Magnetohydrodynamic description of plasmas

Part I

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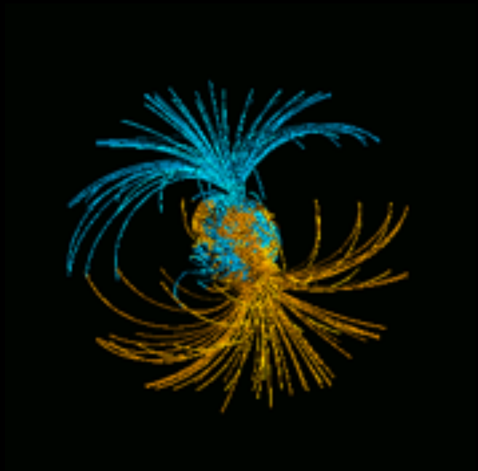
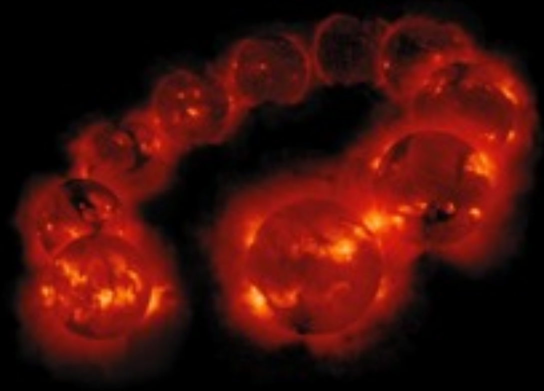
Astrophysical plasmas

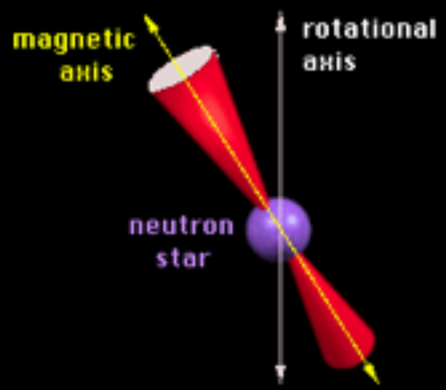


Earth and planets

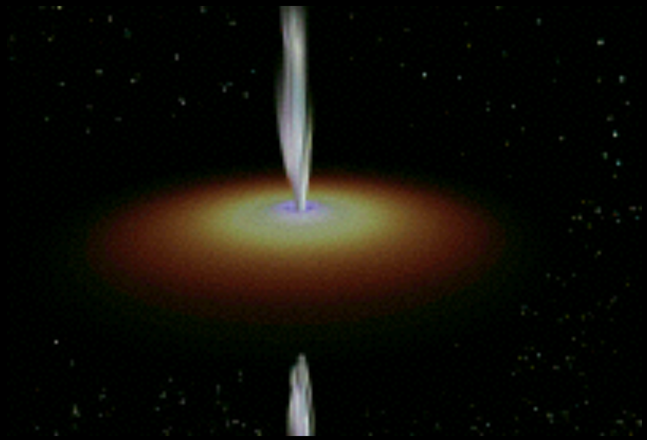


Sun and stars

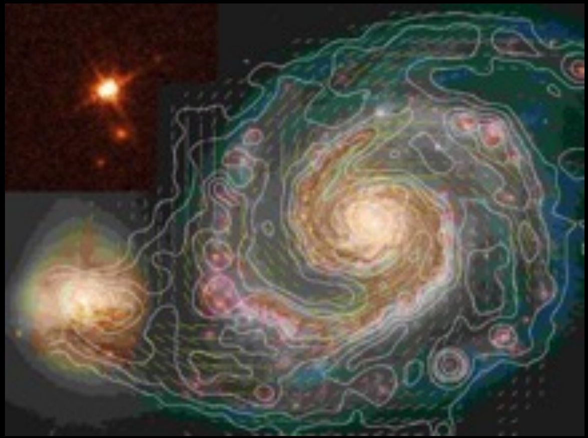
Interstellar medium



Pulsars



Accretion disks



Galaxies

What do we mean by magnetohydrodynamics ?

➡ It is a fluid-like theoretical description for the dynamics of matter

➡ Baryonic matter in the Universe is mostly hydrogen.

➡ At temperatures above 10^4 K it becomes a hydrogen plasma, i.e. a gas made of protons and electrons

➡ The large scale behavior of this gas can be described through fluidistic equations (Navier-Stokes).

➡ This fluid is made of electrically charged particles and therefore it suffers electric and magnetic forces.

➡ Not only that, these charges are sources of self-consistent electric and magnetic fields. Therefore, the fluid equations will couple to Maxwell's equations.

➡ At small spatial scales (and fast timescales) non-fluid or kinetic effects become non-negligible.

MHD equations

➡ The MHD equations are:

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \bullet (\rho \vec{u}) \qquad p = p_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$\rho \frac{\partial \vec{u}}{\partial t} = -\rho (\vec{u} \bullet \vec{\nabla}) \vec{u} - \vec{\nabla} p + \frac{1}{4\pi} (\vec{\nabla} \times \vec{B}) \times \vec{B} + \vec{F}_{ext} + \vec{\nabla} \bullet \overleftrightarrow{\sigma}_{visc}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{B}) + \eta \nabla^2 \vec{B}, \qquad \vec{\nabla} \bullet \vec{B} = 0$$

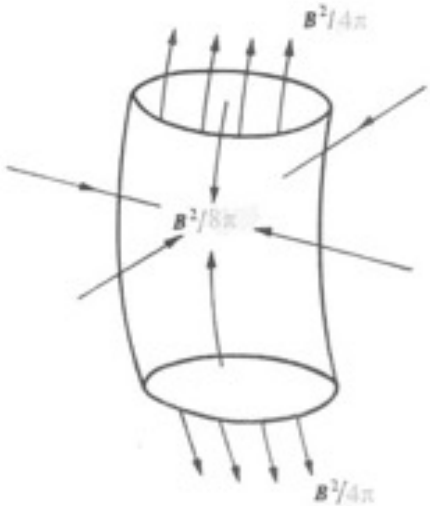
which describe the dynamics of the fluid as well as the evolution of the magnetic field.

➡ The induction equation is the result of Ohm's law

$$\vec{E} + \tfrac{1}{c} \vec{u} \times \vec{B} = \frac{1}{\sigma} \vec{J}, \qquad \eta = \frac{c^2}{4\pi\sigma}$$

and Faraday's equation.

MHD equations



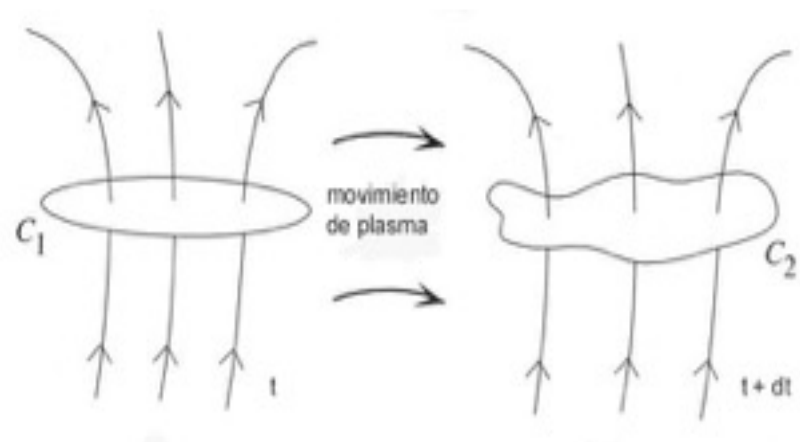
Magnetic pressure
and magnetic tension

➢ The magnetic force can be split into:

$$\frac{1}{4\pi}(\vec{\nabla}\times \vec{B})\times \vec{B} = \frac{1}{4\pi}(\vec{B}\cdot\vec{\nabla})\vec{B} - \vec{\nabla}\left(\frac{B^2}{8\pi}\right)$$

