Multi-scale Stellar Atmospheres, Flows, Heating

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Based On:

1. S.M. Mahajan, R.Miklaszevski, K.I. Nikol'skaya & N.L. Shatashvili. ***Phys. Plasmas***. **8,** 1340 (2001); ***Adv. Space Res*., 30,** 345 (2002).
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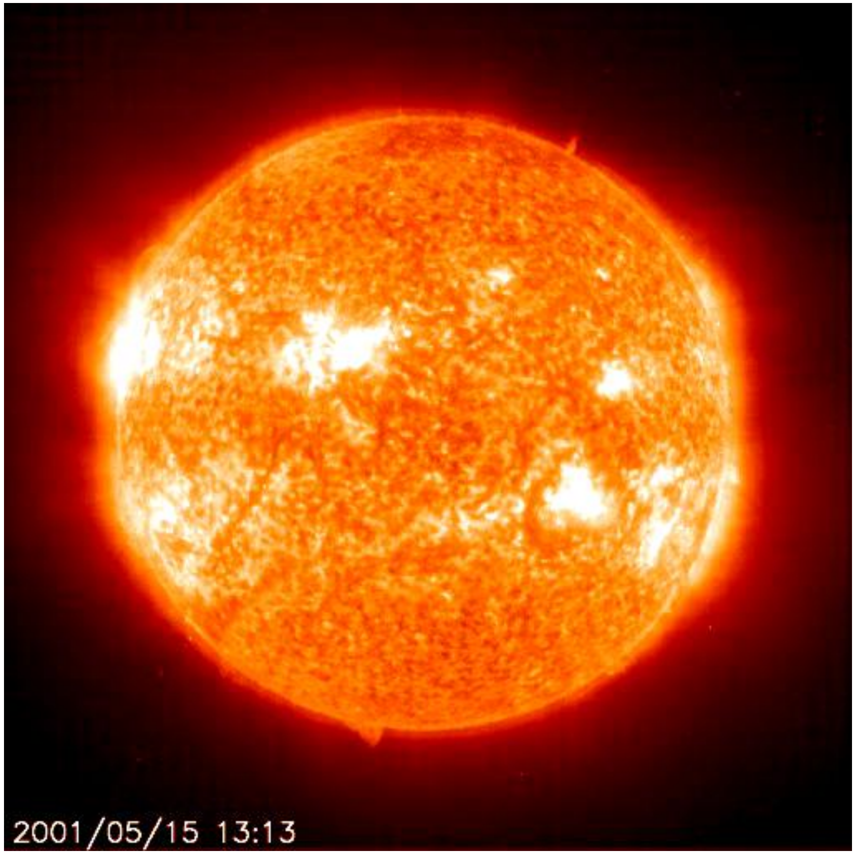
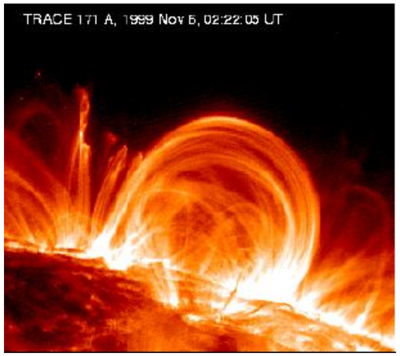
Outline

- **Dynamic Multi-scale Solar Atmosphere**
- **Beltrami-Bernoulli States – Magneto Fluid Coupling – *Solar Atmosphere***
- **New class of Double Beltrami Equlibria sustained by Electron Degeneracy Pressure**
- **Stellar Atmospheres with Degenerate Electrons & Positrons & Ion Fractions**
- **Quadruple Beltrami System – formation of Macro Scale**
- **Triple Beltrami System – formation of Meso Scale**
- **Illustrative Examples - *White Dwarfs***
- **Scale Hierarchy**
- **Discussion & Summary**

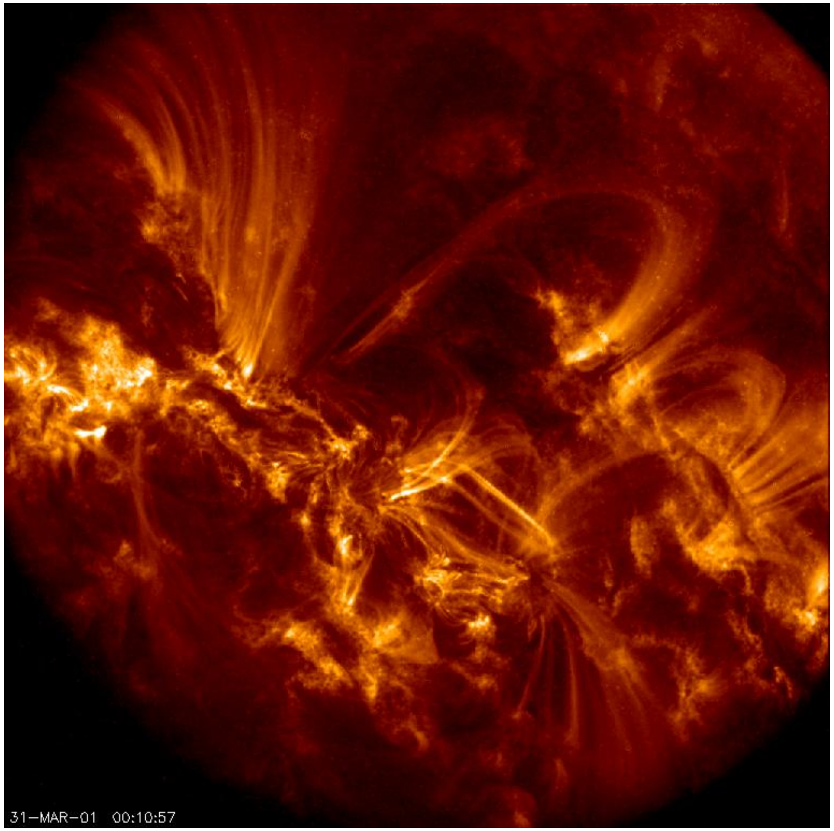
Dynamic multi-scale Solar Corona



- **The solar corona – a highly dynamic arena replete with multi-species multiple–scale spatiotemporal structures.**
- **Magnetic field was always known to be a controlling player.**
- **Strong flows are found everywhere in the low Solar atmosphere — in the sub-coronal (*chromosphere*) as well as in coronal regions (loops)** – *recent observations from HINODE (De Pontieu et al. 2011-2014).*



Active region of the corona with:



Co-existing dynamic structures:

- Flares
- Spicules
- Different-scale dynamic closed/open structures

Message:

- Different temperatures
- Different life-times

Indication:

- Any particular mechanism may be dominant in a specific region of parameter space.

Equally important: ***the plasma flows may complement the abilities of the magnetic field in the creation of the amazing richness observed in the Atmosphere***

Loops at different temperatures exist in the same general region and may be co–located to within their measured diameters.

Challenge – to develop a theory of energy transformations for understanding the **quiescent and eruptive/explosive events in solar atmosphere.**

Recently developed **theory that the formation and heating of coronal structures may be simultaneous** **&**

directed flows may be the carriers of energy.

How does the Solar Corona gets to be so hot ($\geq 10^6$K)? Still an unsolved problem?!

- Is it the ohmic or the viscous dissipation?
- Or is it the shocks or the waves that impart energy to the particles
- And do the observations support the ”other consequences” of a given model?

Heating due to the viscous dissipation of the flow vorticity:

$$\left[\frac{d}{dt}\left(\frac{m_i \mathbf{V}^2}{2}\right)\right]_{\text{visc}} = -m_i n \nu_i \left(\frac{1}{2}(\nabla \times \mathbf{V})^2 + \frac{2}{3}(\nabla \cdot \mathbf{V})^2\right).$$

Towards a General Unifying Model:

Conjecture:

Formation and primary heating of coronal structures as well as the more violent events (flares, erupting prominences and CMEs**) are the expressions of different aspects of the same general global dynamics that operates in a given coronal region.**

- The **plasma flows**, the source of both the particles and energy (part of which is converted to heat), ***interacting with the magnetic field, become dynamic determinants of a wide variety of plasma states*** $\Longrightarrow$
- **the immense diversity of the observed coronal structures.**

Construction of a Typical Coronal structure

Solar Corona — $T_c = (1 \div 4) \cdot 10^6 \, K$ $n_c \leq 10^{10}$ cm^{-3}.

Standard picture – Corona is first formed & then heated. 3 principal heating mechanisms:

- By Waves / Alfven Waves,
- By Magnetic reconnection in current sheets,
- MHD Turbulence

All of these attempts fall short of providing a continuous energy supply that is required to support the observed coronal structures.

New concept:

Formation and heating are contemporaneous – primary flows are trapped & a part of their kinetic energy dissipates during their trapping period.

It is the Initial & Boundary conditions that define the characteristics of a given structure $T_c >> T_{0f} \sim 1eV$.

Observations → there are strongly separated scales both in time and space in the solar atmosphere. *And that is good.*

A closed coronal structure – 2 distinct eras:

1. **A hectic dynamic period when it acquires particles & energy (accumulation + primary heating)**

Full description needed: time dependent dissipative two-fluid equations are used. Heating takes place while particles accumulate (get trapped) in a curved magnetic field (*viscosity is taken local as well as the radiation is local),*

2. **Quasistationary period when it "shines" as a bright, high temperature object — a reduced equilibrium description suffices**

collisional effects and time dependence are ignored.

