Some aspects of Vorticity fields in Relativistic and Quantum Plasmas

Felipe A. Asenjo[1]
Universidad Adolfo Ibáñez,
Chile

- ***Part I: Non-relativistic and Special relativistic Plasmas***
- ***Part II: General relativistic Plasmas***
- ***Part III: Quantum and Quantum Relativistic Plasmas***

[1]felipe.asenjo@uai.cl

Part I:
VORTICITY IN NON-RELATIVISTIC AND SPECIAL RELATIVISTIC PLASMAS

Felipe A. Asenjo

Today...

- We explore the concept of vorticity fields in electromagnetism
- We introduce the concept of vorticity fields in a plasmas
- We study the generation of vorticity
- We introduce the concept of helicity

Maxwell equations

Maxwell equations

Dynamics

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} \\ \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} &= \nabla \times \mathbf{B} \end{aligned}$$

Constraints

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{E} &= \rho \end{aligned}$$

$\mathbf{E}, \mathbf{B}$ electric and magnetic fields

$\rho, \mathbf{J}$ charge and current densities (sources)

Maxwell equations

The dynamics is consistent with the constraints

$$\nabla\cdot \rightarrow \frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \qquad \Longrightarrow \nabla\cdot\mathbf{B} = 0$$

$$\nabla\cdot\frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} = \nabla\times\mathbf{B} \qquad \Longrightarrow \frac{\partial}{\partial t}\nabla\cdot\mathbf{E} = \frac{\partial \rho}{\partial t} = -\nabla\cdot\mathbf{J}$$

via the continuity equation

And they produce the wave-like equations

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{B} = \nabla\times\mathbf{J}$$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{E} = -\frac{\partial \mathbf{J}}{\partial t} - \nabla\rho$$

Electromagnetic fields and potentials

$$\nabla \cdot \mathbf{B} = 0 \Longrightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \Longrightarrow \nabla \times \left(\frac{\partial \mathbf{A}}{\partial t} + \mathbf{E} \right) = 0 \Longrightarrow \frac{\partial \mathbf{A}}{\partial t} + \mathbf{E} = -\nabla \phi$$

Electromagnetic fields and potentials

$$\nabla \cdot \mathbf{B} = 0 \Longrightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

The magnetic field is the vorticity of the electromagnetic field

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \Longrightarrow \nabla \times \left(\frac{\partial \mathbf{A}}{\partial t} + \mathbf{E} \right) = 0 \Longrightarrow \frac{\partial \mathbf{A}}{\partial t} + \mathbf{E} = -\nabla \phi$$

The no-sources Maxwell equations become indetically satisfied

The sources Maxwell equations are written as

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) \mathbf{A} = \mathbf{J} - \nabla \left(\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} \right)$$

$$\nabla \cdot \left(\nabla \phi - \frac{\partial \mathbf{A}}{\partial t} \right) = \rho$$

If Lorentz gauge is used $\partial_t \phi + \nabla \cdot \mathbf{A} = 0$, then

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) \mathbf{A} = \mathbf{J} \, , \qquad \left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi = \rho$$

Vorticity

The vorticity field is any psedovector that is the rotational (curl) of a vector field (potential).

Magnetic helicity

The vorticity field has associated a quantity called helicity

$$h = \int \mathbf{A} \cdot \mathbf{B} \; d^3x$$

such that

$$\begin{aligned}
\frac{\partial h}{\partial t} &= \int \frac{\partial \mathbf{A}}{\partial t} \cdot \mathbf{B} \; d^3x + \int \mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial t} \; d^3x \\
&= \int (-\mathbf{E} - \nabla\phi) \cdot \mathbf{B} \; d^3x - \int \mathbf{A} \cdot \nabla \times \mathbf{E} \; d^3x \\
&\equiv -2 \int \mathbf{E} \cdot \mathbf{B} \; d^3x - \int (\phi\mathbf{B} + \mathbf{E} \times \mathbf{A}) \cdot \; d^2\mathbf{x} \\
&\equiv -2 \int \mathbf{E} \cdot \mathbf{B} \; d^3x
\end{aligned}$$