➢ In the asymptotic limit of negligible resistivity:

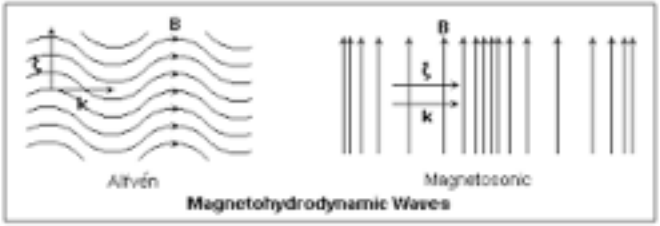
$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}\times\vec{B})$$



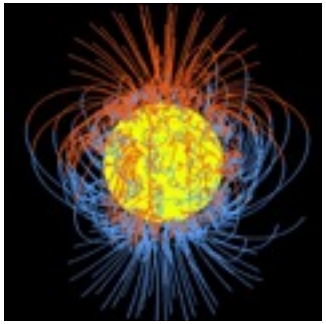
Frozen-in condition

Applications of MHD

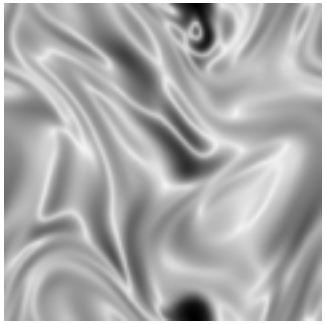
➡Within this level of description (which is adequate at large spatial scales) there is a variety of important plasma processes that have traditionally been addressed:



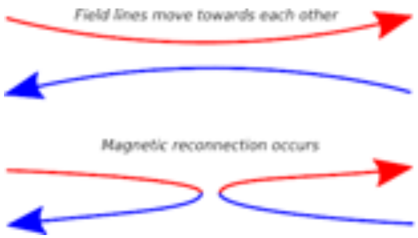
- Instabilities and wave propagation (Alfven and magnetosonic)



- Dynamo mechanisms to generate magnetic fields



- MHD turbulence



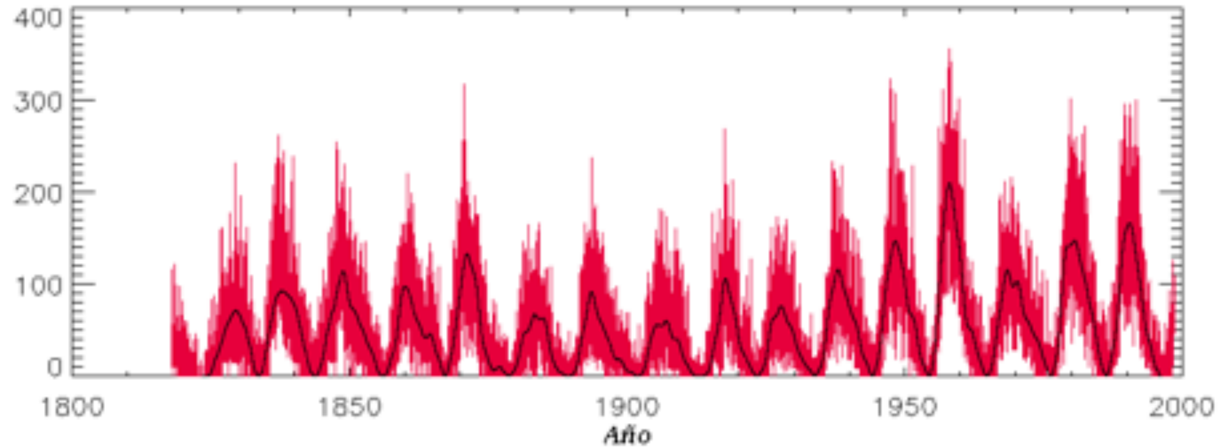
- Magnetic reconnection

Magnetic field of the Sun

➢Number of sunspots vs. time

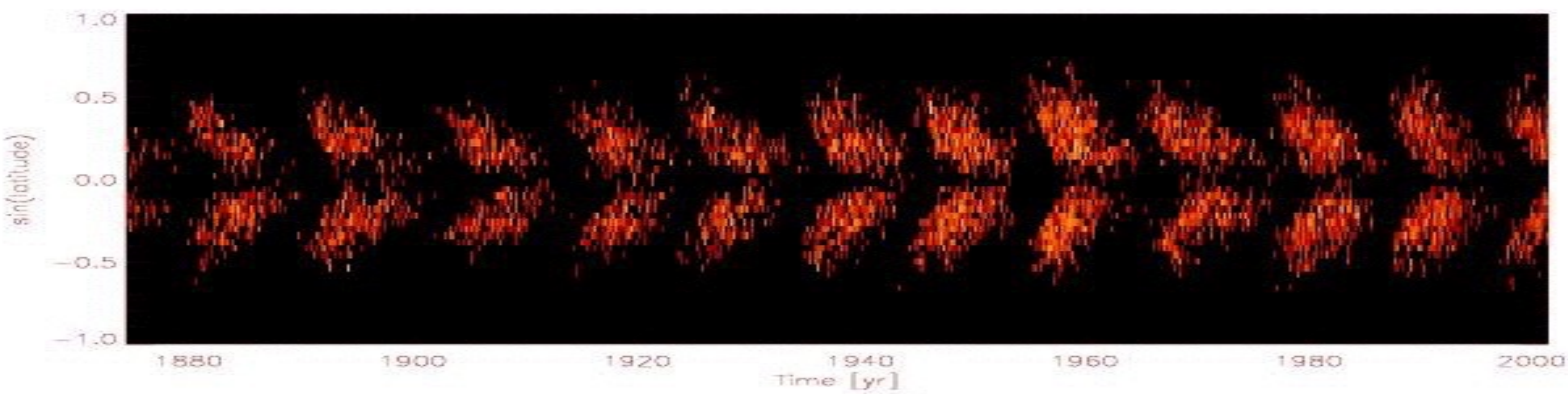
➢It clearly shows an 11 yr period with irregularities in its maxima, its periods and rise-fall times.

➢Area covered by spots as a function of latitude and time.



➢At the beginning of each cycle, sunspots are born at latitudes of $\pm 30^{\circ}$ and migrate to the Equator.

➢ Magnetic polarities are reversed from one cycle to the next and are different at different hemispheres (Hale´s law)



Kinematic dynamos

➢ If we assume the magnetic field **B** to be very small, the MHD equations decouple. We can first solve the equations of motion. For instance, in the incompressible limit

$$\frac{\partial \vec{u}}{\partial t} = -(\vec{u} \bullet \vec{\nabla})\vec{u} - \frac{1}{\rho}\vec{\nabla} p + \nu \nabla^2 \vec{u} \quad , \qquad \vec{\nabla} \bullet \vec{u} = 0$$

➢ Now that we know $\vec{u}(\vec{x},t)$, we can solve the induction equation to obtain $\vec{B}(\vec{x},t)$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{B}) + \eta \nabla^2 \vec{B}, \qquad \vec{\nabla} \bullet \vec{B} = 0$$

➢ This particular and convenient approximation is known as the kinematic dynamo. Note that the induction equation is linear in $\vec{B}(\vec{x},t)$ for any given $\vec{u}(\vec{x},t)$. For stationary flows, there will be a dynamo solution whenever

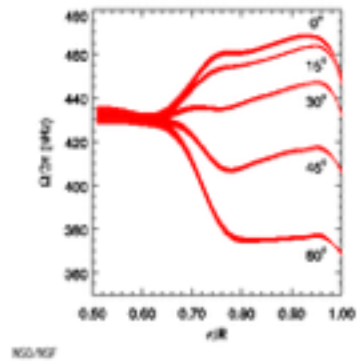
$$\vec{B}(\vec{x},t) = \vec{B}_0(\vec{x}) e^{\gamma t} \quad , \qquad \gamma > 0$$

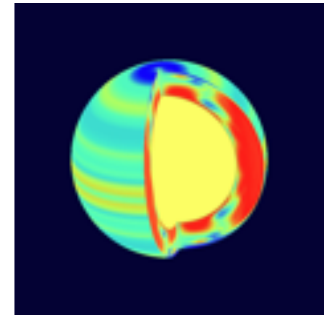
What kind of permanent flows are ubiquitous in astrophysical objects ?

