Heat Transport in a Stochastic Magnetic Field

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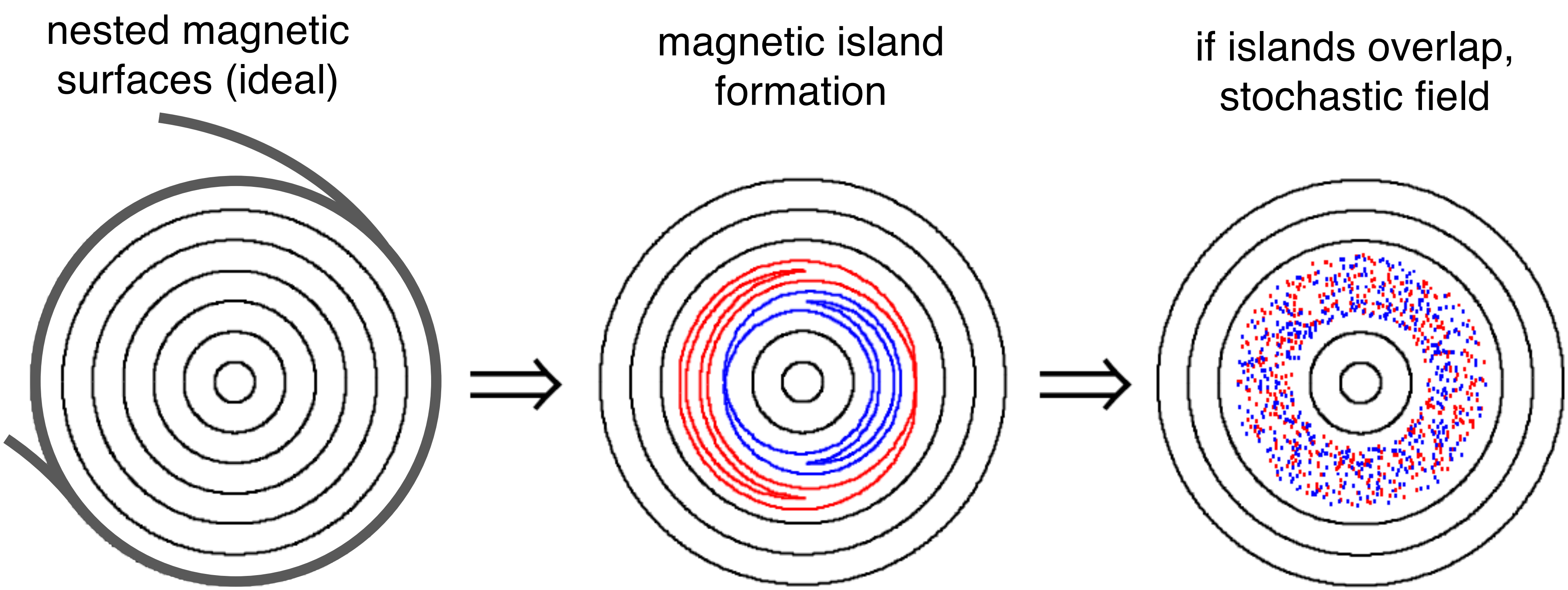
Magnetic perturbations can destroy the nested-surface topology desired for magnetic confinement

- Stochastic instability occurs when magnetic islands overlap, causing the field lines to wander randomly throughout the plasma volume.
- Parallel streaming along the stochastic field leads to radial transport.
- Astrophysical plasmas have weak ordered field (naturally "tangled")



(**B** perturbations from instability or "error" components)

Stochastic transport often appears in fusion plasmas

- Through instability:
  - Large-scale resistive MHD instabilities, e.g., tearing modes with overlapping magnetic islands
  - Electromagnetic microinstabilities

- Externally sourced magnetic perturbations:
  - “Resonant magnetic perturbations” in the edge region of tokamak plasmas to control the stability of the H-mode transport “pedestal” and edge-localized modes (ELMs)
  - Magnetic field errors arising from finite precision in magnets

- Stellarators:
  - Limitations in the control of the magnetic field using realistic magnets
  - Induced through finite plasma pressure and current, which affects the magnetic equilibrium

Outline

- Model for stochastic transport
- Comparisons with experimental measurements (mostly from the RFP)

Projection of radial field yields intuitive estimate of stochastic transport

Recall parallel heat transport

$$\frac{\partial T}{\partial t} = \chi_{||}(\hat{\mathbf{b}} \cdot \nabla)^2 T \qquad \text{where} \quad \hat{\mathbf{b}} = \mathbf{B}/B$$

If $\mathbf{B} = \mathbf{B}_0 + \tilde{B}_r \hat{\mathbf{r}}$ where $\mathbf{B}_0$ = well-ordered field, forming nested magnetic surfaces

$$\frac{\partial T}{\partial t} = \chi_{||}(\hat{\mathbf{b}} \cdot \nabla)^2 T = \chi_{||} \left(\frac{\widetilde{B}_r}{B_0} \right)^2 \frac{\partial^2 T}{\partial^2 r}$$

(not quite rigorous, ok for fluid limit)

effective perpendicular transport

Small fluctuation amplitudes can yield large radial transport

Recall for classical electron transport

$$\frac{\chi_{\parallel}}{\chi_{\perp}} \sim \frac{\lambda_{mfp}^2 \nu_c}{\rho^2 \nu_c} \geq 10^6$$

Small magnetic fluctuation amplitude yields substantial transport

$$\chi_{\parallel}\left(\frac{\tilde{B}_r}{B_0}\right)^2 \sim \chi_{\perp} \qquad \text{for} \quad \frac{\tilde{B}_r}{B_0} \sim 10^{-3}$$

Model for stochastic magnetic transport

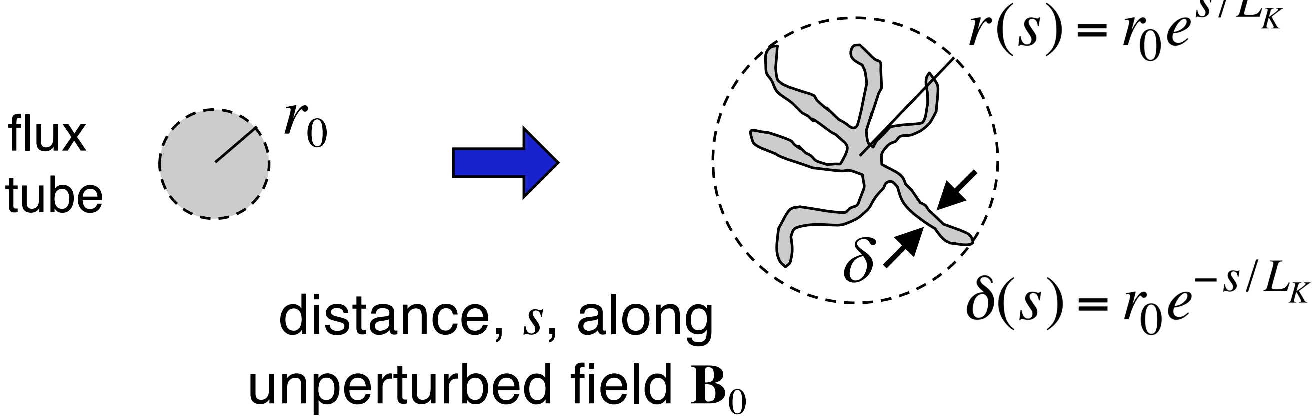
- Very few self-consistent models for magnetic fluctuation induced transport have been developed

- Most analysis has been for a static, imposed set of magnetic fluctuations
  - Error fields from misaligned magnets and other stray fields
  - Low frequency turbulence

- Stochastic magnetic transport is described by a double diffusion process
  1. Random walk of the magnetic field lines
  2. Collisional or other cross-field transport process is required for particles to “lose memory” of which field line they follow

Magnetic diffusion

Divergence of neighboring field lines:



L_K = Kolmogorov-Lyaponov length

Magnetic diffusion

Magnetic diffusion coefficient:

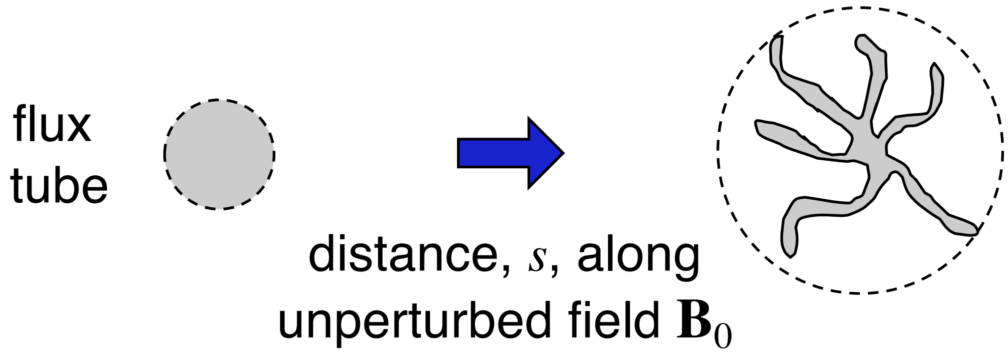
$$D_m = \frac{\langle (\Delta r)^2 \rangle}{\Delta s} = \frac{\int_0^\infty \tilde{B}_r(0)\tilde{B}_r(s)ds}{B_0^2}$$ (units of length)

$$= L_{ac} \langle \tilde{B}_r^2 \rangle / B_0^2$$ $L_{ac} =$ auto-correlation length for $\tilde{B}$

L_{ac} is related to the width of the $k_{||}$ spectrum, $L_{ac} \approx \pi / \Delta k_{||}$ $(\neq L_K)$ in general

Stochastic transport in the collisionless limit

Consider a test particle streaming along the magnetic field



$$\langle (\Delta r)^2 \rangle = D_m \Delta s$$

average radial displacement
associated with field line diffusion

For $\lambda_{mfp} >> L_{ac}$

$$\boxed{\chi_{st} = \frac{\langle (\Delta r)^2 \rangle}{\Delta t} = \frac{D_m \lambda_{mfp}}{\tau_c} = D_m v_T}$$

$v_T = \sqrt{T/m}$ (thermal velocity)

$\tau_c = \lambda_{mfp} / v_T$ (collision time)

Stochastic transport in the collisional limit

For $\lambda_{mfp} << L_{ac}$, test particle must first diffuse $\Delta s \sim L_{ac}$ along the field

The parallel diffusion is given by: $\chi_{\|} = \dfrac{\langle (\Delta s)^2 \rangle}{\Delta t} = \dfrac{\lambda_{mfp}^2}{\tau_c}$

$$\boxed{\chi_{st} = \frac{\langle (\Delta r)^2 \rangle}{\Delta t} = \frac{D_m \Delta s}{\Delta t} = \frac{D_m L_{ac}}{L_{ac}^2 / \chi_{\|}} = D_m v_T \left(\frac{\lambda_{mfp}}{L_{ac}} \right) = \chi_{\|} \frac{\langle \tilde{B}_r^2 \rangle}{B_0^2}}$$

Stochastic transport in the collisional limit

For $\lambda_{mfp} << L_{ac}$, test particle must first diffuse $\Delta s \sim L_{ac}$ along the field

The parallel diffusion is given by: $$\chi_{\parallel} = \frac{\langle (\Delta s)^2 \rangle}{\Delta t} = \frac{\lambda_{mfp}^2}{\tau_c}$$

$$\chi_{st} = \frac{\langle (\Delta r)^2 \rangle}{\Delta t} = \frac{D_m \Delta s}{\Delta t} = \frac{D_m L_{ac}}{L_{ac}^2 / \chi_{\parallel}} = D_m v_T \left(\frac{\lambda_{mfp}}{L_{ac}} \right) = \chi_{\parallel} \frac{\langle \tilde{B}_r^2 \rangle}{B_0^2}$$

Smooth transitional form: $\chi_{st} = v_T L_{eff} \frac{\langle \tilde{B}_r^2 \rangle}{B_0^2}$ with $L_{eff}^{-1} = L_{ac}^{-1} + \lambda_{mfp}^{-1}$

Krommes et al. provided a unifying discussion of various collisional limits with respect to characteristic scale lengths.