Equilibrium: each coronal structure has a nearly constant ***T***,

but different structures have different characteristic ***T*-**s,

i.e. **bright corona seen as a single entity will have considerable *T*–variation**

1st Era – Fast dynamic

Energy losses from corona: $F \sim (5 \cdot 10^5 \div 5 \cdot 10^6)$ erg/cm^2 s.

If the conversion of the kinetic energy in the **Primary Flows** were to compensate for these losses, we would require a radial energy flux

$$\frac{1}{2} m_i n_0 V_0^2 \, V_0 \geq F$$

For initial $V_0 \sim (100 \div 900)$ km/s $n \sim 9 \cdot 10^5 \div 10^7$ cm^{-3}

Viscous dissipation of the flow takes place on a time:

$$t_{\rm visc} \sim \frac{L^2}{\nu_i} \qquad (2)$$

For flow with $T_0 = 3\text{eV} = 3.5 \cdot 10^4$ K , $n_0 = 4 \cdot 10^8$ cm^{-3}

Creating a quiet coronal structure of size $L = (2 \cdot 10^8 \div 10^{10})\,\mathrm{cm}$

$$t_{\mathrm{visc}} \sim (3.5 \cdot 10^8 \div 10^{10}) \text{ s}$$

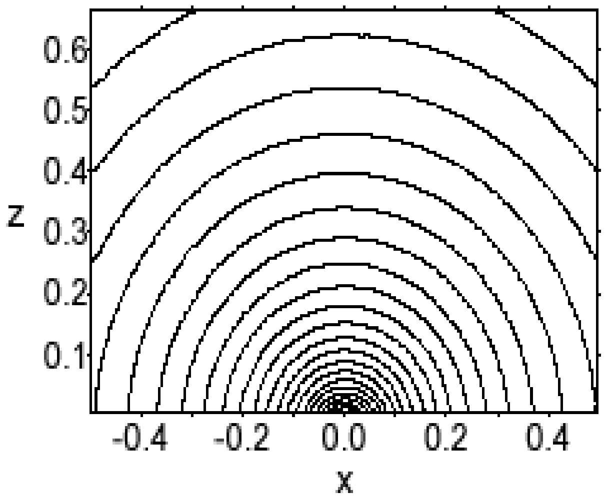
Note: (2) is an overestimate. $t_{\mathrm{real}} \ll t_{\mathrm{visc}}.$

Reasons:

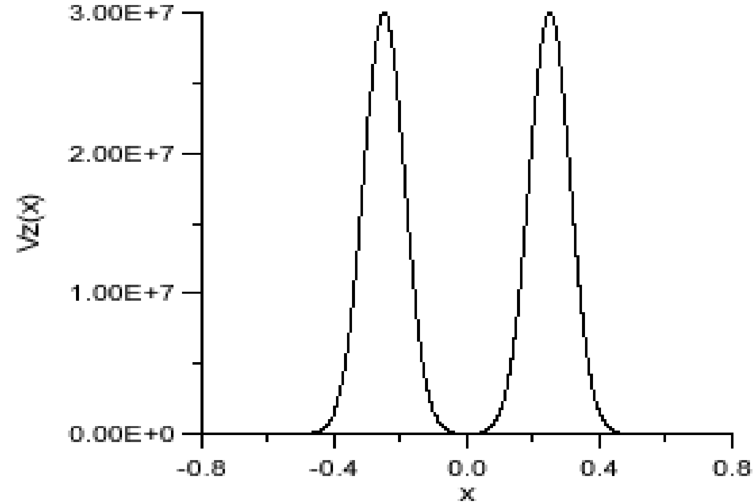
1) $\nu_i = \nu_i(t, \mathbf{r})$ will vary along the structure,

2) the spatial gradients of the ***V***–field can be on a scale much shorter than *L* (defined by the smooth part of ***B***–field).

Initial and Boundary conditions



Contour plots for the vector potential **A** (flux function) in the $x - z$ plane for a **typical arcade- like solar magnetic field**



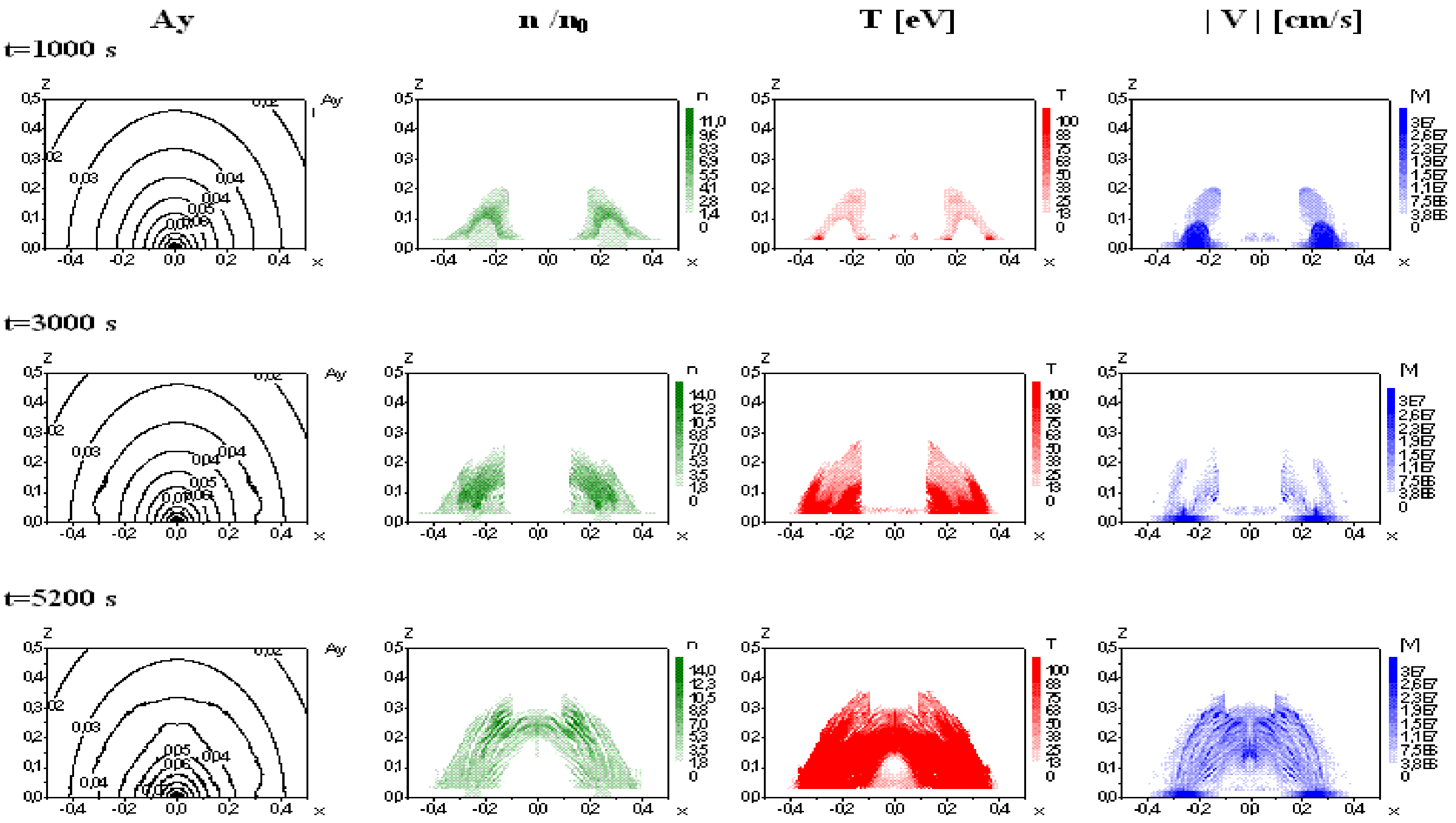
The distribution of the radial component $\mathbf{V_z}$ (with a maximum of **300 km/s at *t*= 0**) for the **symmetric, spatially nonuniform velocity field.**

2.5D numerical simulation of the general two-fluid equations in Cartesian Geometry.
Code: Mahajan et al. PoP (2001), Mahajan et al, ApJ (2005).

Simulation system contains: 1) dissipation (local) and heat flux; 2) plasma is compressible ; 3) ***Radiation*** is local (modified Bremsstahlung) - extra possibility for micro-scale structure creation.

Transport coefficients are taken from Braginskii and are local. Diffusion time of magnetic field > duration of interaction process (would require $T \leq$ a few eV -s).

Hot coronal structure formation

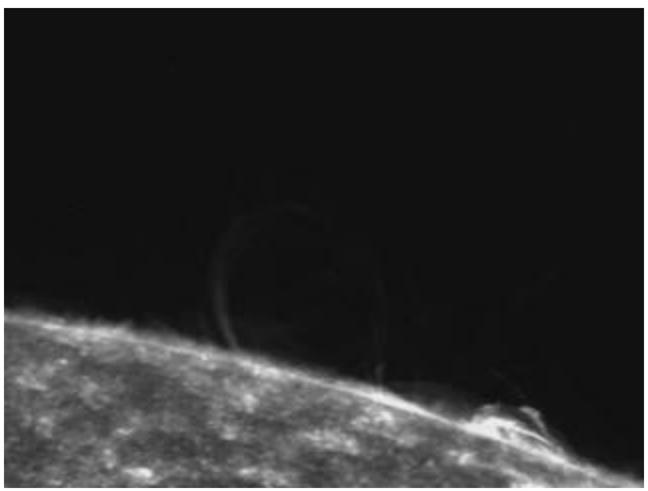


Flow T_0 =3eV, n_0 =$4\cdot10^8$cm^{-3}, initial background density = $2\cdot10^8$cm^{-3} , B_{max} $(x_0 , z_0=0)$ = 20G.