Rotation and Convection

Rotation (macro)

- Radial differential rotation
- Latitudinal differential rotation

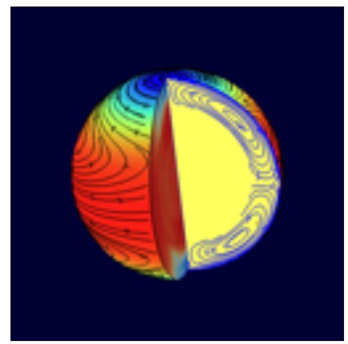




Omega effect

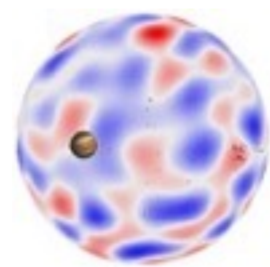
Meridional flow (macro)

- From equator to poles at 20 m/s



Convection (micro)

- Helicoidal convective turbulence
- Giant cells (driven by Coriolis)
- Regular and stochastic components



Alpha effect

1D simulations

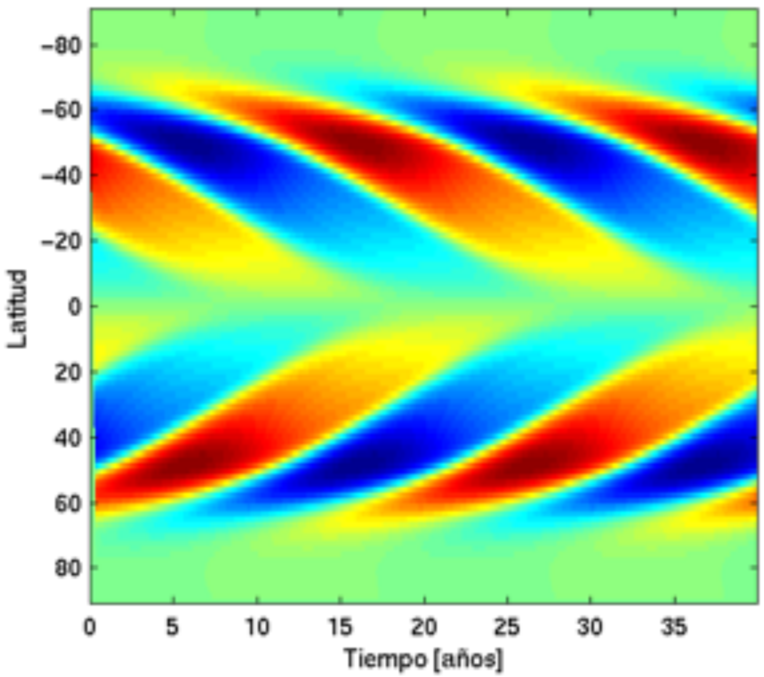
- We integrate the induction equation numerically, assuming axi-symmetry.
- We use empirical profiles of differential rotation and meridional flow. (Mininni & Gómez 2002, ApJ 573, 454).

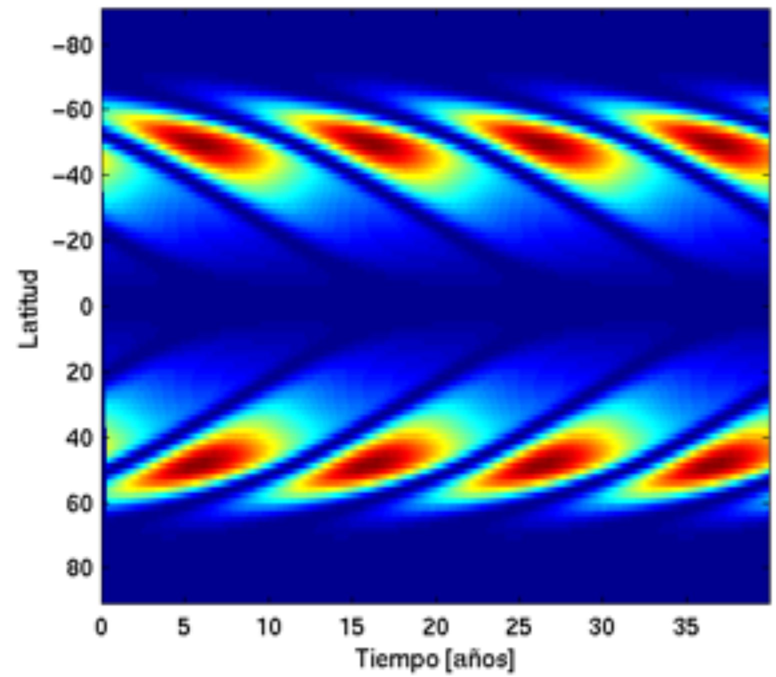
$$\frac{\partial B_\phi}{\partial t} = -(U_r + \varepsilon \frac{\partial U_\theta}{\partial \theta}) B_\phi - \varepsilon U_\theta \frac{\partial B_\phi}{\partial \theta} + \overbrace{(\Delta\omega \cos\theta - \sin\theta \frac{\partial \omega}{\partial \theta}) A + \Delta\omega \sin\theta \frac{\partial A}{\partial \theta}}^{\text{Differential rotation}} + \frac{1}{\Re} \nabla_\theta^2 B_\phi$$

$$\frac{\partial A}{\partial t} = \underbrace{-(U_r + \varepsilon U_\theta \cot\theta) A - \varepsilon U_\theta \frac{\partial A}{\partial \theta}}_{\text{Meridional flow}} + \underbrace{\alpha(B_\phi)\, B_\phi}_{\text{Small-scale convection}} + \underbrace{\frac{1}{\Re} \nabla_\theta^2 A}_{\text{Dissipation}}$$

$$where \quad \Re = \frac{U_0 \delta R}{\eta} \quad , \quad \varepsilon = \frac{\delta R}{R} \quad , \quad \Delta\omega = \omega_{surf}(\theta) - \omega_{core} \quad , \quad \alpha = \frac{\alpha_0 + \delta\alpha}{1 + B_\phi^2 / B_0^2} \sin(\theta)\cos(\theta)$$

Non-stochastic butterfly diagrams





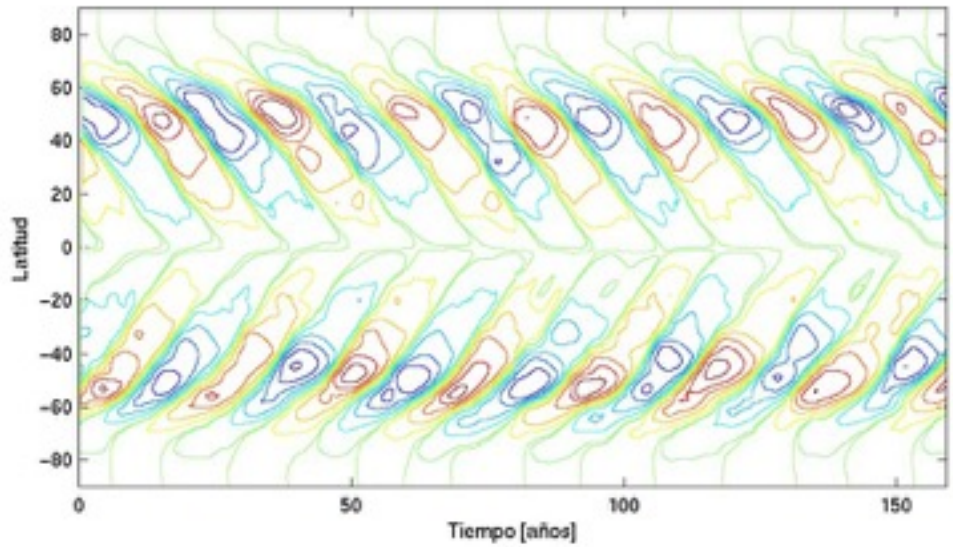
- ➢ Toroidal field vs. latitude and time.
- ➢ Hale´s law can cleary be observed.