How well does the static field model work?

- Few direct measurements of stochastic transport
- Inferences via energetic particles in tokamak plasmas, exploiting expected velocity dependence
- Self-organizing plasmas like the RFP and spheromak provide good opportunity to test expectations, because they exhibit a broad spectrum of low frequency magnetic fluctuations

Fluctuation-induced transport is related to correlated products

- **Electrostatic-fluctuation-induced** particle transport

Particle balance: $$\frac{\partial n}{\partial t} + \nabla \cdot \Gamma = S$$

$$\Gamma = n\mathbf{v} = n(\mathbf{v}_{||} + \mathbf{v}_{\perp}) = nv_{||}\mathbf{B}/B + n\mathbf{v}_{\perp}$$

For radial transport, we need to evaluate $(\nabla \cdot \Gamma)_r$

Suppose there are fluctuations: $n = n_0 + \widetilde{n} \quad \mathbf{E} = \mathbf{E}_0 + \widetilde{\mathbf{E}}$

$$\Longrightarrow \quad \Gamma_r = \langle \widetilde{n}\widetilde{\mathbf{v}}_{\perp} \rangle = \frac{\langle \widetilde{n}\widetilde{\mathbf{E}} \times \mathbf{B}_0 \rangle}{B_0^2} = \frac{\langle \widetilde{n}\widetilde{E}_{\perp} \rangle}{B_0}$$

Angle brackets = spatial average, ensemble average

Fluctuation-induced transport is related to correlated products

- **Magnetic-fluctuation-induced** particle transport

Note: $\nabla \cdot (n v_{||} \mathbf{B}/B) = \nabla \cdot (n v_{||} \mathbf{B})/B + n v_{||} \mathbf{B} \cdot \nabla (1/B)$

Suppose there are fluctuations: $\mathbf{B} = \mathbf{B}_0 + \widetilde{\mathbf{B}}$

$$\Rightarrow \quad (\nabla \cdot \Gamma_{||})_r = \frac{[\nabla \cdot \langle \widetilde{n v}_{||} \widetilde{\mathbf{B}} \rangle]_r}{B_0} + \langle \widetilde{n v}_{||} \widetilde{B}_r \rangle \nabla_r \left(\frac{1}{B_0} \right)$$

$$= \frac{1}{e B_0} \frac{1}{r} \frac{\partial}{\partial r} \left[r \langle \widetilde{J}_{||} \widetilde{B}_r \rangle \right] - \frac{\langle \widetilde{J}_{||} \widetilde{B}_r \rangle}{e B_0^2} \nabla_r B_0$$

Fluctuation-induced heat transport follows similarly to particle transport

Heat balance: $$\frac{\partial W}{\partial t} + \nabla \cdot \mathbf{Q} = S$$ (simplified)

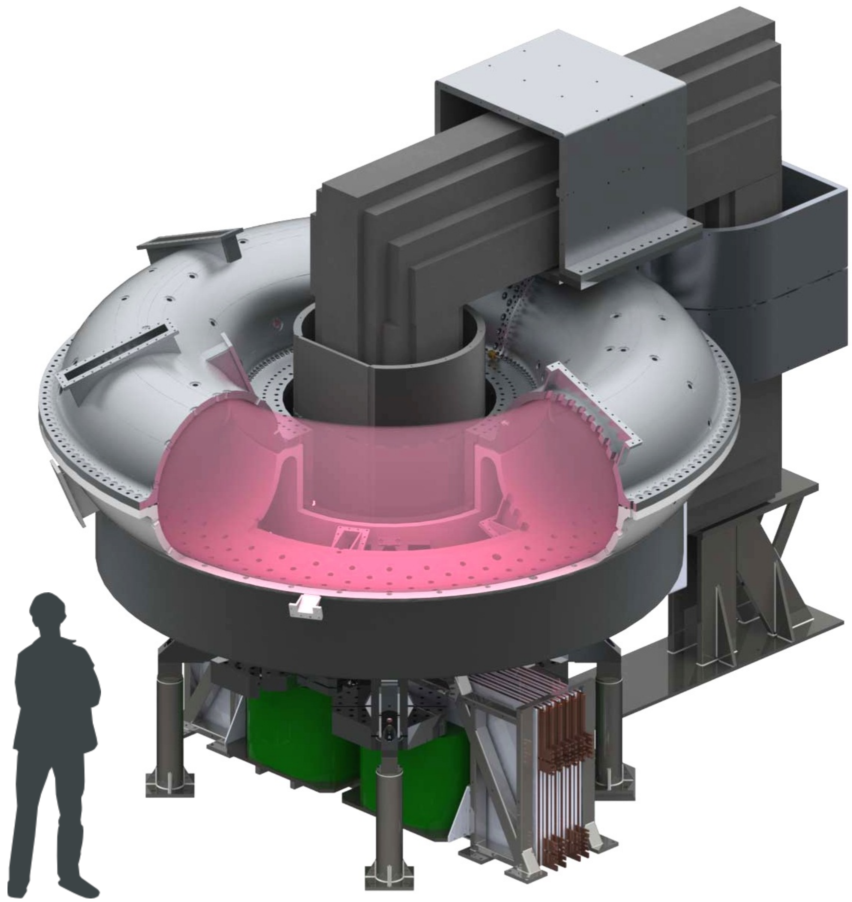
Where $\mathbf{Q} = Q_{||}\mathbf{B}/B + \mathbf{Q}_{\perp}$ is the heat flux

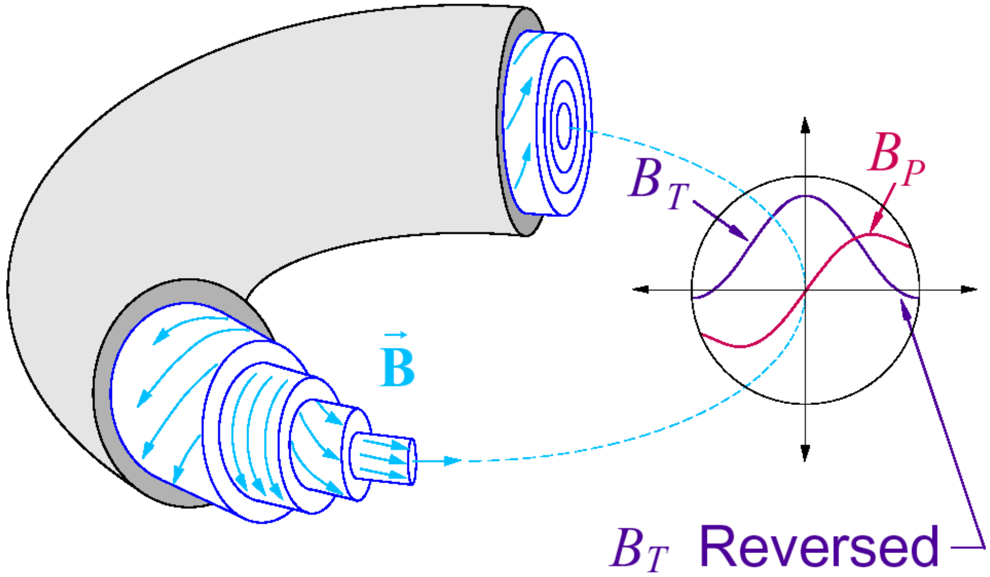
In direct analogy to particle transport:

$$Q_r = \langle \tilde{p}\widetilde{\mathbf{v}}_{\perp} \rangle = \frac{\langle \tilde{p}\widetilde{\mathbf{E}} \times \mathbf{B}_0 \rangle}{B_0^2} = \frac{\langle \tilde{p}\widetilde{E}_{\perp} \rangle}{B_0}$$ (electrostatic)

$$(\nabla \cdot \mathbf{Q}_{||})_r = \frac{1}{B_0}\frac{1}{r}\frac{\partial}{\partial r}\left[r\langle \widetilde{Q}_{||}\widetilde{B}_r \rangle \right] + \frac{\langle \widetilde{Q}_{||}\widetilde{B}_r \rangle}{B_0^2}\nabla_r B_0$$ (magnetic)

The MST reversed field pinch





Typical MST parameters:

