}
u &= \sqrt{\partial_z G \partial_{\bar{z}} \bar{G}} \Big(u + C(z,\bar{z}) \Big) + O\left(\frac{1}{r}\right) \\
r &= O(r) \\
z &= G(z) + O\left(\frac{1}{r}\right) \\
\bar{z} &= \bar{G}(\bar{z}) + O\left(\frac{1}{r}\right)
\end{aligned}$$

The diffeomorphism can be resumed after using 2 tricks

- Use Weyl rescalings [Barnich, Troessaert, 2010]
- Use map to Minkowski foliated by null planes so that $\gamma_{z\bar{z}}$ is taken care of by the Weyl rescaling

The final metric is [G.C, Long, 2016]

$$g_{\mu\nu}(\gamma_{z\bar{z}}, C(z,\bar{z}), G(z), u, r) = g_{\mu\nu}(\gamma_{z\bar{z}}, C_{zz}(u,z,\bar{z}), r)$$

where

$$C_{zz} = -2D_z\partial_z C - (u+C)\left(\frac{\partial_z^3 G}{\partial_z G} - \frac{3(\partial_z^2 G)^2}{2(\partial_z G)^2}\right), \quad C_{z\bar{z}} = 0.$$

The Schwarzian derivative term naturally arises as in $3d$ examples. [Balog, Feher, Palla, 1997] It is the stress-tensor of a free boson $\partial_z G = e^{\psi(z)}$,

$$T_{zz} = -\frac{1}{2}\left(\frac{\partial_z^3 G}{\partial_z G} - \frac{3(\partial_z^2 G)^2}{2(\partial_z G)^2}\right) = \frac{1}{4}(\partial_z\psi)^2 - \frac{1}{2}\partial_z\partial_z\psi, \quad T_{\bar{z}z} = 0.$$

The Bondi mass decreases with retarded time u,

$$\partial_u M = -\frac{1}{2} T^{AB} T_{AB}.$$

$\Rightarrow$ Unbounded negative energy. Discard by imposing the Dirichlet boundary condition $T_{zz} = 0$.

The symplectic structure at $\mathfrak{I}^+$ is

$$\Omega_{\mathfrak{I}^+}[\delta C, \delta\psi; \delta C, \delta\psi] \equiv -\frac{1}{4G} \int_{\mathfrak{I}^+} du d^2\Omega \, \delta C_{AB} \wedge \delta T^{AB}.$$

$\Rightarrow$ The superrotation field is a source conjugated to the supertranslation field.

Conserved superrotation charges for the physical vacua exist, $Q_R \simeq \int_S \partial_z^2 C \partial_{\bar{z}}^2 C$. Similar to AdS prescription [Witten, 1998] : "Turning on a source to compute a vev".

Summary of the results

- The metrics for the Poincaré vacua with supertranslation field in $3d$ and $4d$ gravity have been derived. It is unclear whether or not the $4d$ vacua are physical.
- In the center-of-mass frame, supertranslations are spatial, except the zero mode (time shift).
- Memory effects lead to a different final state of collapse : the Schwarzschild black hole with supertranslation hair. The hair is a large non-linear $O(\hbar^0)$ and $O(M)$ effect which is computable from past history of evolution and collapse.
- Much physics and maths remains to be understood.