- ➢ Magnetic energy vs. latitude and time.
- ➢ It is a proxy of Wolf´s number.

Role of stochasticity

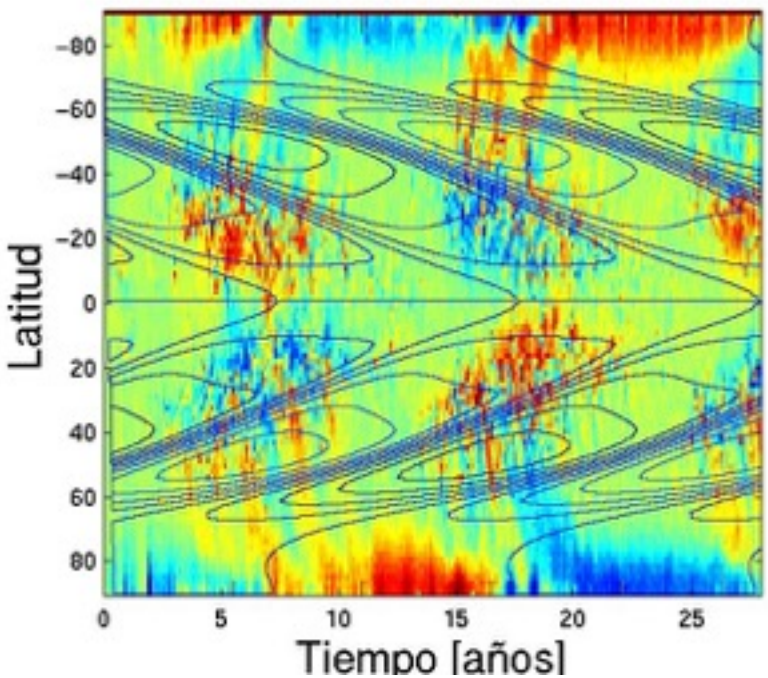
- We model $\delta\alpha$ as a gaussian stochastic process, with spatial and temporal correlations corresponding to typical giant cells.

$$\tau_{corr} \cong 30\, days \quad , \quad \lambda_{corr} \cong 2.10^{5}\, km$$

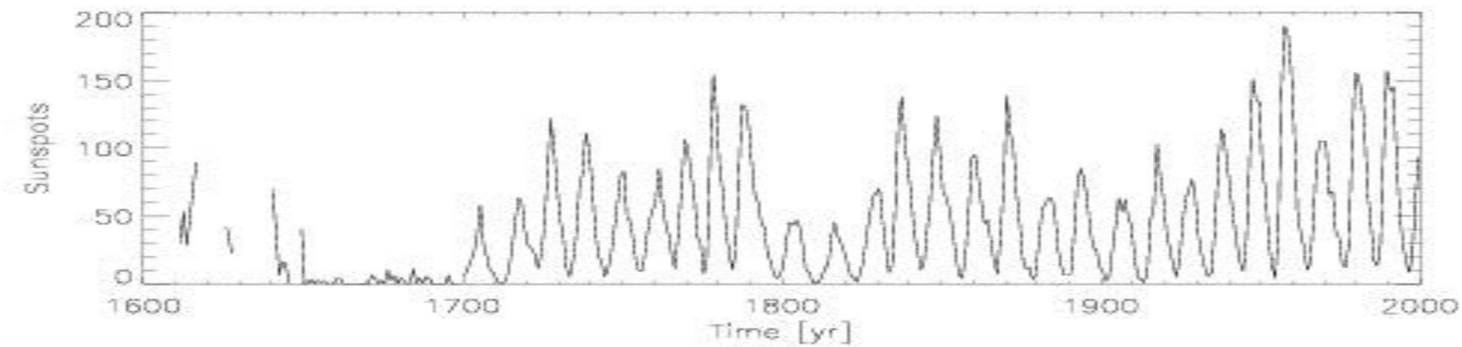


- Toroidal magnetic field obtained from solar magnetograms, displaying the change of polarity in the polar regions.

- Our results correctly reproduce the general behavior, although our butterflies arise at higher latitudes



Maunder minimum

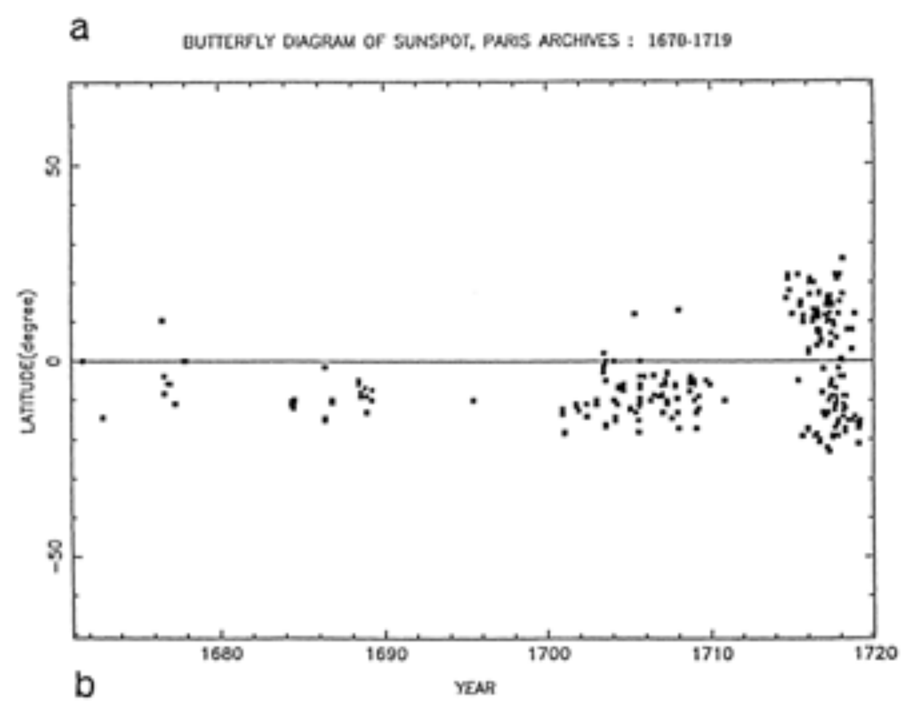


➢Wolf Number vs. time

➢ Maunder minimum lasts from 1650 to 1700.

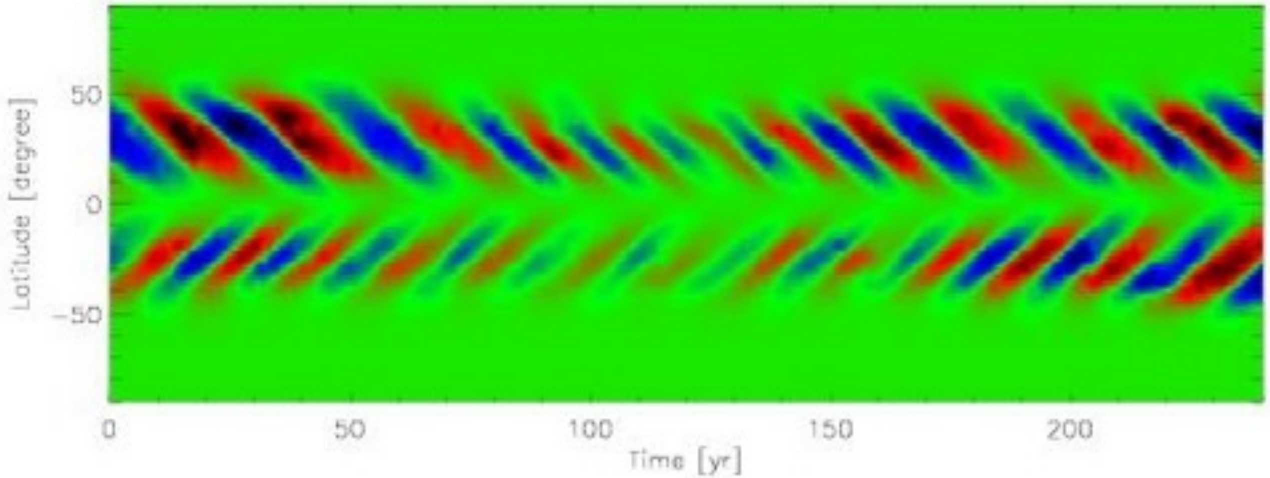
➢ There is evidence of more Maunder-like events (Beer 2000).