$n \sim 10^{13}$ cm^{-3}

$T_e < 2$ keV

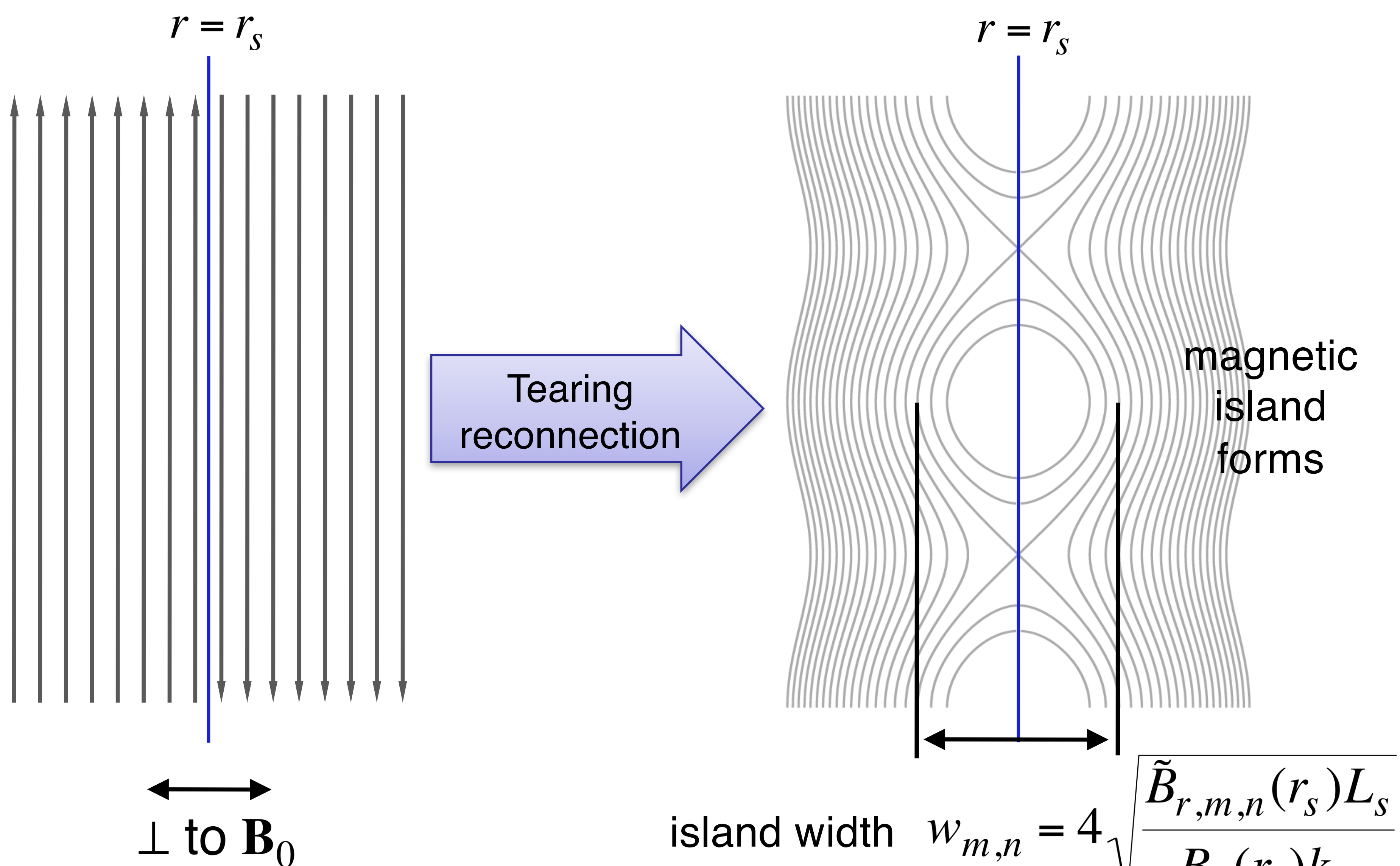
$T_{ion} \sim T_e$

$B < 0.5$ T

$\rho_{ion} \sim 1$ cm

Main source of a symmetry breaking magnetic field in the RFP is MHD tearing instability, which generates magnetic islands

resonant layer $\mathbf{k}\cdot\mathbf{B}_0 \Rightarrow 0$



island width $w_{m,n} = 4\sqrt{\dfrac{\tilde{B}_{r,m,n}(r_s)L_s}{B_0(r_s)k_\perp}}$

Chirikov threshold condition for stochastic instability

If neighboring magnetic islands overlap, the field lines are allowed to wander from island-to-island randomly.

$$s = \frac{1}{2}\frac{w_{n+1} + w_n}{\left|r_{s,n+1} - r_{s,n}\right|}$$

"stochasticity parameter"
(crudely the number of islands overlapping a given radial location)

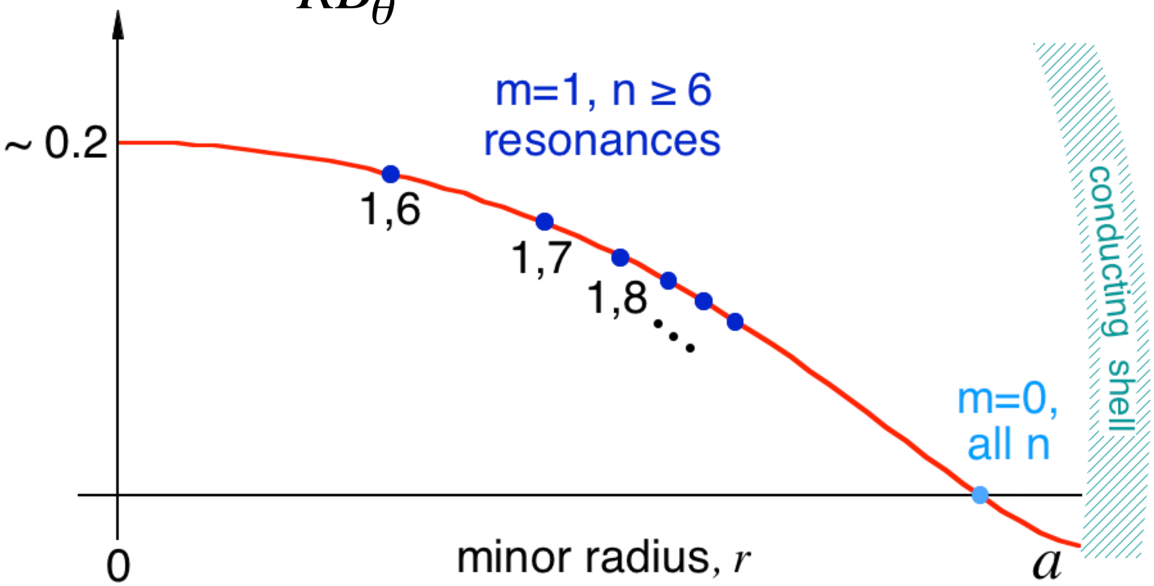
$s < 1$: islands do not overlap, no stochastic transport
(but transport across the island is typically enhanced by its topology)

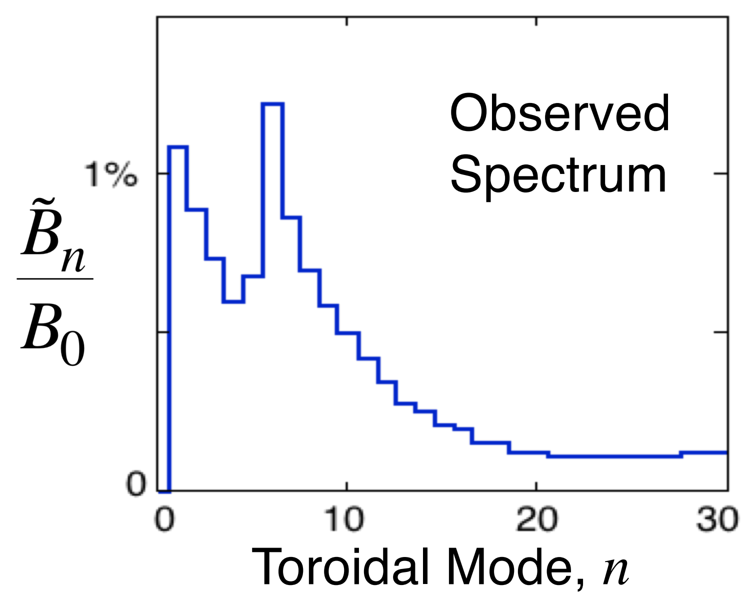
$s \sim 1$: weakly stochastic, magnetic diffusion and transport are transitional
(e.g., as discussed by Boozer and White)

$s >> 1$: magnetic field line wandering is well approximated as a random-walk diffusion process

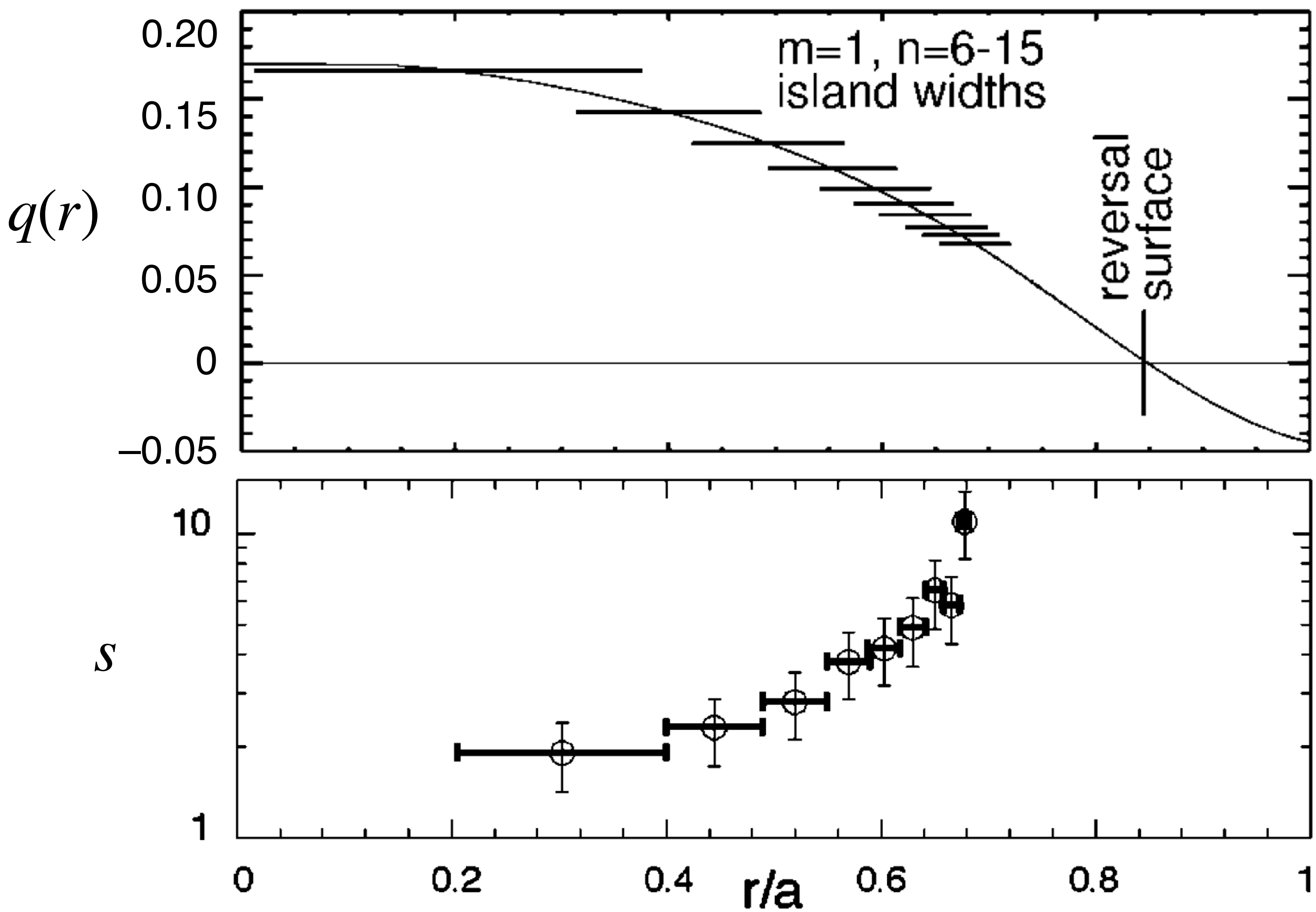
Many possible tearing resonances occur across the radius of the RFP configuration