Much of the primary flow kinetic energy has been converted to heat via shock generation.

Simulations examples –formation & heating of hot structure

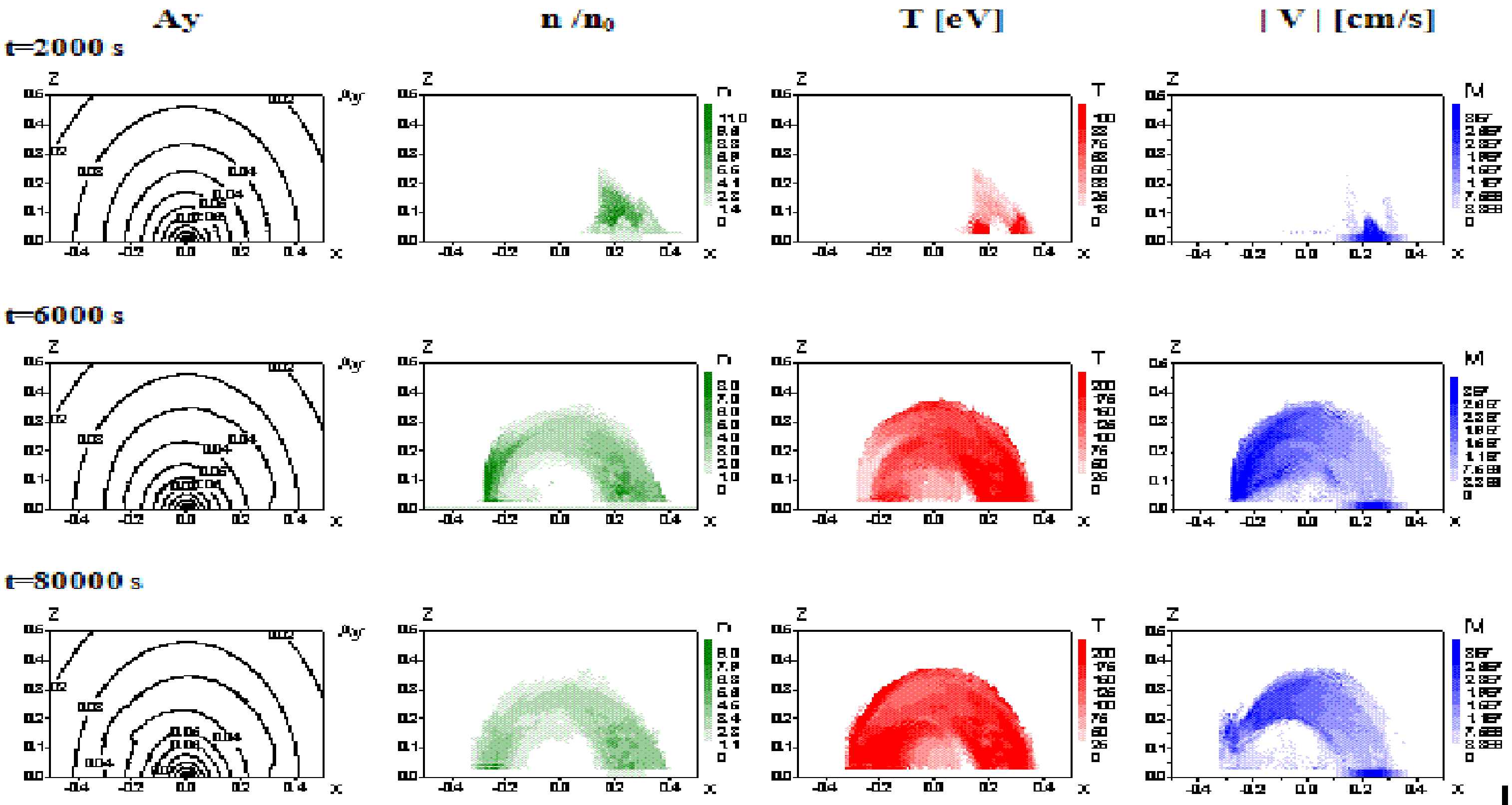
Observations show hot closed structure formation being different for different structures. **In the same region one observes different speeds of formation + heating – we see loop when it is hot.**



Simulation example 1 – symmetric case:
2 identical constant in time flows interact with closed ***B***-field structure. $B_{0max} = 20G$, $V_{z0max} = 300km/s$, $T_0 = 3eV$.
Primary heating is very fast – hot base is created in few 100s of seconds.

Left Column - no resistivity, right column – local resistivity included with coefficient ~ $2 \cdot 10^{-3}$.

Hot coronal structure formation

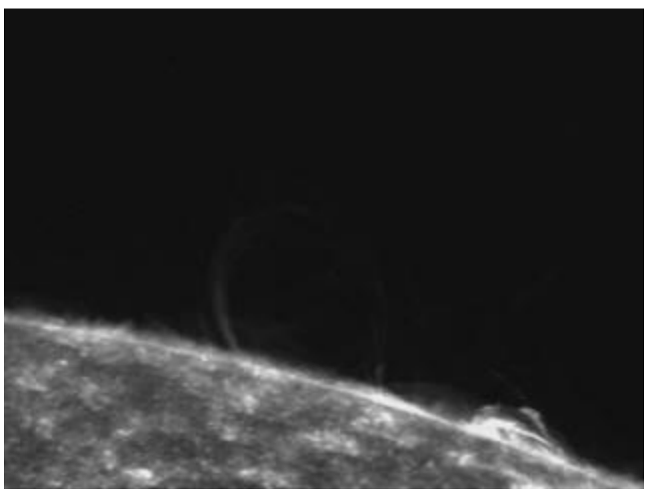


The interaction of an **initially asymmetric, spatially nonuniform primary flow** (*just the right pulse*) with a strong arcade-like magnetic field B_{max} (x_0 , z_0 =0) = 20 G.

Downflows, and the imbalance in primary heating are revealed

Flows found in the loops

Observations show that coronal structure **formation + heating is never a symmetric process; there are flows inside hot loops.**



Simulation example 2 – non-symmetric case:
1 flow (constant in time) interacts with closed ***B***-field structure. $B_{0max} = 20G$, $V_{z0max} = 300km/s$, $T_0 = 3eV$. **Process of formation + heating is slower than in symmetric case.**
Flow remains along loop, just slowed down.

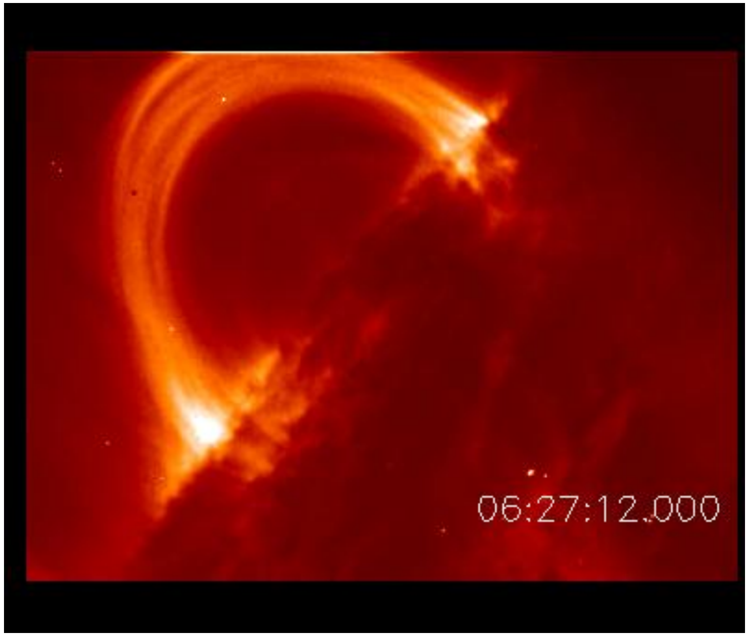
Left Column - no resistivity, right column – local resistivity included with coefficient ~ $2 \cdot 10^{-3}$.

Simulation Results

- **Primary plasma flows are capable of thermalizing during interaction with primary magnetic fields** (that are curved) **to form the hot coronal structure**.
- **Two distinct eras are distinguishable in the life of a hot closed structure** – a fast era of the formation (plus primary heating), and a relatively calm era of in which the hot structure persists in a state of quasi-equilibrium.
- **Parameters of the hot closed structure (in quasi-equilibrium) are fully determined by the characteristics of the primary flow and the ambient magnetic fields**; the greater the primary flow initial velocity and initial magnetic field B_0, the hotter is the coronal base.
- **For the same primary flows the maximum heating is achieved at some height independent of B_0** (in agreement with observations).
- **The greater the resistivity, the shorter is the life-time of the quasi-equilibrium structure**.
- **The formation time of the hot closed structure is strictly dependent on the magnitudes of primary flow & primary magnetic field, as well as their initial time dependence (life-time).**
- The duration of the primary heating is directly determined by the parameters of primary flow and magnetic fields. **Greater the fields, the faster is the primary heating.**

2nd Era – Quasi-Equilibrium

Quasistationary period when closed coronal structure ”shines” as a bright, high temperature object.