➢ N-S asymmetries were enhanced during the Maunder minimum (Ribes & Nesme-Ribes 1993).



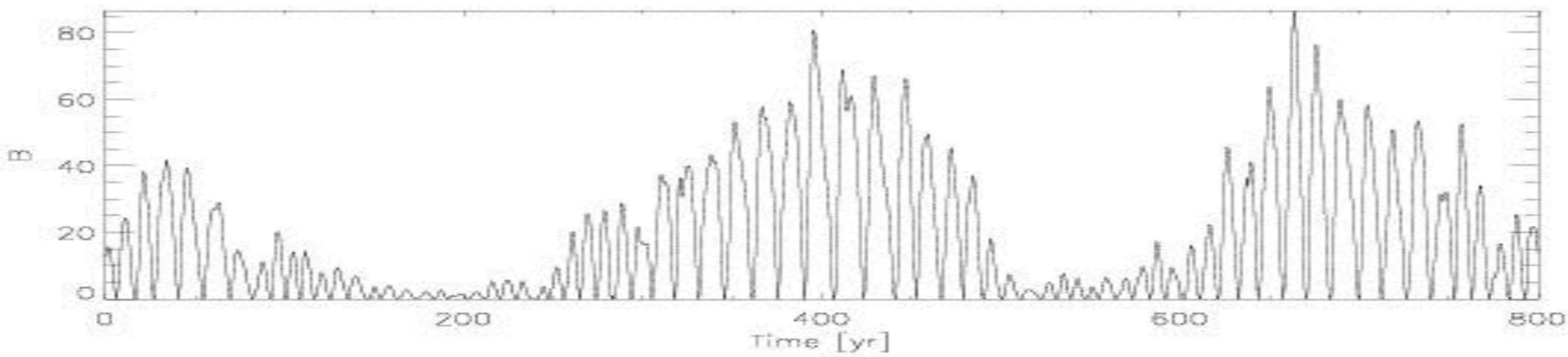
Maunder-like events

- Toroidal magnetic field for a long time integration (Gómez & Mininni 2006).



- A minimum of activity is observed at the center. After a few cycles, normal activity is restablished.

- Magnetic energy at mid-latitudes vs. time. Two Maunder-like events are observed.



Mean field theory

➢ It provides a quantitative expresion for the coefficient alpha. The first assumption is that there is a scale separation between the large scale magnetic field being generated and the small scale convective motions, i.e

$$\vec{B} \rightarrow \vec{B} + \vec{b} \quad , \quad \vec{u} \rightarrow \vec{U} + \vec{u} \quad , \quad < \vec{b} >= 0 =< \vec{u} >$$

where <...> is an average over small scales. To compute the evolution of the mean field, we average the induction equation

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{U} \times \vec{B}) + \vec{\nabla} \times < \vec{u} \times \vec{b} > \quad , \qquad \vec{\nabla} \bullet \vec{B} = 0$$

➢ The extra term can be interpreted as an electromotive force exerted by small scale motions

$$\varepsilon_{EMF} =< \vec{u} \times \vec{b} >$$

➢ We still need to obtain an expresion for the electromotive force, and that requires some assumptions (Steenbeck, Krause & Radler 1966).

Mean field theory

Let us substract the averaged equation from the general induction equation

$$\frac{\partial \vec{b}}{\partial t} = \underbrace{\cancel{\vec{\nabla}\times(\vec{U}\times\vec{b})}}_{[1]} + \vec{\nabla}\times(\vec{u}\times\vec{B}) + \underbrace{\cancel{\vec{\nabla}\times(\vec{u}\times\vec{b} - <\vec{u}\times\vec{b}>)}}_{[2]} \quad , \quad \vec{\nabla}\bullet\vec{b} = 0$$

[1] Can be removed with a Galilean transformation.
[2] It´s a departure from average of a second order quantity (FOSA).

➢ Let us further assume that this system evolves in a typical correlation time of these small scale convective motions.

➢Therefore $$\varepsilon_{EMF} = \tau < \vec{u}\times\vec{\nabla}\times(\vec{u}\times\vec{B}) > = \vec{\alpha}\bullet\vec{B} - \vec{\beta}\bullet\vec{\nabla}\times\vec{B}$$

where we neglected the gradient of the large scale magnetic field.

➢ For an isotropic state of these small scale flows, these tensors become

$$\alpha_{ij} = -\frac{\tau}{3} < \vec{u}\bullet\vec{\nabla}\times\vec{u} > \delta_{ij} \qquad \beta_{ij} = \frac{\tau}{2} < \vec{u}\bullet\vec{u} > \delta_{ij}$$

➢ The kinetic helicity of convective flows is important for dynamo activity.

Simulations

➢ We integrate the MHD equations numerically, using a spectral scheme in all three spatial directions (Gomez, Milano and Dmitruk 2000; also Dmitruk, Gomez & Matthaeus 2003)

➢ We show results from 256x256x256 runs performed in (CAPS), our linux cluster with 80 cores

➢ For the spatial derivatives, we use a pseudo-spectral scheme with 2/3-dealiasing. Spectral codes are well suited for turbulence studies, since they provide exponentially fast convergence.

➢ Time integration is performed with a second order Runge-Kutta scheme. The time step is chosen to satisfy the CFL condition.

caps

Cluster for Astrophysical Plasma Simulations

Simulations: spatial integration

➢ We focus on Fourier-Galerkin methods. Let us illustrate on Burgers equation

$$\partial_t u + u\partial_x u = \nu\, \partial_{xx} u$$

for u(x,t) on the interval $0 \le x < 2\pi$ assuming periodic boundary conditions and the initial condition $u(x,0) = u_0(x)$

➢ We expand in a truncated Fourier expansion ⟹ $u^N(x,t) = \sum_{k=-N/2}^{N/2} u_k(t)\, e^{ikx}$

➢ Demanding zero projection of the solution u(x,t) on the truncated Fourier space

$$\partial_t u_k = -(u\partial_x u)_k - \nu\, k^2\, u_k \quad , \quad (u\partial_x u)_k = \sum_{l+m=k} im\, u_l u_m$$

➢ This truncated expansion $u^N(x,t)$ converges exponentially fast to the exact solution as $N \to \infty$

However, it is computationally very demanding, it involves $O(N^2)$ operations.