$$q(r) = \frac{rB_\phi}{RB_\theta}$$

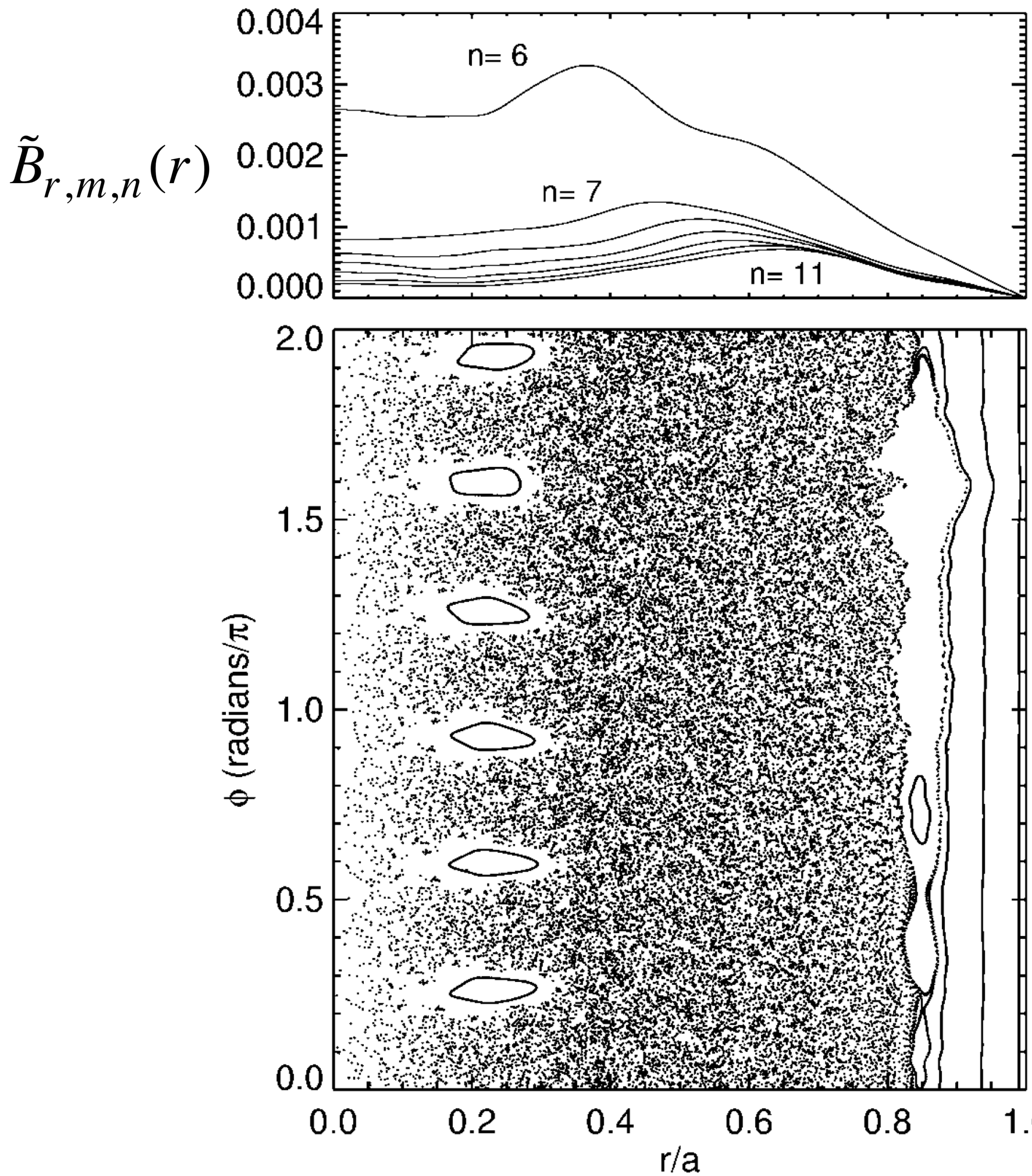




Chirikov threshold is exceeded, particularly in the mid-radius region where the density of rational magnetic surfaces is large



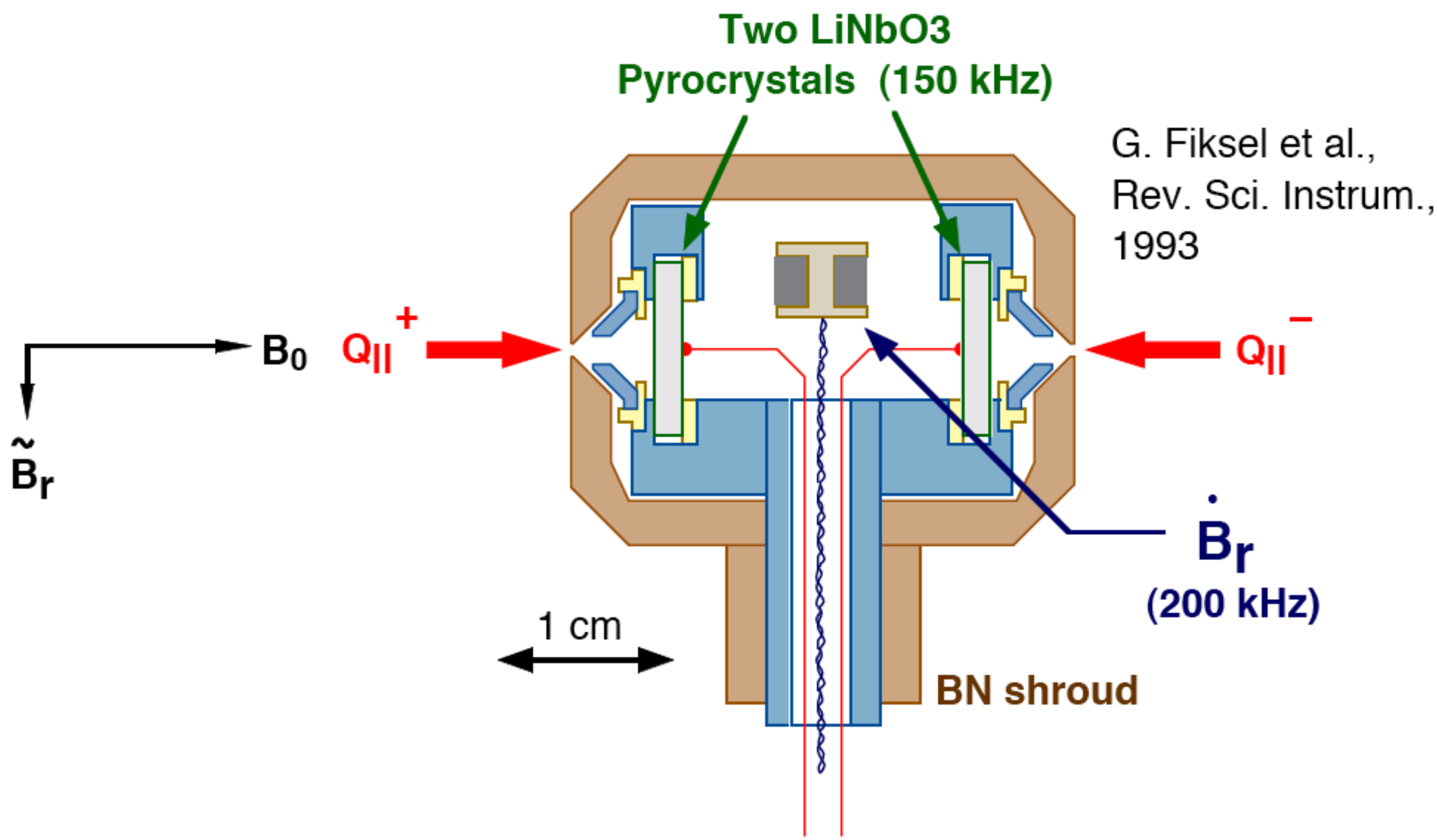
Magnetic puncture plot indicates widespread magnetic stochasticity



Eigenfunctions from nonlinear resistive MHD computation, normalized to measured $\tilde{B}_{m,n}(r=a)$.

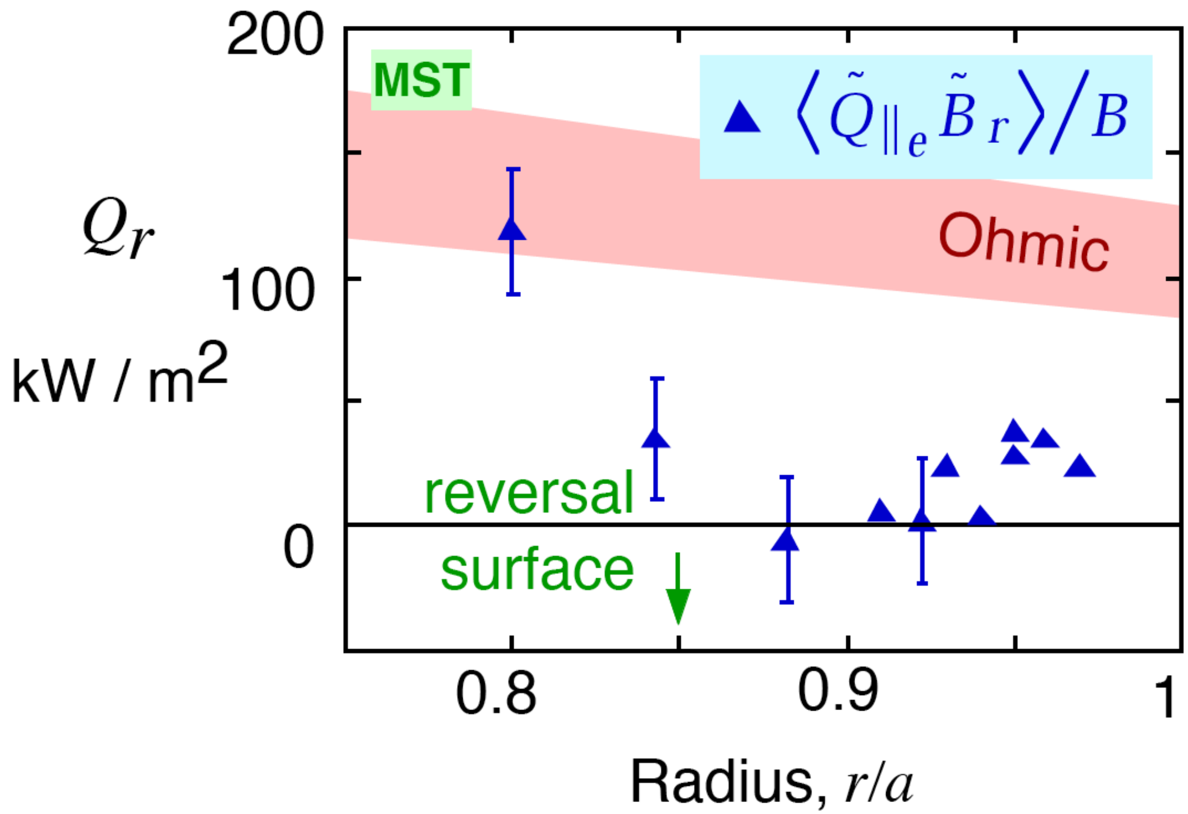
Field is modeled using $\tilde{B}_{m,n}(r)$ eigenfunctions, combined with equilibrium reconstruction that provides $B_0(r)$.