Observations: A loop system may be quiescent for a long time with individual loops living for several hours (2nd era of quasi-equilibrium in the life of the closed coronal structure).

Quiescent periods may be followed by rapid activity (loops are ”turned on”/disappear in ≤ 10 - 40 min).

In equilibrium each coronal structure has a nearly constant T , but **different structures have different characteristic *T*-s,**

i.e. bright corona seen as a single entity will have considerable T –variation.

The familiar magneto hydrodynamics (MHD) theory (*single fluid*) is inadequate.

In a two-fluid description, the velocity field interacting with the magnetic field provides:

new pressure confining states ; the possibility of heating these equilibrium states by dissipation of short scale kinetic energy.

A Quasi-equilibrium Structure

Model: **recently developed magnetofluid theory.**

Assumption: at some distance there exist fully ionized and magnetized plasma structures such that the quasi–equilibrium two–fluid model will capture the essential physics of the system.

Simplest two–fluid equilibria: $T = \mathrm{const} \longrightarrow \mathrm{n}^{-1}\nabla \mathrm{p} \rightarrow \mathrm{T}\nabla \ln \mathrm{n}.$

Generalization to homentropic fluid: $p = \mathrm{const}\cdot \mathrm{n}^{\gamma}$ is straightforward.

The **dimensionless equations:**

$$\frac{1}{n}\nabla \times \mathbf{b} \times \mathbf{b} + \nabla \left(\frac{r_{A0}}{r} - \beta_0 \ln\ n - \frac{V^2}{2} \right) + \mathbf{V} \times (\nabla \times \mathbf{V}) = 0, \quad (3)$$

$$\nabla \times \left[\left(\mathbf{V} - \frac{\alpha_0}{n} \nabla \times \mathbf{b} \right) \times \mathbf{b} \right] = 0, \quad (4)$$

$$\nabla \cdot (n\mathbf{V}) = 0, \quad (5)$$

$$\nabla \cdot \mathbf{b} = 0, \quad (6)$$

Beltrami-Bernoulli States – Magneto Fluid Coupling

Constrained minimization of fluid energy with appropriate helicity invariants has provided a variety of extremely interesting equilibrium configurations that have been exploited and found useful for understanding laboratory as well as astrophysical charged fluid systems.

Two particularly simple manifestations of this genre of equilibria (called **Beltrami States**) are:

1) The ***single Beltrami state***, $\nabla \times \boldsymbol{B} = \alpha\, \boldsymbol{B}$, discussed by ***Woltjer & Taylor*** in the context of **force free single fluid magneto-hydrodynamics (MHD) ,**

&

2) a more general ***Double Beltrami State*** accessible to **Hall MHD** – ***a two-fluid system of ions and inertialess electrons*** - investigated, in depth, by Mahajan,Yoshida, Shatashvili & co-authors (1997 – 2014).

The Beltrami condition implies an alignment of the fluid vorticity and its velocity, and **the characteristic number of a state is determined by the number of independent single Beltrami systems needed to construct it.**

The **Beltrami conditions must be buttressed by an appropriate Bernoulli constraint to fully describe an equilibrium state** – called **Beltrami-Bernoulli (BB) states**.

The system (3-6) allows the following **relaxed state solution**

$$\mathbf{b} + \alpha_0 \nabla \times \mathbf{V} = d\ n\ \mathbf{V}, \qquad \mathbf{b} = a\ n\ \left[\mathbf{V} - \frac{\alpha_0}{n} \nabla \times \mathbf{b}\right] \qquad (7)$$

augmented by the **Bernoulli Condition**

$$\nabla \left(\frac{2\beta_0 r_{c0}}{r} - \beta_0 \ln n - \frac{V^2}{2} \right) = 0 \qquad (8)$$

a and *d* — dimensionless constants related to **ideal invariants**:

the Magnetic and the Generalized helicities

$$h_1 = \int (\mathbf{A} \cdot \mathbf{b})\ d^3x \qquad (9)$$

$$h_2 = \int (\mathbf{A} + \mathbf{V}) \cdot (\mathbf{b} + \nabla \times \mathbf{V}) d^3x \qquad (10)$$

The system is obtained by minimizing **the energy** $E = \int (\mathbf{b} \cdot \mathbf{b} + n\mathbf{V} \cdot \mathbf{V})\ d^3x$ keeping h_1 and h_2 invariant.

Equations (7) yield

$$\frac{\alpha_0^2}{n}\nabla\times\nabla\times\mathbf{V}+\alpha_0\,\nabla\times\left(\frac{1}{a}-d\;n\right)\mathbf{V}+\left(1-\frac{d}{a}\right)\mathbf{V}=0 \qquad (11)$$

which must be solved with (8) for n and $\mathbf{V}$.

Equation (8) is solved to obtain ($g(r) = r_{c0}/r$) .

$$n=\exp\left(-\left[2g_0-\frac{V_0^2}{2\beta_0(T)}-2g+\frac{V^2}{2\beta_0(T)}\right]\right) \qquad (12)$$

The variation in density can be quite large for a low β_0 plasma if the gravity and the flow kinetic energy vary on length scales comparable to the extent of the structure.

Model calculation – temperature varying but density constant ($n = 1$) .

The following still holds (where $\boldsymbol{Q}$ is either $\mathbf{V}$ or $\boldsymbol{b}$):

$$\alpha_0^2\;\nabla\times\nabla\times\mathbf{Q}+\alpha_0\;\left(\frac{1}{a}-d\right)\nabla\times\mathbf{Q}+\left(1-\frac{d}{a}\right)\mathbf{Q}=0 \qquad (13)$$

Analysis of the *Curl Curl* Equation, Typical Equilibria

The existence of **two, rather than one** (as in the standard relaxed equilibria) **parameter in this theory is an indication that we may have found an extra clue to answer the extremely important question:**

why do the coronal structures have a variety of length scales, and what are the determinants of these scales?

$\alpha_0 \sim 10^{-7} - 10^{-8}$ for typical densities $(\sim (10^7 - 10^9\,\mathrm{cm}^{-3})$.

Suppose: a structure has a span $\epsilon R_\odot$, where $\epsilon \ll 1$. For a structure of order 1000 km , $\epsilon \sim 10^{-3}$.

The ratio of the orders of various terms in Eq. (13) are $(|\nabla| \sim L^{-1})$

$$\frac{\alpha_0^2}{\epsilon^2} : \quad \frac{\alpha_0}{\epsilon}\left(\frac{1}{a} - d\right) : \quad \left(1 - \frac{d}{a}\right)$$
$$(1) \qquad (2) \qquad (3)$$

The following two principle balances are representative:

(a) The last two terms are of the same order, and the first << them:

$$\epsilon \sim \alpha_0 \frac{1/a - d}{1 - d/a} \qquad (14)$$

For our desired structure to exist ($\alpha_0 \sim 10^{-8}$ for $n_0 \sim 10^9$ cm^{-3}):

$$\frac{1/a - d}{1 - d/a} \sim 10^5 \qquad (15)$$

which is possible if *d/a* tends to be extremely close to unity.

For the first term to be negligible, we would further need

$$\frac{\alpha_0}{\epsilon} \ll \frac{1}{a} - d \qquad \Rightarrow \qquad \epsilon \gg \frac{10^{-8}}{1/a - d} \qquad (16)$$

easy to satisfy as long as neither of $a \approx d$ is close to unity.

Standard relaxed state: flows are not supposed to play an important part.

Extreme sub–Alfvénic flows: a ~ d >> 1.

The new term introduces a qualitatively new phenomenon:

$\nabla \times (\nabla \times \mathbf{b})$ **is a singular perturbation of the system**; its effect on the standard root (2) ~ (3) >> (1) will be small, but **it introduces a new root** for which th $|\nabla|$ must be large (short length scale!)

For a and d so chosen to generate a 1000km structure

$$d/a \sim 1 + 10^{-4}, \qquad d \simeq a = -10, \qquad |\nabla|^{-1} \sim 10^2 \, \mathrm{cm},$$

an equilibrium root with variation on the scale of 100cm will be automatically introduced by the flows.

Even if flows are weak $(a \simeq d \simeq 10)$, the departure from $\nabla \times \mathbf{B} = \alpha \mathbf{B}$ can be essential: **it introduces a totally different (small!) scale solution** $\Rightarrow$ fundamental importance in understanding the effects of viscosity on the dynamics of structures.

Dissipation of short scale structures $\rightarrow$ primary heating.

(b) **The other balance**: we have a complete departure from conventional relaxed state: **all three terms are of the same order**

$$\epsilon \sim \alpha_0 \frac{1}{1/a - d} \sim \alpha_0 \frac{1/a - d}{1 - d/a} \qquad (17)$$

which translates as:

$$\left(\frac{1}{a} - d\right)^2 \sim 1 - \frac{d}{a}, \qquad \frac{1}{a} - d \sim \alpha_0 \frac{1}{\epsilon} \qquad (18)$$

For a **1000km structure**, $\alpha_0 \cdot 1/\epsilon \sim 10^{-5}$ and ***a ~ d ~ 1***

we would need the flows to be almost perfectly Alfvénic!

Such flow conditions are in the weak magnetic field regions.

(1) Alfvénic flows are capable of creating entirely new kinds of structures – quite different from the ones that we normally deal with.

(2) Though they also have two length scales, these length scales are quite comparable to one another.