Simulations: spatial integration

➢ The FFT algorithm yields the discrete set $\{\hat{u}_k\}$ from the set $\{u(x_j)\}$ after $O(N \log N)$ floating point operations.

$$\left\{u(x_j)\ ,\ x_j = \frac{2\pi}{N} j\ ,\ j = 0,\ldots,N-1\right\} \quad \overrightarrow{FFT} \quad \left\{\hat{u}_k\ ,\ k = -N/2+1,\ldots,N/2\right\}$$

➢ The strategy of computing spatial derivatives in Fourier space and nonlinear terms in physical space, is known as pseudo-spectral, i.e.

$$\partial_t u_k = -(u\partial_x u)_k - \nu k^2 u_k \quad , \quad (u\partial_x u)_k = FFT(FFT^{-1}(u_k)\ FFT^{-1}(iku_k))$$

➢ The relation between discrete Fourier coefficients $\{\hat{u}_k\}$ and the continuous ones is

$$\hat{u}_k = u_k + \sum_{m\neq 0} u_{k+Nm}$$

➢ This sum causes a spurious effect known as aliasing when computing nonlinear terms. Aliasing effects can be suppressed by applying the "two-thirds rule", i.e.

$$\hat{u}_k = 0 \quad , \quad \forall \quad |k| \geq N/3$$

Simulations: temporal integration

➢ We advance the solution through discrete time steps ⟹ $t_i = i\Delta t$

➢ In compact notation, if $\dfrac{dU}{dt} = F(U,t)$

where F is a nonlinear and spatial differential operator, we use a second order Runge-Kutta scheme.

➢ We first advance half a step ⟹ $U^{i+\frac{1}{2}} = U^i + \dfrac{\Delta t}{2} F(U^i, t_i)$

and use $U^{i+\frac{1}{2}}$ to jump the whole step ⟹ $U^{i+1} = U^i + \Delta t\, F(U^{i+\frac{1}{2}}, t_{i+\frac{1}{2}})$

➢ This is second order accurate (i.e. $O((\Delta t)^2)$). The size of the step is limited by

the CFL condition, i.e $\Delta t \leq \Delta x / u_0$ for $\partial_t u = u_0 \partial_x u$

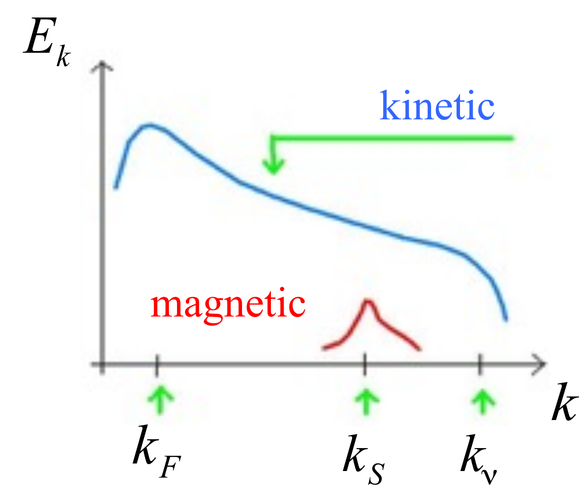
MHD 3D dynamos

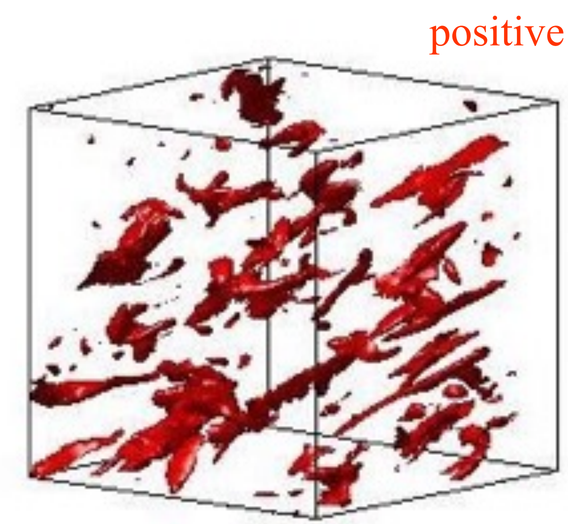
➢ From mean field theory (Krause & Radler 1980), we know that the turbulent generation of magnetic fields (the **alpha effect**) is proportional to the **kinetic helicity** of the flow.

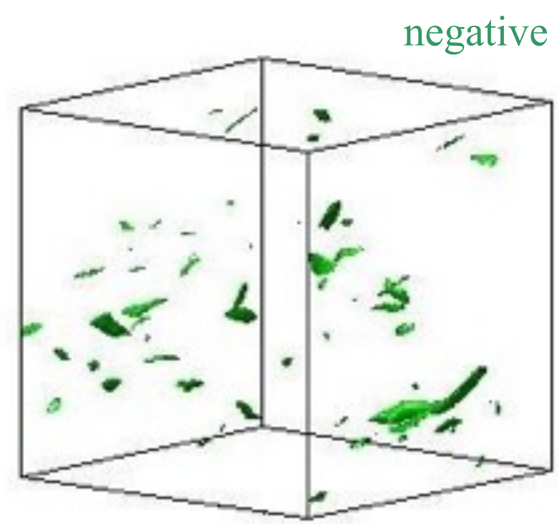
$$H = \frac{1}{2} \langle \vec{u} \bullet \vec{\nabla} \times \vec{u} \rangle$$

➢ To study this mechanism through direct simulations, we externally drive the flow with a helical force at large scales (an ABC pattern), until a stationary turbulent state is reached (Mininni, Gómez & Mahajan, 2003, ApJ, 587, 472; Mininni, Gómez & Mahajan, 2005, ApJ, 619, 1019)

➢ At that point, a magnetic seed is implanted at small scales and the 3D MHD equations are evolved (Meneguzzi, Frisch & Pouquet 1981).