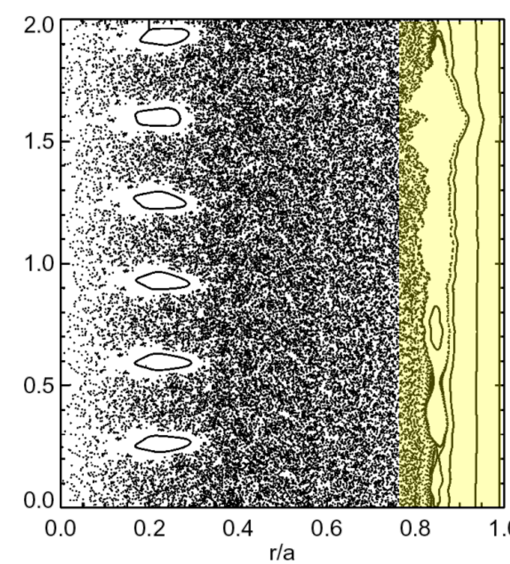
Direct measurement of magnetic fluctuation-induced stochastic transport



Measurements were made in MST (RFP), CCT (tokamak), and TJ-II (stellarator)

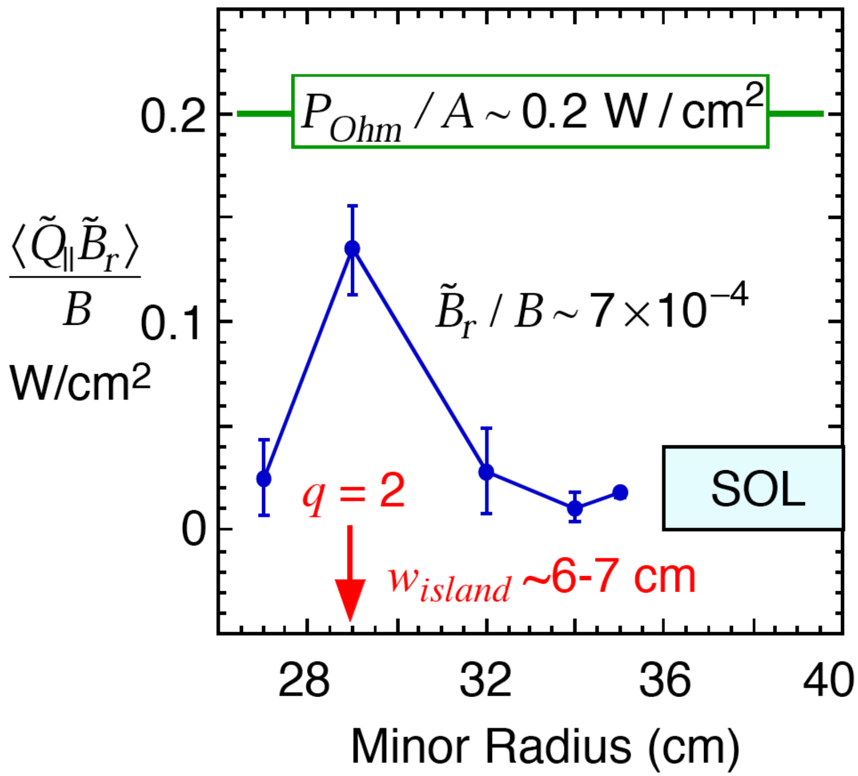
Measured electron heat flux in the edge of MST plasmas

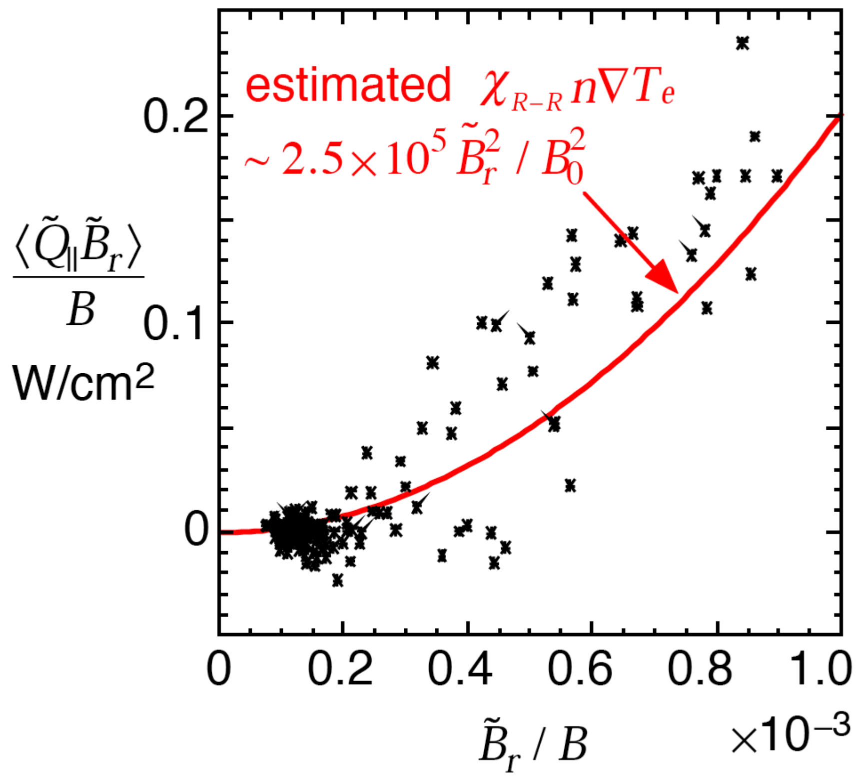




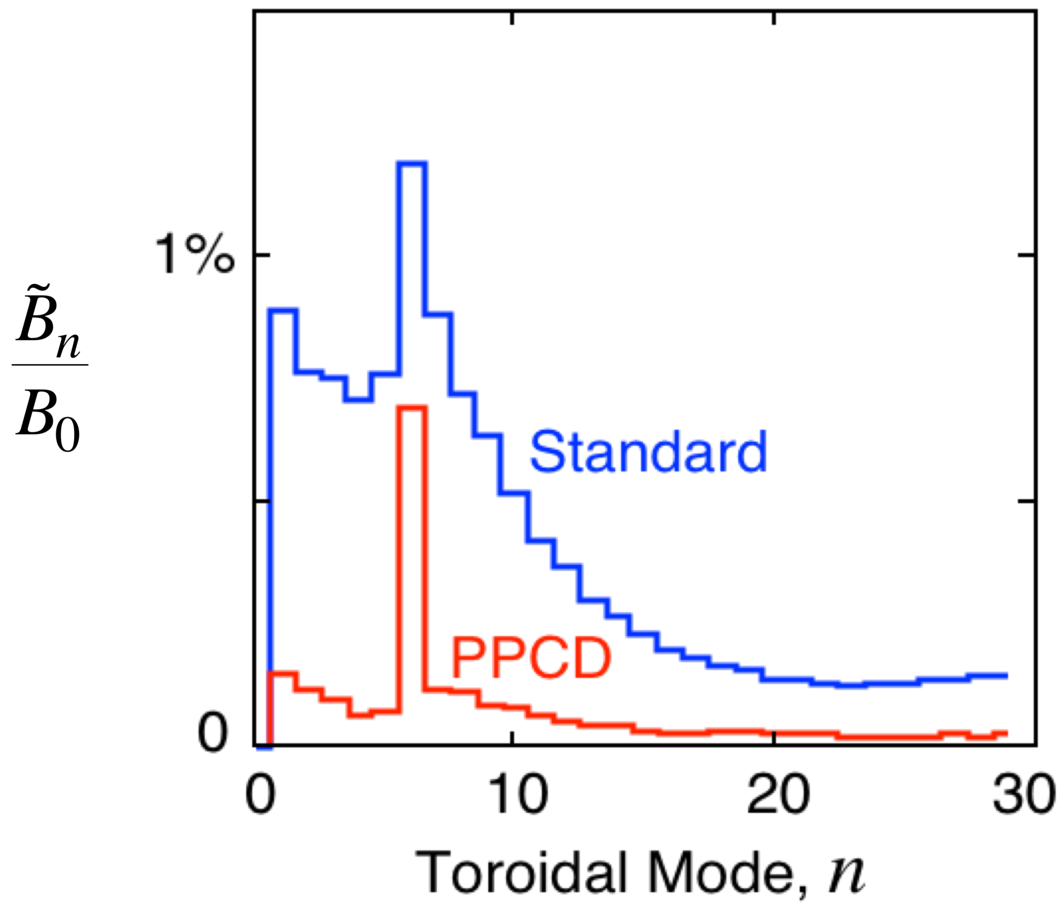
Measured island-induced heat flux in CCT (former tokamak at UCLA)

Heat flux in the magnetic island scales as if stochastic





The amplitude of the tearing fluctuations in the RFP can be reduced using current profile control (PPCD)