(3) Two length scales can become complex conjugate giving rise to fundamentally different structures in ***b*** and **V** .

Curl Curl Equation – Double-Beltrami states

With $p = (1/a - d)$ and $q = (1 - d/a)$, Eq. (13) $\Longrightarrow$

$$(\alpha_0 \nabla \times - \lambda)(\alpha_0 \nabla \times - \mu)\mathbf{b} = 0 \qquad (19)$$

where λ (λ_+) and μ(λ_-) are the solutions of the quadratic equation

$$\alpha_0 \lambda_\pm = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}. \qquad (20)$$

If G_λ is the solution of the **Beltrami Equation** (a_λ and a_μ are constants)

$$\nabla \times \mathbf{G}(\lambda) = \lambda \mathbf{G}(\lambda), \qquad then \qquad (21)$$

$$\mathbf{b} = a_\lambda \mathbf{G}(\lambda) + a_\mu \mathbf{G}(\mu) \qquad (22)$$

is the general solution of the double *curl* equation. Velocity field is:

$$\mathbf{V} = \frac{\mathbf{b}}{a} + \alpha_0 \boldsymbol{\nabla} \times \mathbf{b} = \left(\frac{1}{a} + \alpha_0 \lambda\right) a_\lambda \mathbf{G}(\lambda) + \left(\frac{1}{a} + \alpha_0 \mu\right) a_\mu \mathbf{G}(\mu) \qquad (23)$$

Double curl equation is fully solved in terms of the solutions of Eq. (21).

Double Beltrami States

- **There are two scales in equilibrium** unlike the standard case.

- **A possible clue for answering the extremely important question:** **why do the coronal structures have a variety of length scales, and what are the determinants of these scales?**

- **The scales could be vastly separated** – are determined by the constants of the motion – the original preparation of the system.
  These constants also determine the relative kinetic & magnetic energy in quasi-equilibrium.

- **These vastly richer structures can & do model the quiescent solar phenomena** rather well – construction of coronal arcades fields, slow acceleration, spatial rearrangement of energy etc.

An Example of structural richness

Closed Coronal structure: the magnetic field is relatively smooth but the velocity field must have a considerable short–scale component if its dissipation were to heat the plasma. Can a DB state provide that?

Sub–Alfvénic Flow: $a \sim d >> 1 \implies \lambda \sim (d - a) / \alpha_0 d a$; $\mu = d / \alpha_0$.

$$\mathbf{V} = \frac{1}{a} a_\lambda \mathbf{G}_\lambda + d a_\mu \mathbf{G}(\mu) \tag{24}$$

$$\mathbf{b} = a_\lambda \mathbf{G}_\lambda + a_\mu \mathbf{G}(\mu) \tag{25}$$

while, the slowly varying component of velocity is smaller by a factor ($a^{-1} \approx d^{-1}$) compared to similar part of ***b***-field, the fast varying component is a factor of d larger than the fast varying component of ***b***-field!

Result: for an extreme sub–Alfvénic flow (e.g. $|\mathbf{V}| \sim d^{-1} \sim 0.1$) ,

$$\frac{|\mathbf{V}(\mu)|}{|\mathbf{V}(\lambda)|} \simeq 1 \tag{26}$$

the velocity field is equally divided between slow and fast scales.

Compact Astrophysical Objects with Degenerate Electrons

BB class of equilibria have been studied for both relativistic and nonrelativistic plasmas, most investigations are limited to ”dilute” or **non-degenerate plasmas: the constituent particles are assumed to obey the classical Maxwell-Boltzman statistics**.

Question: **how such states would change/transform if the plasmas were highly dense and degenerate (*mean inter-particle distance is << de Broglie thermal wavelength*) - their energy distribution is dictated by Fermi-Dirac statistics**.

Notice: **at very high densities, particle Fermi Energy can become relativistic & degeneracy pressure may dominate thermal pressure.**

Such **highly dense/degenerate plasmas are found in several astrophysical and cosmological environments as well as in the laboratories devoted to inertial confinement and high energy density physics**; in the latter intense lasers are employed to create such extreme conditions.

Model

The natural habitats for dense/degenerate matter: Compact astrophysical objects like white and brown dwarfs, neutron stars, magnetars with believed characteristic electron number densities $\sim 10^{26} - 10^{32}\ cm^{-3}$, formed under extreme conditions.

We develop the simplest model in which the effect of quantum degeneracy on the nature of the BB class of equilibrium states can be illustrated; fundamental role of another quantum effect – spin vorticity – on BB states was studied in Mahajan *et al* (2011, 2012).

We choose **a model hypothetical system *(relevant to specific aspects of a white dwarf (WD))*** of **a two-species neutral plasma with non-degenerate non relativistic ions, and degenerate relativistic electrons embedded in a magnetic field.**

It **is assumed that, despite the relativistic mass increase, the electron fluid vorticity is negligible compared to the electron cyclotron frequency (*such a situation may pertain, for example, in the pre-WD state of star evolution, and in the dynamics of the WD atmosphere*).** The study of the degenerate electron inertia effects on the Beltrami States in dense neutral plasmas will be shown later.

Model Details

For an **ideal isotropic degenerate Fermi gas of electrons** at temperature T_e the relevant thermodynamic quantities – the pressure $\mathcal{P}_e$ & *the proper internal energy density* $\mathcal{E}_e$ (the corresponding enthalpy $w_e = \mathcal{E}_e + \mathcal{P}_e$), *per unit volume – can be calculated to be*

$$\mathcal{P}_e = \frac{m_e^4 c^5}{3\pi^2\hbar^3} f\left(P_F\right), \qquad \mathcal{E}_e = \frac{m_e^4 c^5}{3\pi^2\hbar^3}\left[P_F^3\left(\left(1+P_F^2\right)^{1/2}-1\right) - f\left(P_F\right)\right], \qquad (27)$$

$$8f\left(P_F\right) = 3\sinh^{-1} P_F + P_F\left(1+P_F^2\right)^{1/2}\left(2P_F^2-3\right) \qquad (28)$$

$P_F = p_F/m_e c$ **is the normalized Fermi momentum of electrons**

Fermi Energy in terms of P_F **is** $\epsilon_F = m_e c^2\left[\left(1+P_F^2\right)^{1/2}-1\right]$.

P_F is related to the **rest-frame electron density** n_e via $p_F = m_e c\left(n_e/n_c\right)^{1/3}$.

$n_c = 5.9 \times 10^{29} cm^{-3}$ **- critical number-density at which the Fermi momentum equals** $m_e c$ - defines the onset of the relativistic regime.

The electron (positron) plasma is treated as the completely degenerate gas – their thermal energy is much lower than their Fermi energy $(n_e T_e/\mathcal{P}_e \ll 1)$.

Distribution function of electrons remains locally Juttner-Fermian which for ***0*-temperature case leads to the just density dependent thermodynamical quantities** $\mathcal{E}_e(n_e)$, $\mathcal{P}_e(n_e)$ & $w_e(n_e)$.

Electron plasma dynamics is isentropic, obeys relation: $d(w_e/n_e) = (d\mathcal{P}_e)/n_e$.

Model Equations

Equation of motion for degenerate electron fluid reduces to:

$$\frac{\partial}{\partial t}\left(\sqrt{1+P_F^2}\,\mathbf{p}_e\right) + m_e c^2 \nabla\left(\sqrt{1+P_F^2}\,\gamma_e\right) =$$

$$= -e\mathbf{E} - \frac{e}{c}\mathbf{V}_e \times \mathbf{B} + \frac{e}{c}\mathbf{V}_e \times \nabla \times \left(\sqrt{1+P_F^2}\,\mathbf{p}_e\right) \qquad (29)$$

With $\mathbf{p}_e = \gamma_e m_e \mathbf{V}_e$ being electron hydrodynamical momentum

Under our assumption of negligible electron fluid vorticity the last term can be negligible.

For the non-degenerate ion fluid we have the equation of motion written as (*m_i* - proton mass):

$$m_i \left[\frac{\partial \mathbf{V}_i}{\partial t} + (\mathbf{V}_i \cdot \nabla)\mathbf{V}_i\right] = -\frac{1}{N_i}\nabla p_i + e\mathbf{E} + \frac{e}{c}\mathbf{V}_i \times \mathbf{B} \,. \qquad (30)$$

Simplest model **- non relativistic ions & inertialess electrons** $N_e \simeq N_i = N$ **- there are two independent Beltrami conditions** (*aligning ion & electron generalized vorticities along their respective velocities*):

$$\mathbf{b} + \nabla \times \mathbf{V} = d\, N\, \mathbf{V} \,, \qquad \mathbf{b} = a\, N \left[\mathbf{V} - \frac{1}{N}\nabla \times \mathbf{b}\right] , \qquad \mathbf{b} = e\mathbf{B}/m_i c \qquad (31)$$

Bernoulli Condition

Density is normalized to N_0 (the corresponding rest-frame density is n_0)

Magnetic field is normalized to some ambient measure B_0

All **velocities are measured in terms of** corresponding Alfvén speed $V_A = B_0/\sqrt{4\pi N_0 m_i}$

All **lengths [times] are normalized to the skin depth** λ_i $[\lambda_i/V_A]$

where $\lambda_i = c/\omega_{pi} = c\sqrt{m_i/4\pi N_0 e^2}$

The ***Beltrami conditions* (31) must be supplemented by the Bernoulli constraint to define an equilibrium state (the stationary solution of the dynamical system):**

$$\nabla\left(\beta_0 \ln N + \mu_0\sqrt{1+P_F^2}\,\gamma + \frac{V^2}{2}\right) = 0 \qquad (32)$$

Where β_0 **is the ratio of thermal pressure to magnetic pressure**, $\mu_0 = m_e c^2/m_i V_A^2$

and **for the electron fluid Lorentz factor we put** $\gamma_e \simeq \gamma(\mathbf{V})$.