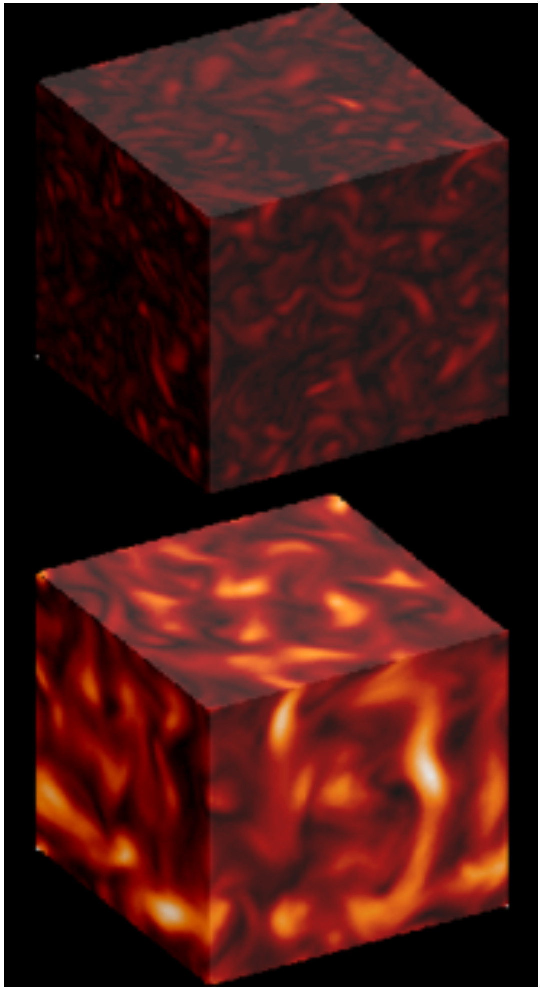


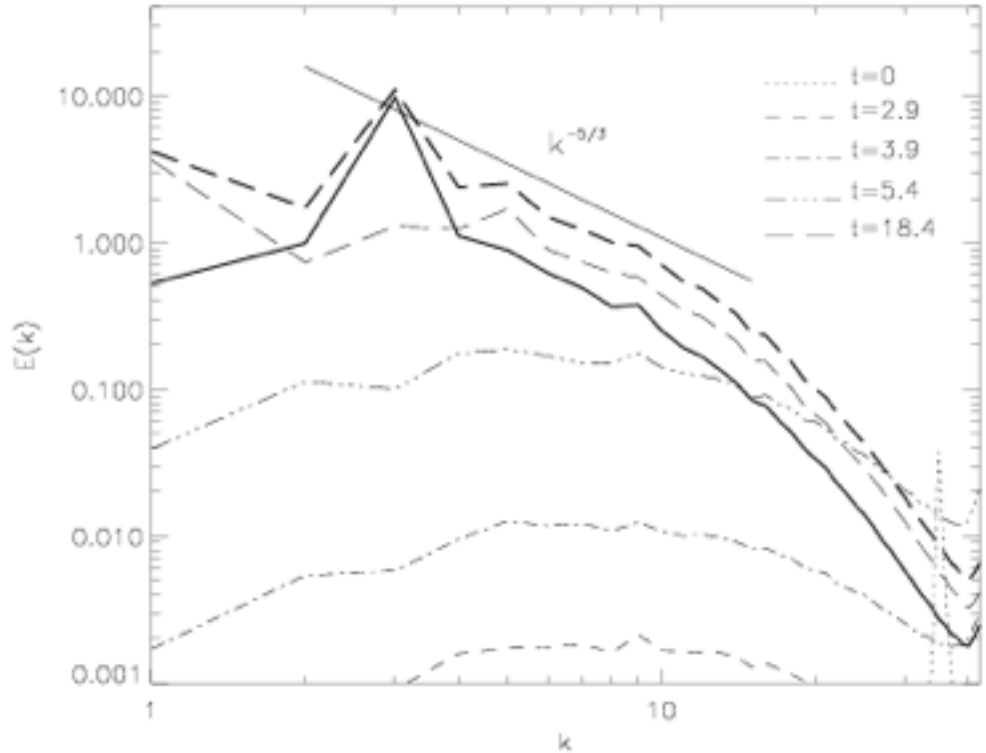




➢ The boxes show the intermittent spatial distribution of positive and negative kinetic helicity H, clearly displaying a net unbalance.

Energy power spectra



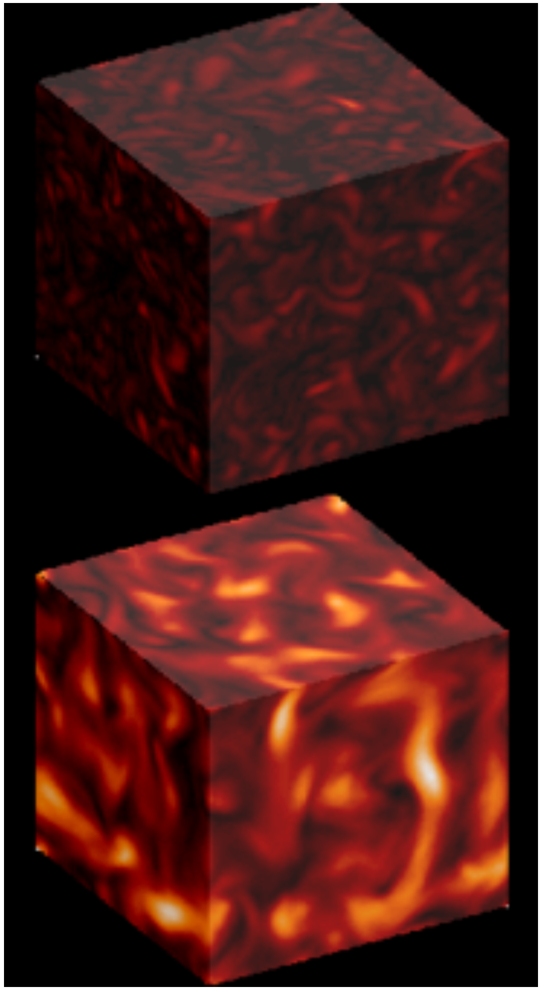


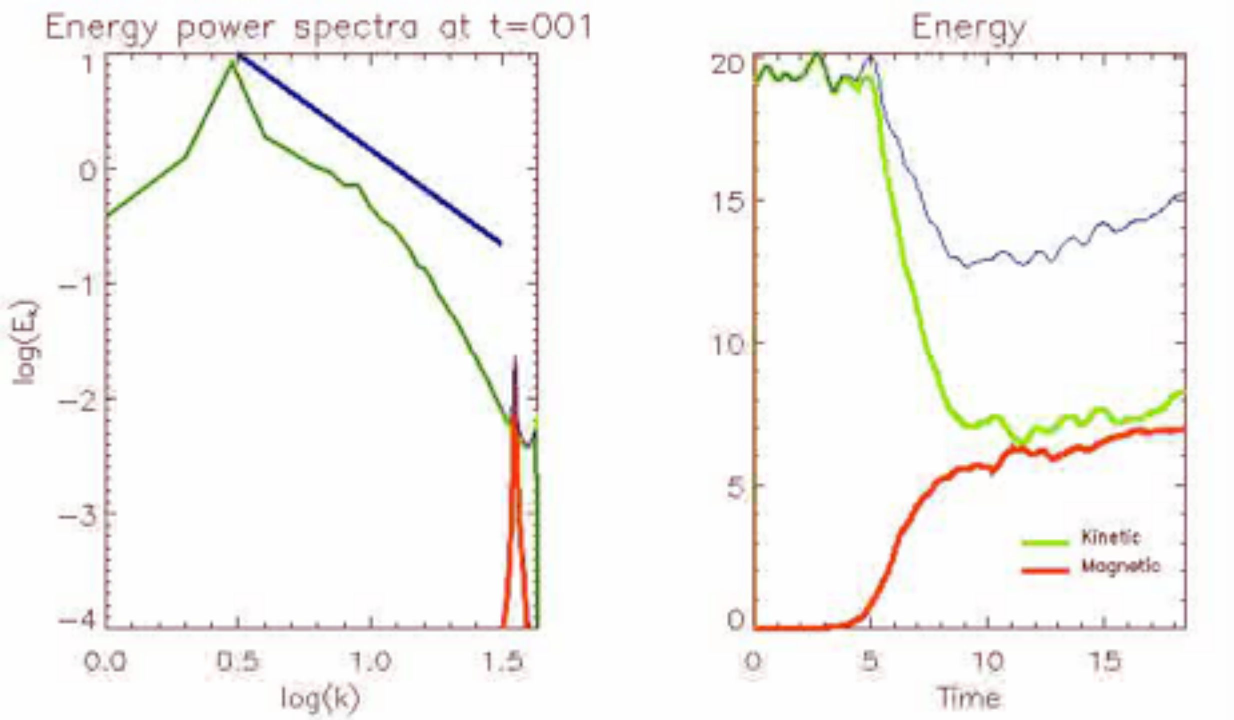
➢ The power spectrum of magnetic energy grows in time until it reaches equipartition at each scale (Brandenburg et al. 2003).

➢ The Kolmogorov slope is also displayed for reference.

➢ The full line is the kinetic energy power spectrum and the dotted line is the total energy.