Non-relativistic plasma

Non-relativistic plasma fluid

Fluid equation

$$m\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\mathbf{v}=q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)-\frac{1}{n}\nabla p$$

Maxwell equations

$$\begin{aligned}\frac{\partial\mathbf{B}}{\partial t} &= -\nabla\times\mathbf{E}\\ \frac{\partial\mathbf{E}}{\partial t}+\mathbf{J} &= \nabla\times\mathbf{B}\end{aligned}$$

And an equation of state

Non-relativistic plasma fluid

We re-write the fluid equation as

$$m\frac{\partial \mathbf{v}}{\partial t} - m\mathbf{v} \times (\nabla \times \mathbf{v}) = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \frac{1}{2}\nabla v^2 - \frac{1}{n}\nabla p$$

where we have used $\mathbf{a} \times (\nabla \times \mathbf{b}) = (\nabla \mathbf{b}) \cdot \mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$

$$m\frac{\partial \mathbf{v}}{\partial t} = q\left[\mathbf{E} + \mathbf{v} \times \left(\mathbf{B} + \frac{m}{q}\nabla \times \mathbf{v}\right)\right] - \frac{1}{2}\nabla v^2 - \frac{1}{n}\nabla p$$

It appears the interesting field

$$\Omega = \mathbf{B} + \frac{m}{q}\nabla \times \mathbf{v} = \nabla \times \mathbf{P}$$

that will be a generalized vorticity with the potential [the canonical momentum]

$$\mathbf{P} = \mathbf{A} + \frac{m}{q}\mathbf{v}$$

Generalized vorticity

Taking the curl of the previous equation

$$\frac{m}{q}\frac{\partial \nabla \times \mathbf{v}}{\partial t} = \nabla \times \mathbf{E} + \nabla \times (\mathbf{v} \times \Omega) - \frac{1}{2q}\nabla \times \nabla v^2 - \nabla \times \left(\frac{1}{qn}\nabla p\right)$$

can be written as

$$\frac{\partial \Omega}{\partial t} - \nabla \times (\mathbf{v} \times \Omega) = \frac{1}{qn^2}\nabla n \times \nabla p$$

and

$$\frac{\partial \mathbf{P}}{\partial t} - \mathbf{v} \times \Omega = -\frac{1}{qn}\nabla p - \nabla \phi$$

Fluid helicity

The helicity associated to the fluid is

$$h = \int \mathbf{P} \cdot \Omega \; d^3x$$

which satisfies

$$\begin{aligned}
\frac{\partial h}{\partial t} &= \int \frac{\partial \mathbf{P}}{\partial t} \cdot \Omega \; d^3x + \int \mathbf{P} \cdot \frac{\partial \Omega}{\partial t} \; d^3x \\
&= \int \left(\mathbf{v} \times \Omega - \frac{1}{qn} \nabla p - \nabla \phi \right) \cdot \Omega \; d^3x \\
&\quad + \int \mathbf{P} \cdot \left[\nabla \times (\mathbf{v} \times \Omega) + \frac{1}{qn^2} \nabla n \times \nabla p \right] \; d^3x \\
&\equiv -\int \frac{1}{qn} \nabla p \cdot \Omega \; d^3x + \int \frac{1}{qn^2} \mathbf{P} \cdot \nabla n \times \nabla p \; d^3x \equiv -\int \frac{2}{qn} \nabla p \cdot \Omega
\end{aligned}$$

the helicity is conserved if $p = p(n)$. [2]

[2] Mahajan & Yoshida, Phys. Plasmas **18**, 055701 (2011).