Bernoulli condition (32) is an expression of the balance of all remaining potential forces when Beltrami conditions (31) are imposed on the two-fluid equilibrium equations.

$P_F = p_F/m_e c = (N N_0/n_c\gamma)^{1/3}$ $[= (N n_0/n_c)^{1/3}]$ **is a function of density. (31-32) is a complete system of equations.**

$[\nabla\cdot(N\mathbf{V}) = 0]$ **- Equilibrium Continuity Eq. ,** $[\nabla\cdot\mathbf{b} = 0]$ **are automatically satisfied.**

New class of Double Beltrami Equlibria
sustained by Electron Degeneracy Pressure

1) The Beltrami conditions reflect the simple physics: (i) the inertia-less (despite the relativistic increase in mass) degenerate electrons follow the field lines, (ii) while the ions, due to their finite inertia, follow the magnetic field modified by the fluid vorticity.

The combined field $\mathbf{b} + \nabla \times \mathbf{V}$ **- an expression of magneto-fluid unification, may be seen either as an effective magnetic field or an effective vorticity.**

2) The **Beltrami conditions (31) are not directly affected by the degeneracy effects** in the current approximation neglecting the electron inertia. These are precisely the two conditions that define the Hall MHD states. **In the highest density regimes Fermi momentum** (& hence the Lorentz factor $\gamma(\mathbf{V})$) **may be so large that the effective electron inertia will have to be included in (31).**

3) In this minimal model, **electron degeneracy manifests only through the Bernoulli condition (32).** The degeneracy induced term ~ μ_0 would go to unity (whose gradient is zero), and would disappear in the absence of the degeneracy pressure. **For significant P_F, the degeneracy pressure can be >> thermal pressure (measured by β_0).**

Degenerate electron gas can sustain a qualitatively new state: a nontrivial Double Beltrami – Bernoulli equilibrium at zero temperature. ***In the classical zero-beta plasmas, only the relatively trivial, single Beltrami states are accessible***.

4) It is trivial to eliminate $\boldsymbol{b}$ in Eqs. (31) to obtain

$$\frac{1}{N}\nabla\times\nabla\times\mathbf{V}+\nabla\times\left[\left(\frac{1}{aN}-d\right)N\,\mathbf{V}\right]+\left(1-\frac{d}{a}\right)\mathbf{V}=0\,, \qquad (33)$$

which, coupled with (32), provides us with a closed system of four equations in four variables *(N,**V**).*

Once this is solved with appropriate boundary conditions, one can invoke 1[st] eq. of (31) to calculate $\boldsymbol{b}$. See solution for similar math. problem in Mahajan *et al* (2001).

5) **The Bernoulli condition (32) introduces a brand new player in the equilibrium balance**; the spatial variation in the electron degeneracy energy (~ μ_0) could increase or decrease the plasma $\boldsymbol{\beta_0}$ or the fluid kinetic energy (measured by V^2) in the corresponding region. Thus, **Fermi energy could be converted to kinetic energy; it could also forge a re-adjustment of the kinetic energy from a high-density/low-velocity plasma to a low-density/high-velocity plasma**. Similar energy transformations, mediated through classical gravity, were discussed in Mahajan *et al* (2002, 2005, 2006).

Possible extensions of model:

- **When electron fluid degeneracy is very high and one can not neglect inertia effects in their vorticity, the order of BB states is likely to rise** (the triple BB states have been studied in 2008).
- **For the supper-relativistic electrons extension will be the introduction of Gravity, which could balance the highly degenerate electron fluid pressure**. Gravity (Newtonian) effects in the BB system have been investigated in the solar physics context (e.g Mahajan *et.al* (2002, 2005, 2006).

Illustrative Example - White Dwarfs

A possible application of the "degenerate" BB states may be found in stellar physics.

Star collapses + cools down: the density of lighter elements increases affecting the total pressure / enthalpy of unit fluid element –first order departure from the classical e-i plasma; beyond the hot, pre-white dwarf stage, photon cooling dominates and gravitational contraction is dramatically reduced as the interior equation of state hardens into that of a strongly degenerate electron gas. Mechanical and thermal properties separate; the degenerate electrons provide the dominant pressure, while the thermal motions of the ions make a negligible contribution to the mechanical support; the roles of electrons and ions are reversed in their contribution to the overall energy.

Recent studies show that **a significant fraction of White Dwarfs are found to be magnetic with typical fields strengths below 1KG. Massive and cool White Dwarfs,** interestingly, **are found with much higher fields detected.** Recent investigations have uncovered **several cool, magnetic, polluted hydrogen atmosphere (DAs) white dwarfs.**

A simple example: **if degenerate BB states could shed some light on the physics of WDs?**

Considering High B-field WDs, we assume: degenerate electrons densities ~ $(10^{25}$-$10^{29})$ cm^{-3} ; Magnetic fields ~ $(10^5$ - $10^9)$ G , Temperatures ~ (40000-6000) K . Alfvén speed $V_A \sim (10^4$-$10^6)$ cm/s,

➔ $\beta_0 \sim (10^6 - 10^0)$ & $\mu_0 \sim (10^{10} - 10^6) >> 1$. **Ion skin-depth** $\lambda_i \sim (10^{-5} - 10^{-7})\,cm$ - very short.

Illustrative Example - White Dwarfs

For this class of systems, **2nd term (*degeneracy pressure*) in (32) >> 1st term (*thermal pressure*).** Neglecting the 1st term, and remembering that for non relativistic flows (*essential at ion speeds*) $\gamma(V) \sim 1$, **Bernoulli Condition with inclusion of classical (Newtonian) gravity (justified by observations for WDs) implies**

$$\mu_0 \sqrt{1+P_F^2} - \frac{R_A}{R} + \frac{V^2}{2} = const \qquad (34)$$

***const* measures the main energy content of the fluid**; the Beltrami conditions (31) remain the same;

R - radial distance from the center of WD normalized to its radius R_W **[~ (0 .008-0.02)*R*$_{sun}$]** ;
$R_A = GM_W / R_W V_A^2$ (here G is the gravitational constant and M_W - WD mass).

Since P_F is a function of Fermi energy (and hence, of density), **we assume that at some distance R_*** (corresponding to density maximum), P_F **reaches its maximum value** P_{F*} .
Taking the corresponding minimum velocity to be zero ($V_* \sim 0$), we find $const = \mu_0 \sqrt{1+P_{F*}^2} - R_A/R_*$.
Magnitude of velocity is now determined to be

$$|\mathbf{V}| \sim \sqrt{2\mu_0}\, \kappa(P_F) \qquad (35)$$

with

$$\kappa(P_F) = \left[\left(\sqrt{1+P_{F*}^2} - \sqrt{1+P_F^2} \right) - \frac{R_A}{\mu_0} \left(\frac{1}{R_*} - \frac{1}{R} \right) \right]^{1/2} .$$

Results

Dimensionless coefficient $R_A / \mu_0 << 1$ measures relative strength of gravity versus degenerate pressure term.

For WD-s with Mass $M_W \sim (0.8 - 0.25)\, M_{sun}$ & radius $R_W \sim (0.013 - 0.02) R_{sun}$,

$R_A / \mu_0 \sim (0.2 - 0.04) << 1$; **less massive the WD, the smaller is the coefficient .**

DB structure scales are small compared to R_W in outer layers of the WD (where model applies).

The gravity contribution to the flow velocity can be neglected

at specific distance of outer layers of WD-s with $R \geq R_*$ & $(R - R_*)/ R_* << 1$, $R_* \leq 1$.

Gravity contribution determines the radial distance in WD 's outer layer over which the "catastrophic" acceleration of flow may appear (due to the magneto-fluid coupling).

In the regions where the flows are insignificant (at very short distances from the WD 's surface) gravity controls the stratification but as we approach the flow "blow-up "distances (the flow becomes strong) the self-consistent magneto-Bernoulli processes take over & control density / velocity stratification.

Calculating the maximum flow velocity, occurring at $\kappa\,(P_F)$ maximum (density minimum), needs a detailed knowledge of the system.

If □ $\sqrt{2\mu_0}\,\kappa(P_F) > 1$ **the generated flow is locally super-Alfvénic in contradistinction to the non-degenerate, thermal pressure dominated plasma,** when the maximal velocity due to the magneto-Bernoulli mechanism be locally sub-Alfvénic (when local plasma beta <1 as in the Solar Atmosphere).

This example shows that **the electron degeneracy effects can be both strong, and lead to interesting predictions like the anticorelation between the density and flow speeds.**

The richness introduced by electron-degeneracy to the Beltrami-Bernoulli states could help us better understand compact astrophysical objects. When star contracts, its outer layers keep the multi-Structure character although density in structures becomes defined by electron degeneracy pressure.

Important conclusion for future studies - when studying the evolution of the atmospheres/outer layers of compact objects, flow effects can not be ignored. Knowledge of the effects introduced by flows (observed in stellar outer layers) acquired for classical plasmas can be used when investigating the dynamics of White Dwarfs and their evolution.