~5X reduction of most modes

allows tests of

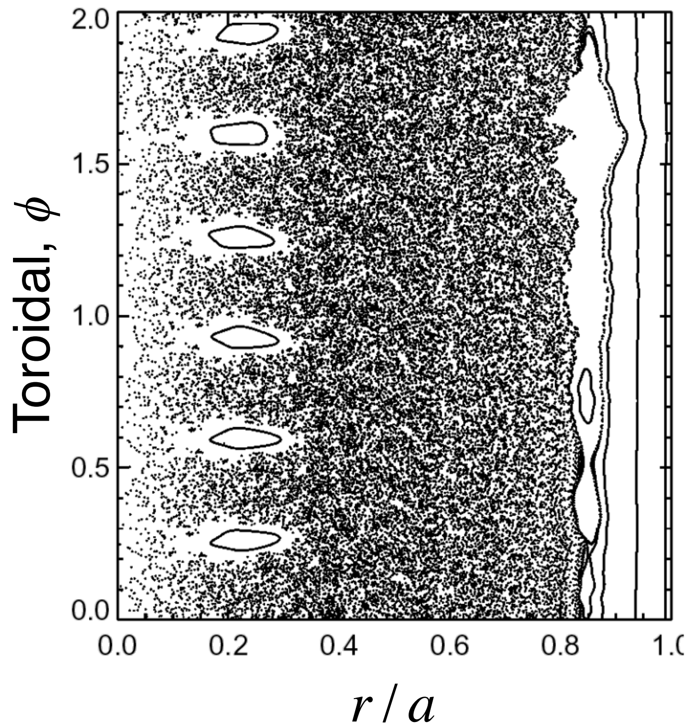
$\chi_{st} \sim \tilde{B}_r^2$ scaling

and dependence

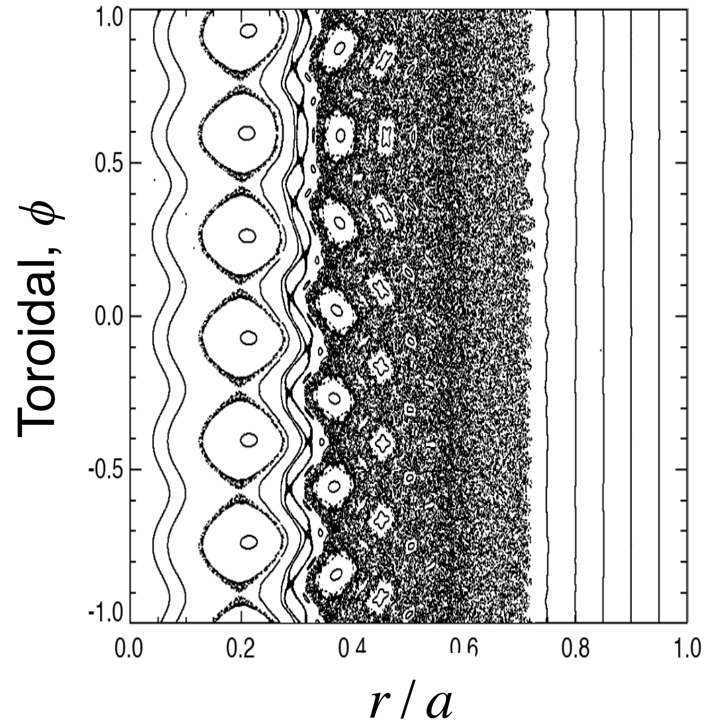
on spectral features

Region of stochastic field shrinks with current profile control

Standard



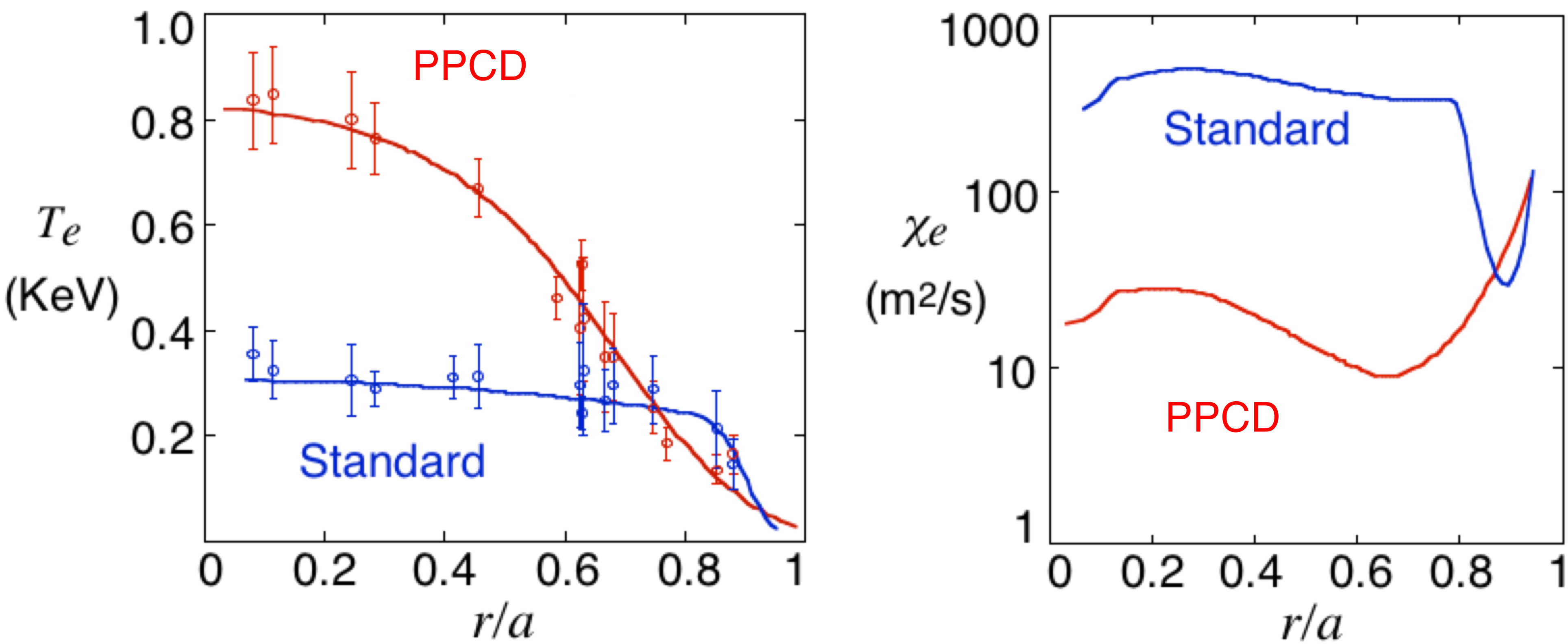
PPCD



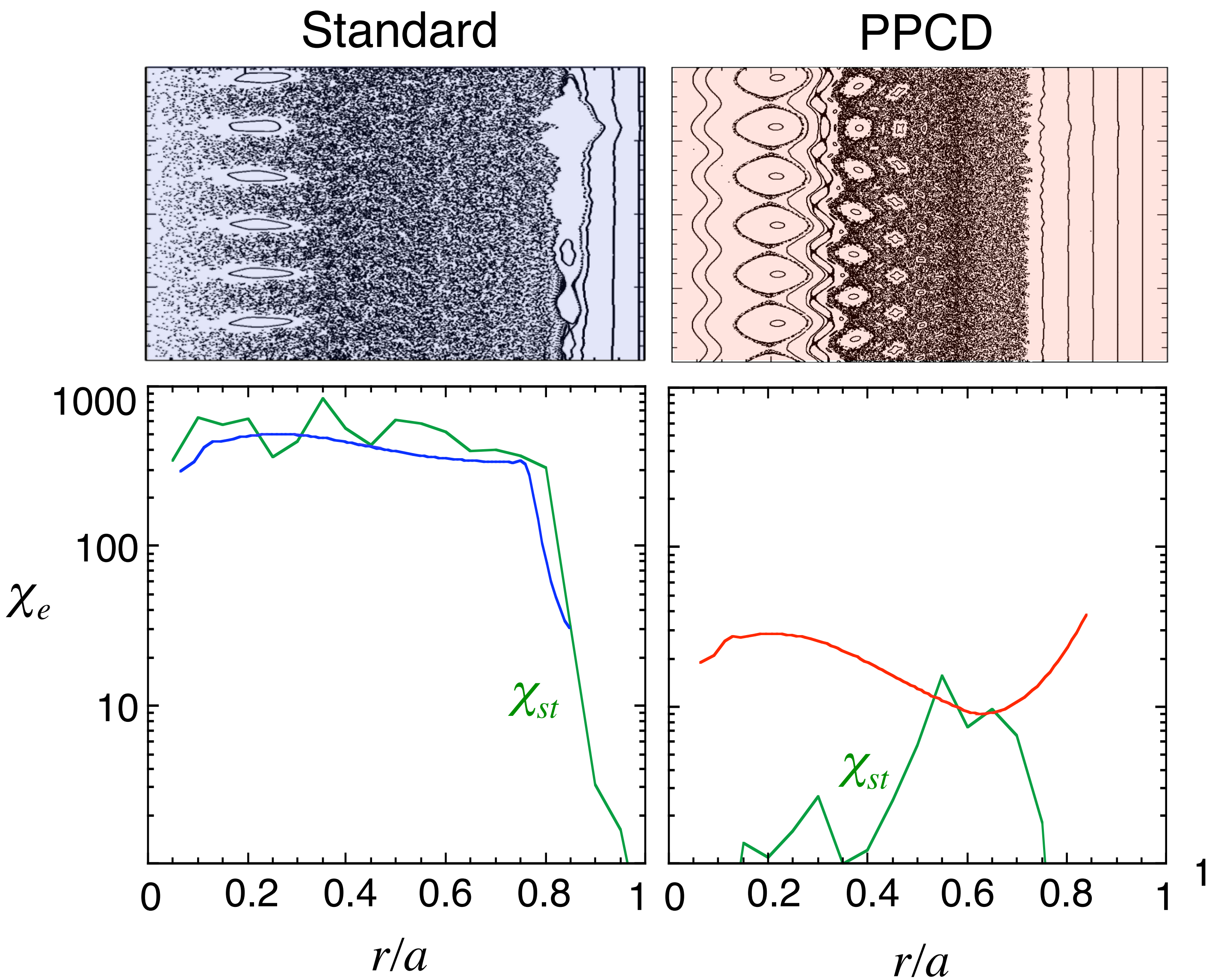
Power balance measurements provide the experimental electron heat conductivity profile

Electron heat flux

$$Q_e = \chi_e n \nabla_r T_e$$



Measured heat diffusivity consistent with collisionless stochastic transport model (where the field is stochastic)



Magnetic diffusivity is evaluated directly from an ensemble of magnetic field lines.