Sources for Generalized vorticity $\partial_t \Omega - \nabla \times (\mathbf{v} \times \Omega) = \frac{1}{n^2} \nabla n \times \nabla p$

If $p = p(n)$, then $\nabla n \times \nabla p = 0$

$$\frac{\partial \Omega}{\partial t} - \nabla \times (\mathbf{v} \times \Omega) = 0$$

Therefore, if initiallhy the vorticity is null, it remains null for all times

If $\nabla n \times \nabla p \neq 0$, then the term

$$\frac{1}{n^2} \nabla n \times \nabla p$$

is so-called Biermann battery. It can generate vorticity from plasma thermodynamical inhomogenities.

- The conservation of helicity establishes topological constraints. It can forbid the creation (destruction) of vorticity in plasmas.
- We can see that the generalized helicity remains unchanged in ideal dynamics. This conservation implies serious contraints on the origin and dynamics of magnetic fields.
- Otherwise, the nonideal effects can change the helicity. For example, if gradients of pressure and temperature have different directions [Biermann battery].
- An anisotropic pressure tensor may also generate vorticity.

Special relativistic plasma

Special Relativistic plasma fluids

For relativistic plasmas there exist also a generalized voticity and a fluid helicity. Now the relativistic plasma fluid is a little more complicated. We have to consider:

- the rest-frame density of the lfuid n.
- the energy density of the fluid ϵ.
- the pressure of the fluid p.
- the enthalpy density of the fluid $h = \epsilon + p$.
- the relativistic velocity, through the Lorentz factor $\gamma = (1 - \mathbf{v}^2)^{-1/2}$.
- coupled to Maxwell equations via the current density $n\gamma\mathbf{v}$.

Special Relativistic plasma fluids - covariant form

The relativistic ideal plasma description can be obtained from the conservation of the ideal fluid energy-momentum tensor $\partial_\nu T^{\mu\nu} = 0$, with

$$T^{\mu\nu} = (\epsilon + p) U^\mu U^\nu + p\,\eta^{\mu\nu}$$

with

$$\eta_{\mu\nu} = (-1, 1, 1, 1) \qquad U_\mu U^\mu = -1$$

such that in the rest-frame, where $U^\mu = (1, 0, 0, 0)$, we find

$$T^{00} = \epsilon$$

$$T^{0i} = 0$$

$$T^{ij} = p\delta^{ij}$$

The equation for the plasma fluid is

$$U^\nu \partial_\nu \left(mfU^\mu \right) = qF^{\mu\nu} U_\nu - \frac{1}{n} \partial^\mu p$$

with

$$f = \frac{\epsilon + p}{mn}$$

Special Relativistic plasma fluids - covariant form

Also we have the continuity equation

$$\partial_\mu (nU^\mu) = 0$$

and Maxwell equations

$$\partial_\nu F^{\mu\nu} = qnU^\mu$$

Magnetofluid Unification[3]

Instead of solving the previous equations, let us look the big picture. The covariant fluid equation can be cast in the form

$$qU_\nu M^{\mu\nu} = T\partial^\mu \sigma$$

where the magnetofluid tensor is

$$M^{\mu\nu} = F^{\mu\nu} + \frac{m}{q} S^{\mu\nu}$$

with

$$S^{\mu\nu} = \partial^\mu (fU^\nu) - \partial^\nu (fU^\mu)$$

and the entropy density follows

$$\partial^\mu \sigma = \frac{1}{nT} \left(\partial^\mu p - mn\partial^\mu f \right)$$

For an ideal relativistic gas

$$f = K_3(m/T)/K_2(m/T)$$

[3] Mahajan PRL **90**, 035001 (2003); Mahajan & Yoshida, PoP **18**, 055701 (2011).

Magnetofluid tensor (why is important)

$$M^{\mu\mu} \equiv 0$$

$$M^{0i} \to \xi = \mathbf{E} - \frac{m}{q}\partial_t(f\gamma\mathbf{v}) - \frac{m}{q}\nabla(f\gamma)$$

$$M^{ij} \to \Omega = \mathbf{B} + \frac{m}{q}\nabla \times (f\gamma\mathbf{v})$$

The magnetofluid tensor is the natural extension to the covariant form of the plasma vorticity.