We found the possibility of the existence of DB relaxed states in plasmas with degenerate electrons (met in astrophysical conditions). **Non degenerate double BB states guarantee scale separation phenomenon**

➔ **provide energy transformation pathways for various astrophysical phenomena** (erruptions, fast / transient outflow & jet formation, ***B***-field generation, structure formation, heating & etc.), **such pathways could be explored for degenerate case with degeneracy pressure providing an additional energy source.**

Astrophysical Objects with Degenerate Electrons & Positrons *& Ion Fraction*

Magnetospheres of rotating neutron stars are believed to contain e-p plasmas produced in the cusp regions of the stars due to intense electromagnetic radiation. Since protons or other ions may exist in such environments, **three-component e-p-i plasmas can exist in pulsar magnetospheres**.

Positron component could have a variety of origins:

(1) positrons can be created in the interstellar medium due to interaction of atoms & cosmic ray nuclei,

(2) they can be introduced in a Tokamak e-i plasma by injecting bursts of neutral positronium atoms ($e+e-$), which are then ionized by plasma.

The annihilation usually occurs at much longer characteristic time scales compared with the time in which the collective interaction between the charged particles takes place.

The natural habitats for dense/degenerate matter: Compact astrophysical objects like white and brown dwarfs, neutron stars, magnetars with believed characteristic electron number densities $\sim 10^{26} - 10^{32}\ cm^{-3}$, formed under extreme conditions.

Model

We develop the simplest model in which the effect of quantum degeneracy as well as the mobility of heavier ions on the nature of the BB class of equilibrium states can be illustrated.

We choose **a model hypothetical system *(relevant to specific aspects of a white dwarf (WD))*** of **a three-species neutral plasma of degenerate relativistic electron-positron plasma with small fraction of non-degenerate classical mobile ions.**

The new BB equilibrium is defined by: two relativistic Beltrami conditions (one for each dynamic degenerate species), one non-relativistic Beltrami condition for ion fluid, an appropriate Bernoulli condition, and Ampere's law to close the set. This set of equations will lead to what may be called a *quadruple Beltrami system.*

The ions, though a small mobile component, play an essential role, they create an asymmetry in the electron-positron dynamics (to maintain charge neutrality, there is a larger concentration of electrons than positrons) and that asymmetry introduces a new and very important dynamical scale. This scale, though present in a classical non-degenerate plasma, turns out to be degeneracy dependent and could be vastly different from its classical counterpart.

Presence of mobile ions leads to "effective mass" asymmetry in electron and positron fluids, which, coupled with degeneracy-induced inertia, manifests in the existence of Quadruple Beltrami fields.

Model Equations - 1

Charge neutrality in an e-p-i plasma of degenerate electrons (−), positrons (+) and a small mobile ion component, forces the following density relationships

$$N_0^- = N_0^+ + N_{i0} \implies \frac{N_0^+}{N_0^-} = 1 - \alpha \qquad \text{with} \quad \alpha = \frac{N_{i0}}{N_0^-}, \tag{1}$$

The equation for ion dynamics is standard. **The e(p) dynamics** will be described by the relativistic degenerate fluid equations: ***the continuity***

$$\frac{\partial N^\pm}{\partial t} + \nabla \cdot (N^\pm \mathbf{V}_\pm) = 0, \tag{2}$$

and ***the equation of motion***

$$\frac{\partial}{\partial t}(G^\pm \mathbf{p}_\pm) + m_\pm c^2 \nabla (G^\pm \gamma_\pm) = q_\pm \mathbf{E} + \mathbf{V}_\pm \times \mathbf{\Omega}_\pm \tag{3}$$

where $\mathbf{p}_\pm = \gamma_\pm m_\pm \mathbf{V}_\pm$ - is hydrodynamic momentum, $n^\pm = N^\pm/\gamma_\pm$ is the rest-frame particle density ($N^\pm$ denotes laboratory frame number density), the degeneracy effects manifest through the “effective mass” $G^\pm = w_\pm / n^\pm m_\pm c^2$, where $w_\pm$ is the enthalpy per unit volume. For fully degenerate relativistic e(p) plasma its general expression transfers to

$w_\pm \equiv w_\pm(n)$ and $w_\pm / n^\pm m_e c^2 = (1 + (R^\pm)^2)^{1/2}$ $R^\pm = p_{F\pm}/m_\pm c$

The mass factor is then $G^\pm = [1 + (n^\pm/n_c)^{2/3}]^{1/2}$ for arbitrary $n_\pm / n_c$.

Model Equations - 2

On taking the *curl of these equations, one can cast them* into an ideal vortex dynamics

$$\frac{\partial}{\partial t}\mathbf{\Omega}_{\pm} = \nabla \times (\mathbf{V}_{\pm} \times \mathbf{\Omega}_{\pm}), \qquad \textit{where} \quad \mathbf{\Omega}_{\pm} = (q_{\pm}/c)\,\mathbf{B} + \nabla \times (\tilde{G}^{\pm}\mathbf{p}_{\pm}). \qquad (4)$$

We emphasize that the so called plasma approximation for a degenerate e(p) assembly is valid if their average kinetic energy $(\sim\epsilon_F^{\pm})$ is larger than the interaction energy $(\sim e^2(n_0^{\pm})^{1/3})$. This condition is fulfilled for a sufficiently dense fluid when $n_0^{\pm} \gg (2m_- e^2/(3\pi^2)^{2/3}\hbar^2)^3 =$ 6 3×10^{22} cm^{-3} ; such a condition would imply $R^{\pm} \gg 4.76 \times 10^{-3}$.

The low frequency dynamics is, now, closed with Ampere's law

$$\nabla \times \mathbf{B} = \frac{4\pi e}{c}\left[(1-\alpha)N^{+}\,\mathbf{V}_{+} - N^{-}\,\mathbf{V}_{-} + \alpha N_i\,\mathbf{V}_i\right], \qquad (5)$$

The small static/mobile ion population, represented by α and $\boldsymbol{V}_{\mathrm{i}}$, creates an asymmetry between the currents contributed by the electrons and positrons. This will be the source of a new scale-length that turns out to be much larger than the intrinsic electron and positron scale lengths (skin depths).

Equilibrium States in Relativistic E-P-I Degenerate Plasma

Density is normalized to electrons N_0 (the corresponding rest-frame density is n_0)

Magnetic field is normalized to some ambient measure B_0

All **velocities are measured in terms of** corresponding Alfvén speed $V_A = V_A^- = B_0/\sqrt{8\pi n_0^- m^- G_0^-}$

All **lengths [times] are normalized to the skin depth** $\lambda_{\text{eff}}[\lambda_{\text{eff}}/V_A]$,

where $$\lambda_{\text{eff}} \equiv \lambda_{\text{eff}}^- = \frac{1}{\sqrt{2}}\frac{c}{\omega_p^-} = c\sqrt{\frac{m^- G_0^-}{8\pi n_0^- e^2}} \quad , \quad G_0^\pm(n_0^\pm) = [1 + (R_0^\pm)^2]^{1/2} \quad , \quad R_0^\pm = \left(\frac{n_0^\pm}{n_c}\right)^{1/3}$$

The intrinsic skin depths, the natural length scales of the dynamics, are generally much shorter compared to the system size. For the degenerate electron fluid, the effective mass goes to

$G_0^-(n_0^-) = 1 + \frac{1}{2}(\frac{n_0^-}{n_c})^{2/3}$ for $(R_0^- \ll 1)$ and to $G_0^-(n_0^-) = (\frac{n_0^-}{n_c})^{1/3}$ for $(R_0^- \gg 1)$

Following the well-known procedure we obtain the set of equilibrium equations for the degenerate system (the primary difference is in the physics of G .

The ***Beltrami conditions***:
$$\mathbf{B} \pm \nabla \times (G^\pm \gamma_\pm \mathbf{V}_\pm) = a_\pm \frac{n^\pm}{G^\pm}(G^\pm \gamma_\pm \mathbf{V}_\pm), \qquad (6)$$

aligning the Generalized vorticities along their velocity fields, and the ***Bernoulli conditions***
$$\nabla(G^\pm \gamma_\pm \pm \varphi) = 0 \quad \Longrightarrow \quad G^+\gamma_+ + G^-\gamma_- = const. \qquad (7)$$

And ***Ion fluid Beltrami Condition***
$$\mathbf{B} + \zeta \nabla \times \mathbf{V}_i = \alpha a_i n_i \mathbf{V}_i, \quad \text{where } \zeta = \left[G_0^- \frac{m^-}{m_i}\right]^{-1} \qquad (8)$$

The Quadruple Beltrami System

An appropriate tedious manipulation of the set Eqs. (5)–(7), leads us to an explicit **quadruple Beltrami equation** obeyed by the Ion Fluid Velocity $\mathbf{V}_i$ ***(the Beltrami index is measured by the highest number of curl operators)****. Written schematically as*

$$\nabla\times\nabla\times\nabla\times\nabla\times\mathbf{V}_i - b_1'\nabla\times\nabla\times\nabla\times\mathbf{V}_i + b_2'\nabla\times\nabla\times\mathbf{V}_i - b_3'\nabla\times\mathbf{V}_i + b_4'\mathbf{V}_i = 0. \quad \textbf{(9)}$$

Equation (9) was derived in the incompressible approximation, and for $\gamma_+ \sim \gamma_- \equiv 1$

The *b* coefficients are functions of effective masses, Beltrami & system characteristic parameters.