$$L_{ac} << \lambda_{mfp}$$

~ 1 m   ~ 30 m

$$\chi_{st} = D_m v_T$$

collisionless limit

Magnetic diffusivity as expressed by Rechester-Rosenbluth, PRL '78

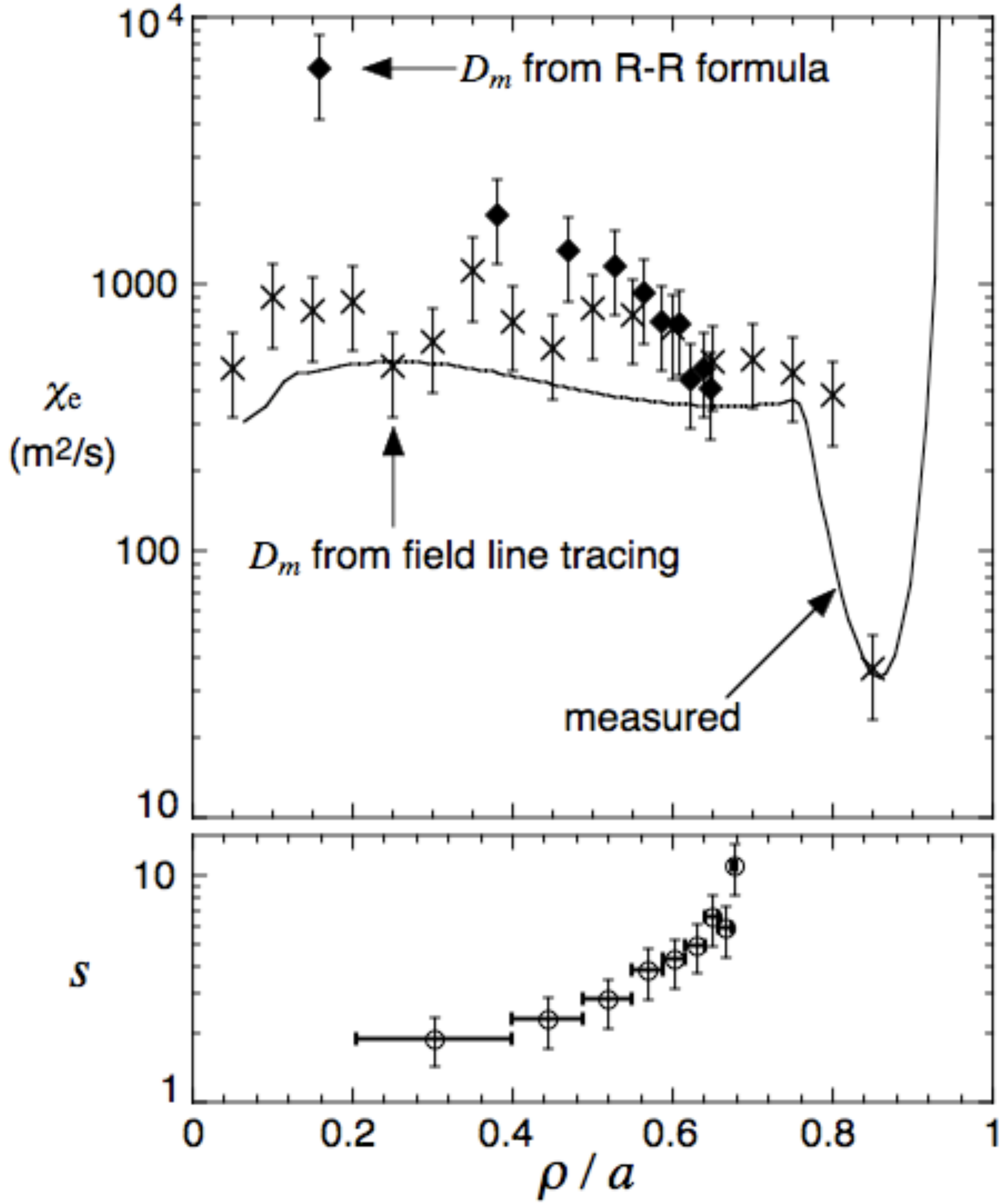
$$D_m = \pi R \sum_{m,n} \frac{\left| B_{r,m,n}(r) \right|^2}{B_z^2} \delta[m/n - q(r)]$$

auto-correlation length, L_{ac}

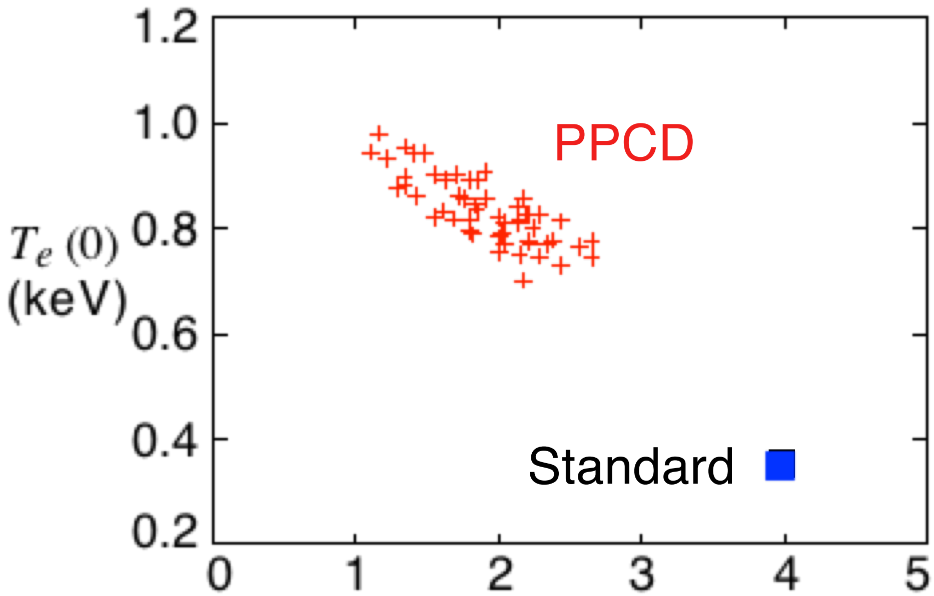
RMS fluctuation amplitude^2

… but **only** $k_{\|} = 0$ modes resonant nearby r

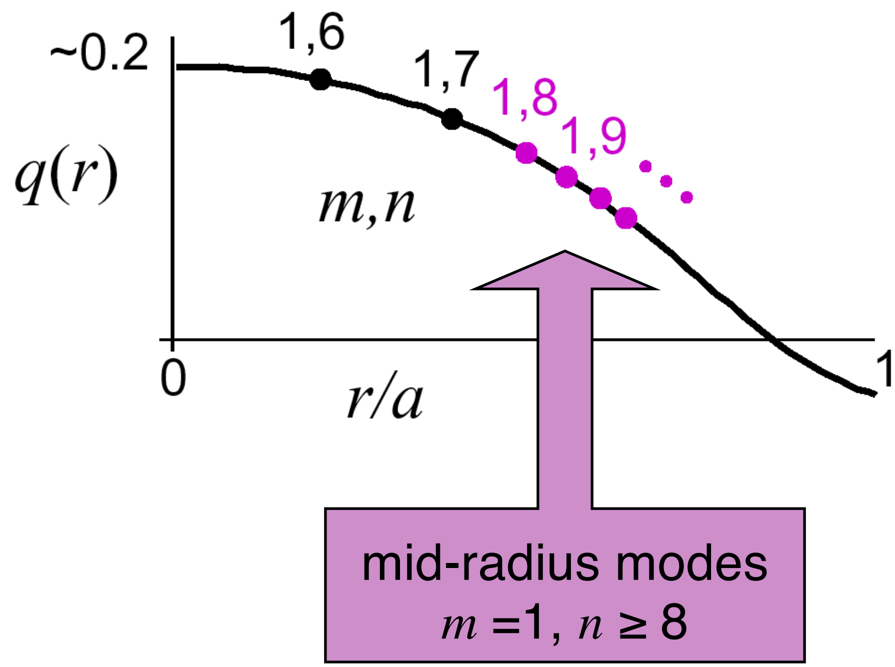
Rechester-Rosenbluth magnetic diffusivity overestimates χ_{st} for regions with low Chirikov parameter, s



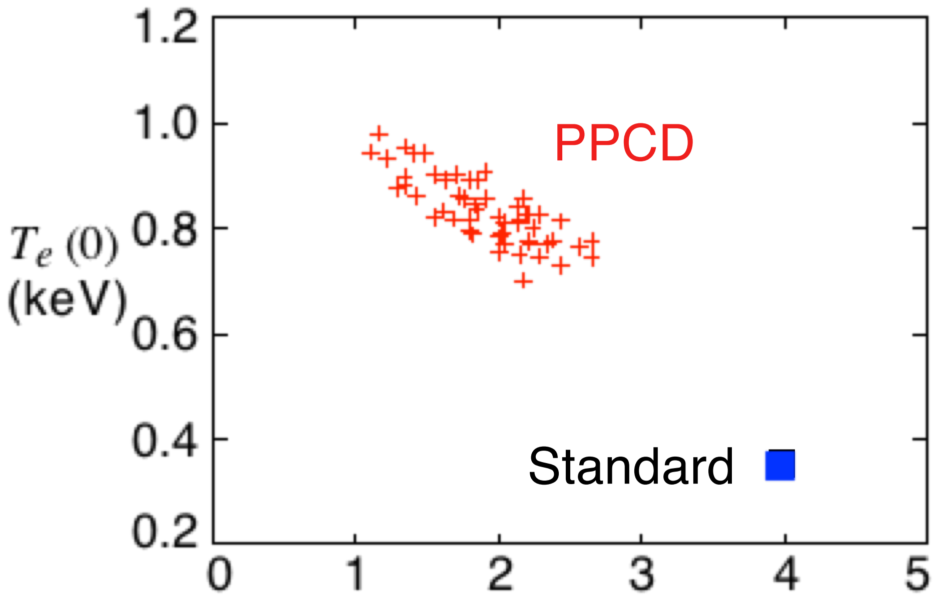
Electron temperature gradient correlates with amplitude of tearing modes resonant at mid-radius



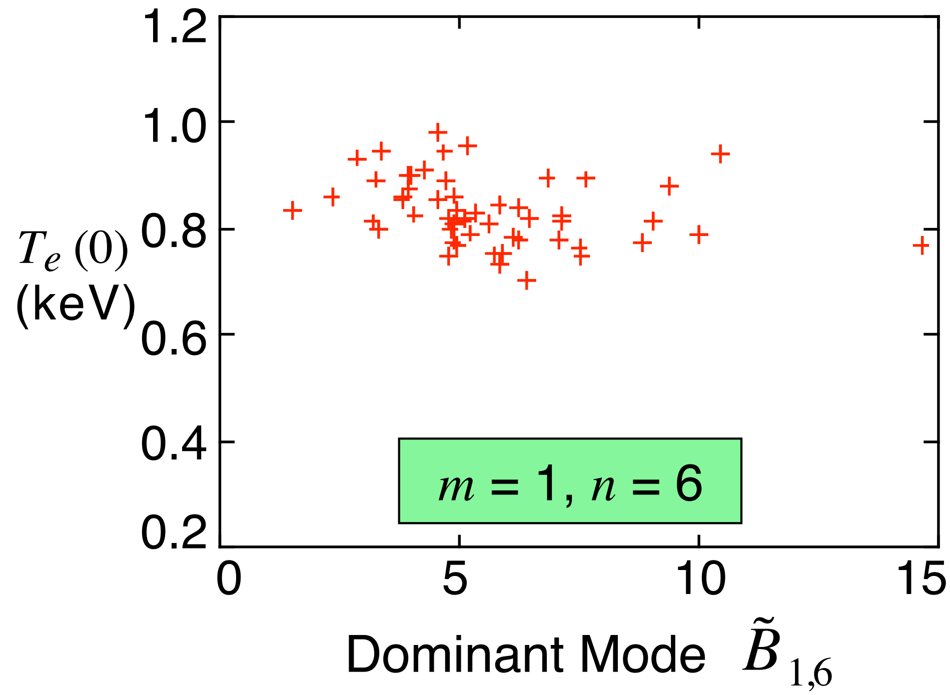
$$\tilde{B}_{rms} = \sqrt{\sum_{n=8}^{15} \tilde{B}_n^2(a)}$$