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The magnetofluid tensor is the natural extension to the covariant form of the plasma vorticity.

Equation $qU_\nu M^{\mu\nu} = T\partial^\mu\sigma$ is the covariant vorticity equation for the plasma.

$$(\text{For } \mu = 0) \Longrightarrow \mathbf{v}\cdot\xi = -\frac{T}{q\gamma}\frac{\partial\sigma}{\partial t}$$

$$(\text{For } \mu = i) \Longrightarrow \xi + \mathbf{v}\times\Omega = \frac{T}{q\gamma}\nabla\sigma$$

Defining the potential (generalized canonical momentum)

$$\mathcal{P}^\mu = A^\mu + \frac{m}{q} f U^\mu = (\mathcal{P}^0, \mathcal{P})$$

then

$$M^{\mu\nu} = \partial^\mu \mathcal{P}^\nu - \partial^\nu \mathcal{P}^\mu$$

In this way

$$\xi = -\frac{\partial \mathcal{P}}{\partial t} - \nabla \mathcal{P}^0 \,, \qquad \Omega = \nabla \times \mathcal{P}$$

$$\Longrightarrow \nabla \times \xi = -\frac{\partial \Omega}{\partial t} \Longleftrightarrow \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}\partial^\beta M^{\mu\nu} = 0$$

Defining the potential (generalized canonical momentum)

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then

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In this way

$$\xi = -\frac{\partial \boldsymbol{\mathcal{P}}}{\partial t} - \nabla \mathcal{P}^0 \,, \qquad \Omega = \nabla \times \boldsymbol{\mathcal{P}}$$

$$\Longrightarrow \nabla \times \xi = -\frac{\partial \Omega}{\partial t} \Longleftrightarrow \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu} \partial^\beta M^{\mu\nu} = 0$$

$$(\text{For } \mu = 0) \Longrightarrow \mathbf{v} \cdot \xi = -\frac{T}{q\gamma} \frac{\partial \sigma}{\partial t}$$

$$(\text{For } \mu = i) \Longrightarrow \frac{\partial \boldsymbol{\mathcal{P}}}{\partial t} - \mathbf{v} \times \Omega = -\frac{T}{q\gamma} \nabla \sigma - \nabla \mathcal{P}^0$$

This last equation is the potential equation for the vortical dynamics!

Generalized relativistic vorticity and its dynamics

$$\Omega = \nabla \times \mathcal{P} = \mathbf{B} + \frac{m}{q}\nabla \times (f\gamma\mathbf{v})$$

$$\frac{\partial \Omega}{\partial t} - \nabla \times (\mathbf{v} \times \Omega) = -\nabla \left(\frac{T}{q\gamma}\right) \times \nabla\sigma$$

- The Generalized voticity has both kinematical and thermal relativistic corrections [NR limit $\gamma \to 1, f \to 1$].
- The vortical dynamics contains those corrections. It appears a more general battery.

Generalized relativistic helicity [4]

$$K^{\mu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}P_{\nu}M_{\alpha\beta}$$

[4] Mahajan PRL **90**, 035001 (2003)

Generalized relativistic helicity [4]

$$K^\mu = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}P_\nu M_{\alpha\beta}$$

$$\partial_\mu K^\mu = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}\partial_\mu P_\nu M_{\alpha\beta} + \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}P_\nu \partial_\mu M_{\alpha\beta} = \varepsilon^{\mu\nu\alpha\beta}\partial_\mu P_\nu M_{\alpha\beta}$$

the Generalized helicity

$$h \equiv \int K^0 d^3x = \int \varepsilon^{0ijk} P_i M_{jk} d^3x = \int \mathcal{P}\cdot\Omega d^3x$$

[4] Mahajan PRL **90**, 035001 (2003)

Generalized relativistic helicity [4]

$$K^\mu = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}P_\nu M_{\alpha\beta}$$

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the Generalized helicity

$$h \equiv \int K^0 d^3x = \int \varepsilon^{0ijk}P_i M_{jk} d^3x = \int \mathcal{P}\cdot\Omega d^3x$$

Then

$$\begin{aligned}\int \partial_\mu K^\mu d^3x &= \int \partial_t K^0 d^3x = \partial_t h \\ &= \int \partial_t \mathcal{P}\cdot\Omega d^3x + \int \mathcal{P}\cdot\partial_t\Omega d^3x = \int \frac{2}{q\gamma}\nabla\sigma\cdot\Omega d^3x\end{aligned}$$

There is room for generation!