Incompressibility assumption is expected to be adequate for outer layers of compact objects, though, compressibility effects can be significant e.g. in the atmospheres of pre-compact stars (Berezhiani et al. 2015). Ion fluid velocity & the magnetic field are related to

e-p plasma average bulk fluid velocity $\mathbf{V} = \frac{1}{2}[(1-\alpha)\mathbf{V}_+ + \mathbf{V}_-]$. (10)

through
$$\mathbf{V} = \eta\left(2\beta G_0^+\nabla\times\nabla\times\mathbf{B} - \left[a_+(1-\alpha)\beta - a_-\right]\nabla\times\mathbf{B}\right) + \eta\left(\left[1+(1-\alpha)\beta\right]\mathbf{B}\right) - \alpha\beta\nabla\times\mathbf{V}_i + \frac{\alpha}{2}\left[a_+(1-\alpha)\beta - a_-\right]\mathbf{V}_i \quad (11)$$

with
$$\eta \equiv \left[a_+(1-\alpha)\beta + a_-\right]^{-1} \text{ and } \beta = G_0^-/G_0^+. \quad (12)$$

The **quadruple Beltrami (9) can be factorized as**

$$(curl - \mu_1)(curl - \mu_2)(curl - \mu_3)(curl - \mu_4)\mathbf{V}_i = 0, \qquad (13)$$

where μ_i -s define coefficients in Eq. (14) & are functions of α , β , n_0^- and the degeneracy-determined G_0^+ .

The general solution of Eq. (13) is a sum of four Beltrami fields $\boldsymbol{F_k}$ (solutions of Beltrami Equations $\nabla \times \mathbf{F}_k = \mu_k \mathbf{F}$) while eigenvalues (μ_k) of the *curl operator are* the solutions of the fourth order equation

$$\mu^4 - b_1'\mu^3 + b_2'\mu^2 - b_3'\mu + b_4' = 0. \qquad (14)$$

An examination of the various *b* coefficients of (14), for the most relevant limit ***α << 1*** , reveal:

Though the inverse scales, determined by b_1' , b_2', and b_3' , do get somewhat modified by α<<1 corrections, it is the inverse scale associated with b_4' that is most profoundly affected; being $\sim \alpha$, it tends to become small, i.e., ***the corresponding scale length becomes large as α approaches zero; this scale length becomes strictly* infinite for α=0, and disappears reducing (14) to a triple Beltrami system**.

Thus, **the ion contamination-induced asymmetry may lead to the formation of macroscopic structures through creating an intermediate/large length scale, much larger than the intrinsic scale skin depths, and less than the system size.** The possible significance and importance of this somewhat natural mechanism (a small ion contamination is rather natural) for creating Macro-structures in astrophysical objects, could hardly be overstressed.

Notice: this mechanism operates for all levels of degeneracy **(the range of R_0^- was irrelevant).**

Illustrative Examples – White Dwarfs - Large Scale

This **new macroscopic scale can be "determined" by dominant balance arguments**; as scale gets larger, $|\nabla|$ gets smaller, and the dominant balance will be between the last terms of (14), yielding ($\zeta \gg 1$):

$$L_{\mathrm{macro}} = \frac{|b_3'|}{|b_4'|} = \frac{A}{\alpha} \qquad \text{where} \qquad A = \zeta \frac{|(a_+ - a_-)[1 - \frac{\alpha}{\zeta}(G_0^+)] + \frac{\alpha}{\zeta} a_i [(G_0^+) - a_+ a_-]|}{|a_i (a_+ - a_-) - a_+ a_-|} \quad (15)$$

Assuming that: $\alpha G_0^+/\zeta = \alpha\beta(G_0^+)^2 \frac{m^-}{m_i} \leq \alpha \ll 1$ [*e(p)* plasma density is within $(10^{25}\text{–}10^{32})\ \mathrm{cm}^{-3}$]]

we can simplify ***A*** when both $a_+ \ll a_i$ and $a_- \ll a_i$.

(i) When **$a_+ \neq a_-$** the simplified expression

$$L_{\mathrm{macro}} \sim \frac{\zeta}{\alpha} \left| \frac{1}{a_i} + \frac{\alpha}{\zeta} \frac{(G_0^+) - a_+ a_-}{a_+ - a_-} \right| \quad (16)$$

for $a_i \leq \zeta$ satisfies $L_{\mathrm{macro}} \gg 1$.

(ii) When $a_+ \sim a_- = a \neq (G_0^+)^{1/2}$ $\quad L_{\mathrm{macro}} \sim \frac{a_i}{a^2} |(G_0^+) - a^2| \gg 1$ for all $a_i \gg a$. (17)

Without ion contamination (α = 0), the degenerate e-p system is still capable of creating length scales larger than the non-relativistic skin depths through the degeneracy-enhanced inertia of the light particles.

Notice **that even with equal effective masses ($G^- = G^+ \equiv G(n)$ at equal electron-positron temperature), inertia change due to degeneracy can cause asymmetry in e(p) fluids.**

Illustrative Examples – Meso Scale

Even in the absence of ions (**L_{macro} → infinity**), the Beltrami states could be characterized by what could be called meso-scales—the temperature and degeneracy-boosted effective skin depths $\lambda^{\pm}_{eff}$ larger than λ

[$\lambda_{eff}^{\pm}/\lambda = \sqrt{G_0^{\pm}} > 1$ and $1 < \sqrt{G_0^{\pm}} < 5.6$ for densities (10^{25} –10^{32})cm^{-3}].

For pure compressible e-p plasma, if $\nabla[G^{\pm}(n^{\pm})]$ is at a much slower rate than the spatial derivatives of ***B*** and $\boldsymbol{V_{\pm}}$**,** we can write following relation:

$$\left(\frac{G}{n}\right)\nabla\times\left(\frac{1}{n}\right)\nabla\times\nabla\times\mathbf{V}-\kappa_1\left(\frac{1}{n}\right)\nabla\times\nabla\times\mathbf{V}+\kappa_2\nabla\times\left(\frac{G}{n}-a_+a_-\right)\left(\frac{n}{G}\right)\mathbf{V}-\kappa_3\mathbf{V}=0. \quad (18)$$

Estimation for the large scale l_{meso} in case of pure degenerate e-p plasma, derived from the dominant balance, gives:

$$l_{\rm meso}=\frac{|\kappa_2|}{|\kappa_3|}|(G/n)-a_+a_-|=2\frac{|(G/n)-a_+a_-|}{|a_+-a_-|}\gg 1 \quad \text{if} \quad a_+=a_-=a\neq\left(\frac{G(n)}{n}\right)^{1/2}. \quad (19)$$

Hence, ***whenever the local density satisfies this condition there is a guaranteed scale separation in the degenerate e-p plasma with at least one large scale present.***

At the same time: for larger scale to exist we do need an entirely different mechanism — a dynamic ion-species with a much lower density and higher rest mass (justified by observations for many astrophysical objects plasmas) — this scale corresponds to the ion skin depth enhanced, dramatically, by low density $[\lambda_i=(\alpha m_-/m_i)^{-1/2}\,\lambda\gg\lambda]$.

Scale Hierarchy

This work registers a major departure from e-i system leading to the most important result — by studying BB states in an e-p-i (*small dynamic ion contamination added to a primarily e-p plasma*), we demonstrated the creation of a new macroscopic length scale L_{macro} lying between *the system* size and relatively small intrinsic scales (measured by the skin depths) of the system.

(1) For a pure electron-positron plasma, the equilibrium is *triple Beltrami with the following fundamental three scales*: system size L , and the two intrinsic scales (electron and positron *skin depths).*

(2) **The e-p skin depths**, microscopic in a non degenerate plasma, **can become much larger due to degeneracy effects and could be classified as meso-scales,** l_{meso} .

(3) **When a dynamic low density ion species is added, the equilibrium becomes *quadruple Bertrami with a new* additional scale, L_{macro}.** Although the exact magnitude of this scale is complicated, its origin is entirely due to the ion contamination; this scale disappears as the ion concentration α goes to zero. **Both the larger ion mass and low density contribute towards boosting L_{macro} .**

(4) The meso-scale l_{meso} cannot become very large but for some special constraints on the Bertrami parameters, for instance, if $a_- \neq a_+$ and both $a_\pm \ll 1$ or the condition (19) is satisfied.

Discussion & Summary

We derived *Quadruple [Triple] Beltrami* relaxed states in e-p-i plasma with classical ions, and degenerate electrons and positrons. Such a mix is often met in both astrophysical and laboratory conditions.

The presence of the mobile ion component has a striking qualitative effect; it converts, what would have been, a *triple Beltrami state to a new quadruple Beltrami state. In* the process, it adds structures at a brand new macroscopic scale L_{macro} (absent when ion concentration is zero) that is much larger than the intrinsic skin depth ($\lambda = c\sqrt{\frac{m^-}{8\pi n_0^- e^2}}$) of the lighter components.

Though primarily controlled by the mobile ion concentration, L_{macro} **also takes cognizance of the electron and positron inertias that could be considerably enhanced by degeneracy**.

The creation of these new intermediate scales (between the system size, and λ) adds immensely to the richness of the structures that such an e-p-i plasma can sustain; many more pathways become accessible for energy transformations. Such pathways could help us better understand a host of quiescent as well as explosive astrophysical phenomena — eruptions, fast/transient outflow and jet formation, magnetic field generation, structure formation, heating etc.