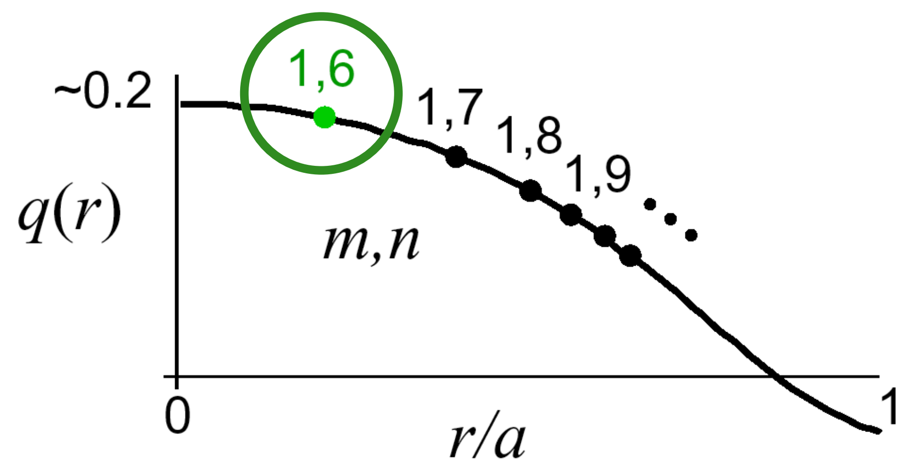


Electron temperature gradient does **not** correlate with largest mode, resonant in the core

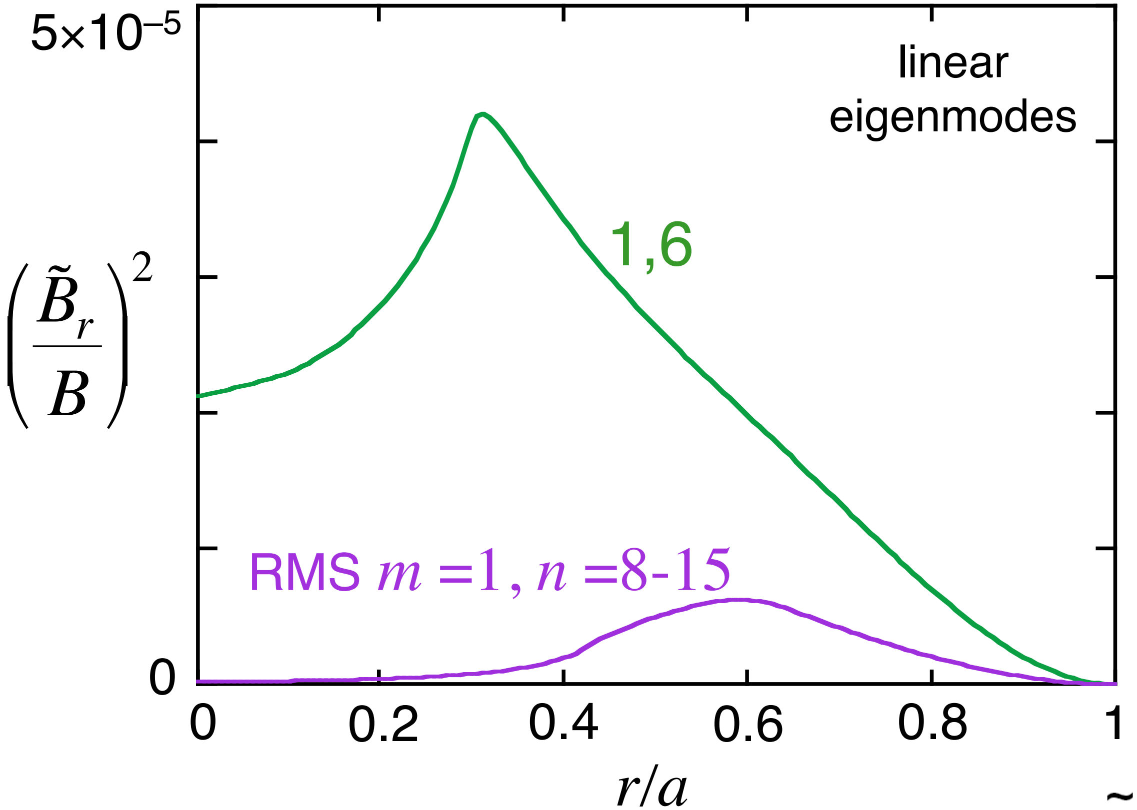


$$\tilde{B}_{rms} = \sqrt{\sum_{n=8}^{15} \tilde{B}_n^2(a)}$$



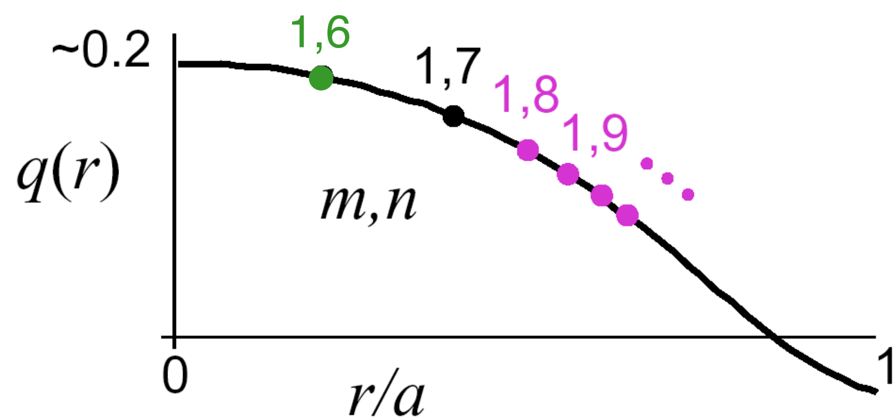


Though parallel streaming transport is nonlocal, the tearing reconnection process is local



$$D_m \sim \sum_{m,n} \frac{\left|B_{r,m,n}(r)\right|^2}{B_z^2}\,\delta[m/n - q(r)]$$

illustrates importance of $k_{\parallel} = 0$



In astrophysical plasmas, stochastic field can reduce heat transport

Reflects large transport anisotropy in a magnetized plasma.

Consider collisionless limit $L_{ac} << \lambda_{mfp}$:

$$\chi_{st} = D_m v_T = \chi_{\parallel} \frac{D_m}{\lambda_{mfp}} = \chi_{\parallel} \underbrace{\frac{L_{ac}}{\lambda_{mfp}} \left(\frac{\tilde{B}_r}{B_0} \right)^2}_{< 1, \text{ even for } \tilde{B} \sim B_0}$$

Has been applied to cooling flows in galactic clusters to argue small heat conduction.

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Main source of symmetry breaking magnetic field in the RFP is MHD tearing instability

- Linear stability analysis using force balance $\mathbf{J} \times \mathbf{B} = \nabla p$ yields

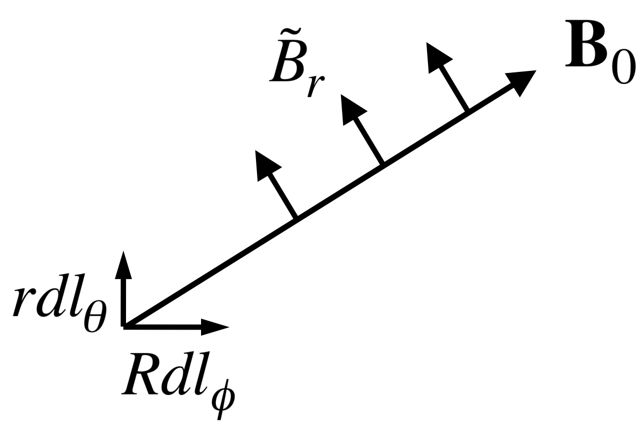
$$\tilde{B}_r'' + \left(\frac{F''}{F} + k^2 \right) \tilde{B}_r \approx 0 \qquad \mathbf{k} = \frac{m}{r}\hat{\theta} - \frac{n}{R}\hat{\phi} \qquad F = \mathbf{k} \cdot \mathbf{B}_0$$

$$\frac{F''}{F} \sim \frac{B_0''}{B_0} \sim \nabla_r \left(\frac{J_\parallel}{B} \right)$$

- Growth rate depends on $\nabla_r J_\parallel$ and the plasma's resistivity

- Mode resonance appears at the minor radius where $\mathbf{k} \cdot \mathbf{B}_0 \Rightarrow 0$

$$\mathbf{k} \cdot \mathbf{B}_0 = 0 \quad \Rightarrow \quad \frac{m}{n} = \frac{rB_\phi}{RB_\theta} = q(r)$$



Estimate of the auto-correlation length from the spectral width

For a tokamak $B_\phi >> B_\theta$ $\quad k_{||} = \dfrac{\mathbf{k}\cdot\mathbf{B}}{B} \approx \dfrac{1}{B_\phi}\left(\dfrac{m}{r}B_\theta + \dfrac{n}{R}B_\phi\right) = \dfrac{1}{R}\left(\dfrac{m}{q} + n\right)$

$$\Delta k_{||} \sim \underbrace{\Delta r}_{\text{mode radial width}} \left|\frac{\partial k_{||}}{\partial r}\right|_{r_s} = \frac{n}{R}\underbrace{\Delta r \left|\frac{1}{q}\frac{dq}{dr}\right|_{r_s}}_{\sim 1} \sim \frac{1}{R}$$

($n=1$ typically dominant)

Fluctuation-induced transport fluxes.

Linearizing the drift kinetic equation $\dfrac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f = 0 \qquad f = f_0 + \tilde{f}$

$$\frac{\partial f_0}{\partial t} = -\tilde{\mathbf{v}}\cdot\nabla\tilde{f} = -\left(\frac{\tilde{\mathbf{E}}\times\mathbf{B}_0}{B_0^2} + v_{\|}\frac{\tilde{\mathbf{B}}}{B_0}\right)\cdot\nabla\tilde{f}$$

drift associated with electrostatic fluctuations

streaming associated with magnetic fluctuations

Fluctuation-induced transport fluxes.

Moments of the d.k.e. lead to the fluctuation-induced transport fluxes

particle $$\Gamma_r = \langle \int dv \tilde{f} \left(\frac{\tilde{\mathbf{E}} \times \mathbf{B}_0}{B_0^2} + v_{\parallel} \frac{\tilde{\mathbf{B}}}{B_0} \right) \cdot \hat{\mathbf{r}} \rangle = \langle \tilde{n} \tilde{E}_{\perp} \rangle / B_0 + \langle \tilde{J}_{\parallel} \tilde{B}_r \rangle / e B_0$$

electrostatic magnetic

energy $$Q_r = \langle \int dv\, v^2 \tilde{f} \left(\frac{\tilde{\mathbf{E}} \times \mathbf{B}_0}{B_0^2} + v_{\parallel} \frac{\tilde{\mathbf{B}}}{B_0} \right) \cdot \hat{\mathbf{r}} \rangle = \langle \tilde{p} \tilde{E}_{\perp} \rangle / B_0 + \langle \tilde{Q}_{\parallel} \tilde{B}_r \rangle / B_0$$

where $\langle \dots \rangle$ denotes an appropriate average, e.g., over an unperturbed magnetic flux surface