[4] Mahajan PRL **90**, 035001 (2003)

Spacetime dynamics $\Longleftrightarrow$ Generation of magnetic fields! [5]

$$\Omega = \nabla \times \mathcal{P} = \mathbf{B} + \frac{m}{q}\nabla \times (f\gamma\mathbf{v})$$

$$\begin{aligned}
\frac{\partial\Omega}{\partial t} - \nabla \times (\mathbf{v} \times \Omega) &= -\nabla\left(\frac{T}{q\gamma}\right) \times \nabla\sigma \\
&= -\nabla\left(\frac{1}{q\gamma n}\right) \times \nabla p + \frac{m}{q}\nabla\left(\frac{1}{\gamma}\right) \times \nabla f \\
&= \frac{\nabla n}{q\gamma n^2} \times \nabla p + \frac{\nabla\gamma}{q\gamma^2 n} \times \nabla p - \frac{m}{q\gamma^2}\nabla\gamma \times \nabla f
\end{aligned}$$

[5] Mahajan & Yoshida, PRL **105**, 095005 (2010)

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\frac{\partial \Omega}{\partial t} - \nabla \times (\mathbf{v} \times \Omega) &= -\nabla\left(\frac{T}{q\gamma}\right) \times \nabla\sigma \\
&= -\nabla\left(\frac{1}{q\gamma n}\right) \times \nabla p + \frac{m}{q}\nabla\left(\frac{1}{\gamma}\right) \times \nabla f \\
&= \frac{\nabla n}{q\gamma n^2} \times \nabla p + \frac{\nabla\gamma}{q\gamma^2 n} \times \nabla p - \frac{m}{q\gamma^2}\nabla\gamma \times \nabla f
\end{aligned}$$

- The first one is the “relativistic-corrected” Biermann battery
- The second and third one are the RELATIVISTIC DRIVES. The third one is a kinematically and thermally relativistic correction.

[5] Mahajan & Yoshida, PRL **105**, 095005 (2010)

Special relativistic drives (pure relativistic batteries)

$$\frac{\partial \Omega}{\partial t} - \nabla \times (\mathbf{v} \times \Omega) \quad = \quad \frac{\nabla T}{q\gamma} \times \nabla \sigma + \frac{T \nabla \gamma}{q\gamma^2} \times \nabla \sigma$$

They can generate a generalized vorticity (a magnetic field) from the relativistic plasma interaction between its kinematics and its thermodynamics.

Special relativistic drives (pure relativistic batteries)

$$\frac{\partial \Omega}{\partial t} - \nabla \times (\mathbf{v} \times \Omega) = \frac{\nabla T}{q\gamma} \times \nabla \sigma + \frac{T \nabla \gamma}{q\gamma^2} \times \nabla \sigma$$

They can generate a generalized vorticity (a magnetic field) from the relativistic plasma interaction between its kinematics and its thermodynamics.

In most astrophysical settings $p = p(n)$ and $\nabla n \times \nabla p = 0$.
in this situations the only possible source for a vorticity is the relativistic drive.
Even so, if

$$\frac{T|\nabla \gamma|}{\gamma |\nabla T|} \sim \frac{T|\nabla (v^2/c^2)|}{|\nabla T|(1 - v^2/c^2)} \gg 1$$

then the relativistic drive is more relevant than Biermann battery (for $v \to c$ or very inhomogenous hot plasmas).

That’s all (for now).
Thanks!