Energy power spectra





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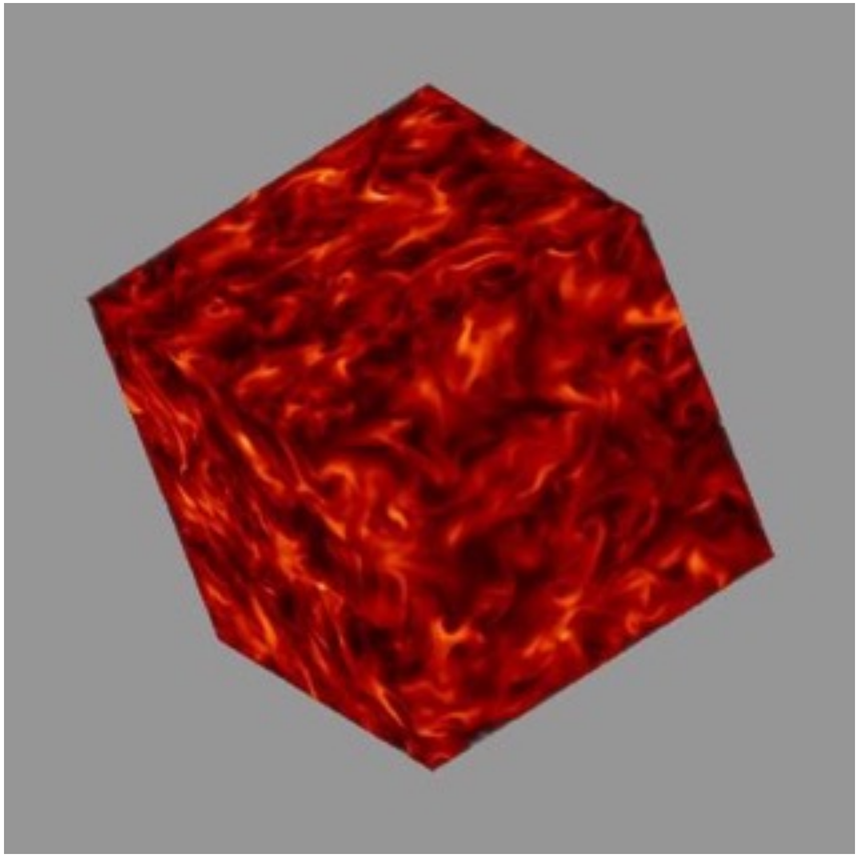
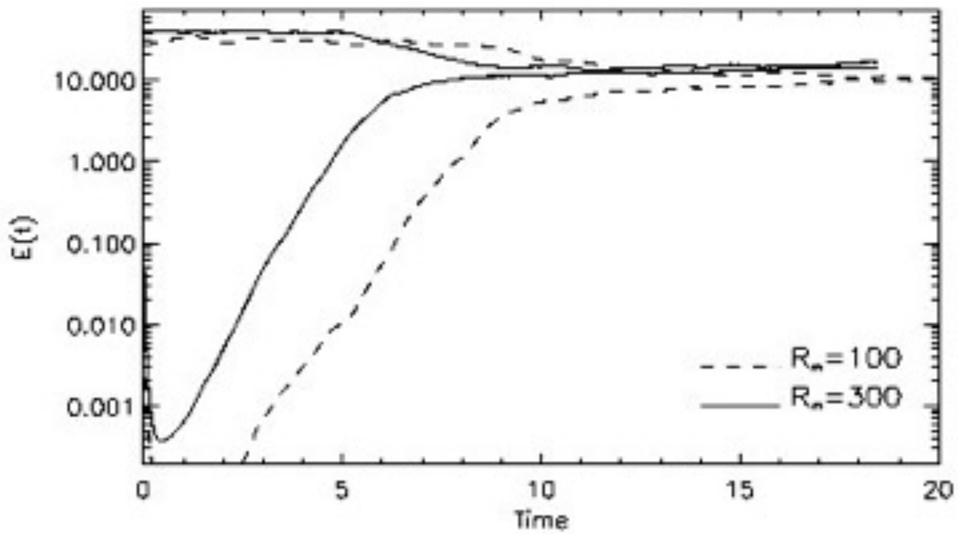
Turbulent dynamos

➢ The image on the right shows the spatial distribution of magnetic energy.

➢ The image below shows an initial exponential growth stage (kinematic dynamo) for the total magnetic energy. At later times it saturates when it reaches approximate equipartition with the total kinetic energy of the turbulent flow.





➢ As predicted by MFT (Steenbeck et al. 1966), kinematic helicity (H) at the microscale produces magnetic field at macroscopic scales (large-scale dynamos).

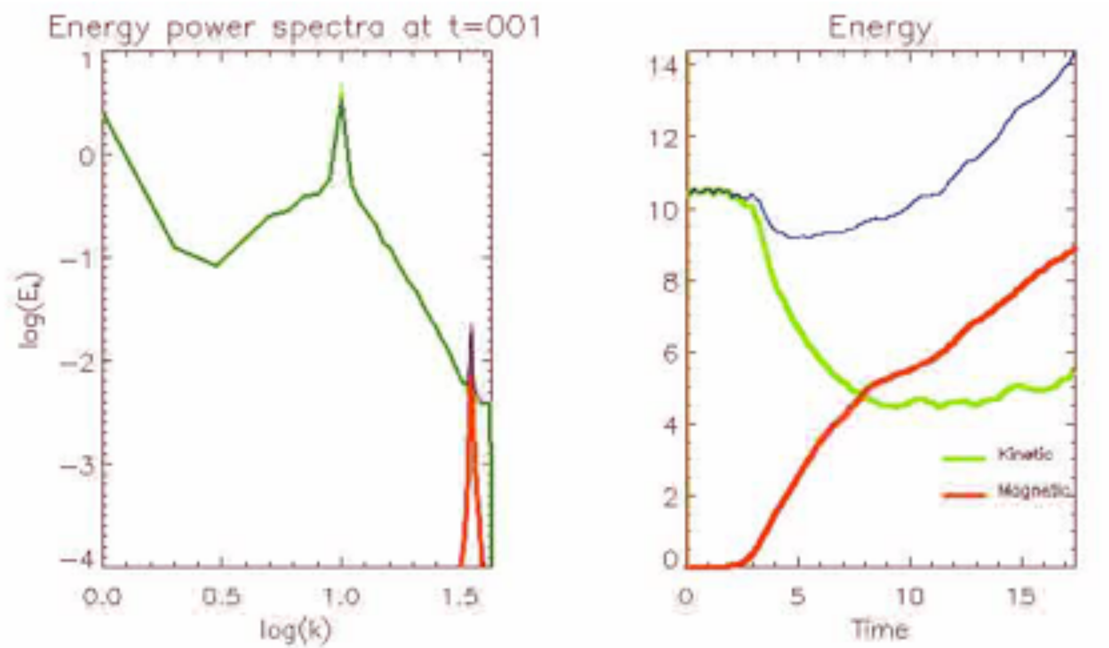
Force-free equilibria

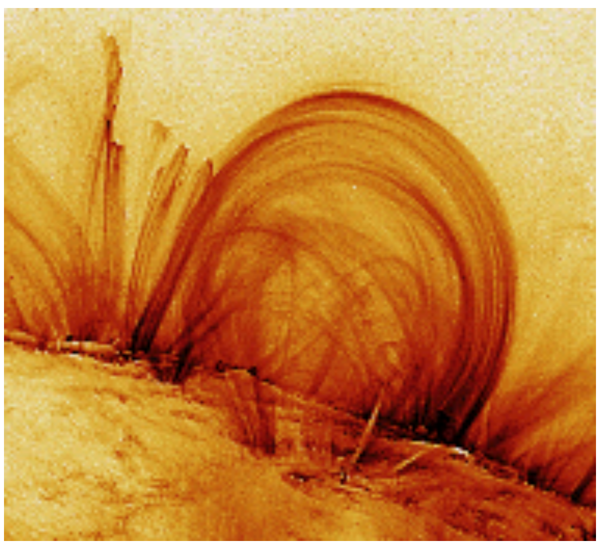
➢ When forcing is applied at intermediate scales, an accumulation of magnetic energy is observed at the largest scales.

➢ This behavior is caused by the inverse cascade of magnetic helicity.

➢ The magnetic field at large scales is approximately force-free, i.e.

$$\vec{\nabla} \times \vec{B} \ // \ \vec{B}$$





➢ Small scales, however, are consistent with a strongly turbulent MHD regime.

➢ This configuration can be representative of active regions of the solar corona, which are approximately force-free at large scales and at the same time are being heated by a strong MHD turbulence at smaller scales (Gómez & F.Fontán 1988)

Conclusions

- Today we presented the MHD equations as a valid description of the large-scale behavior of astrophysical plasmas.
- As a first application, we presented the Alpha-Omega dynamos to describe the basic features of the solar dynamo.
- Using empirical profiles of differential rotation and meridional flows, we manage to reproduce various observed aspects of the solar cycle, such as its period, rise-fall asymmetry and sunspot migration toward the Equator.
- Moreover, considering a stochastic part for the Alpha effect, we not only reproduce the irregularities observed in the cycle, but also the potential occurrence of Maunder-like events where magnetic activity on the Sun switches off for several decades.
- Finally, we numerically show a turbulent dynamo in action. An initial magnetic seed grows to equipartion with kinetic energy, provided that the flow is helical.