Magnetohydrodynamic description of plasmas

Part II

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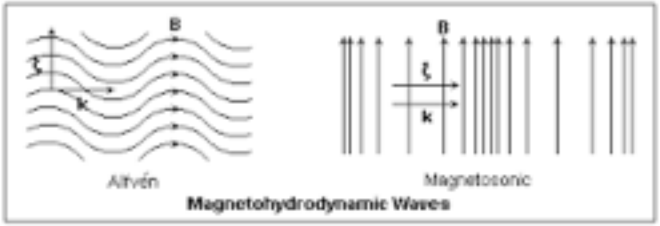
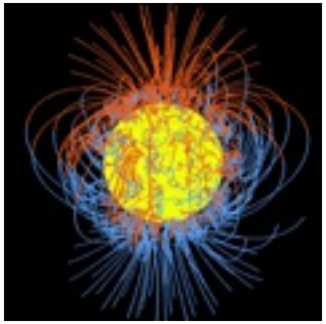
Applications of MHD

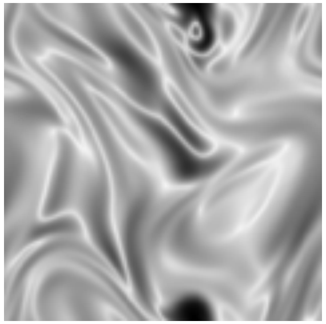
➢Within this level of description (which is adequate at large spatial scales) there is a variety of importante plasma processes that have traditionally been addressed:



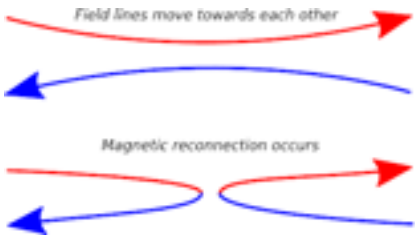
- Instabilities and wave propagation (Alfven and magnetosonic)



- Dynamo mechanisms to generate magnetic fields



- MHD turbulence



- Magnetic reconnection

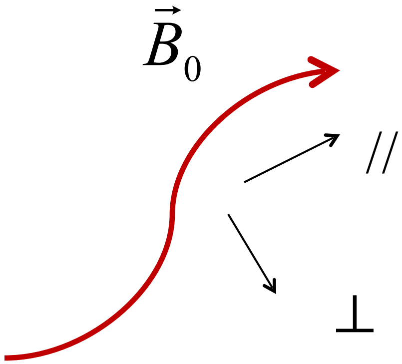
Two-fluid MHD

- MHD is a fluidistic approach to describe the large scale dynamics of plasmas.

- The standard approach is also known as one-fluid MHD.

- We are going to start from a somewhat more general approach known as two-fluid MHD, which acknowledges the presence of ions and electrons and considers kinetic effects such as Hall, electron pressure and electron inertia.

- Physical processes that can be addressed with MHD include:
  - Magnetic reconnection
  - Magnetic confinement
  - Magnetic dynamo
  - MHD turbulence



- We will also address the case of plasmas embedded in strong external magnetic fields, which allow for an approximation known as reduced MHD, both for one-fluid MHD (RMHD) and two-fluid MHD (RHMHD).

Fluid equations for multi-species plasmas

- For each species s we have (Goldston & Rutherford 1995):
  - Mass conservation $$\frac{\partial n_s}{\partial t} + \vec{\nabla} \cdot (n_s \vec{U}_s) = 0$$
  - Equation of motion $$m_s n_s \frac{d\vec{U}_s}{dt} = q_s n_s (\vec{E} + \frac{1}{c}\vec{U}_s \times \vec{B}) - \vec{\nabla} p_s + \vec{\nabla} \bullet \vec{\sigma}_s + \sum_{s'} \vec{R}_{ss'}$$
  - Momentum exchange rate $$\vec{R}_{ss'} = -m_s n_s \upsilon_{ss'} (\vec{U}_s - \vec{U}_{s'})$$
- These moving charges act as sources for electric and magnetic fields:
  - Charge density $$\rho_c = \sum_s q_s n_s \approx 0$$
  - Electric current density $$\vec{J} = \frac{c}{4\pi} \vec{\nabla} \times \vec{B} = \sum_s q_s n_s \vec{U}_s$$

Two-fluid MHD equations

- For a fully ionized plasma with ions of mass m_i and massless electrons (since $m_e << m_i$):
  - Mass conservation: $$0 = \frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\vec{U}) \quad , \qquad n_e \cong n_i \cong n$$
  - Ions: $$m_i n \frac{d\vec{U}}{dt} = en(\vec{E} + \frac{1}{c}\vec{U} \times \vec{B}) - \vec{\nabla} p_i + \vec{\nabla} \bullet \vec{\sigma} + \vec{R}$$
  - Electrons: $$0 = -en(\vec{E} + \frac{1}{c}\vec{U}_e \times \vec{B}) - \vec{\nabla} p_e - \vec{R}$$
  - Friction force: $$\vec{R} = -m_i n \nu_{ie} (\vec{U} - \vec{U}_e)$$
  - Ampere's law: $$\vec{J} = \frac{c}{4\pi} \vec{\nabla} \times \vec{B} = en(\vec{U} - \vec{U}_e) \qquad \Rightarrow \qquad \vec{R} = -\frac{m\upsilon_{ie}}{e} \vec{J}$$
  - Polytropic laws: $$p_i \propto n^{\gamma} \quad , \qquad p_e \propto n^{\gamma}$$
  - Newtonian viscosity: $$\sigma_{ij} = \mu \, (\partial_i U_j + \partial_j U_i)$$

Hall-MHD equations

- The dimensionless version, for a length scale L_0, density n_0 and Alfven speed $v_A = B_0/\sqrt{4\pi m_i n_0}$

$$\frac{d\vec{U}}{dt} = \frac{1}{\varepsilon}(\vec{E} + \vec{U}\times\vec{B}) - \frac{\beta}{n}\vec{\nabla}p_i - \frac{\eta}{\varepsilon n}\vec{J} + \nu\nabla^2\vec{U} \qquad \nu = \frac{\mu}{m_i n v_A L_0}$$

$$0 = -\frac{1}{\varepsilon}(\vec{E} + \vec{U}_e\times\vec{B}) - \frac{\beta}{n}\vec{\nabla}p_e + \frac{\eta}{\varepsilon n}\vec{J} \qquad \text{where} \qquad \vec{J} = \vec{\nabla}\times\vec{B} = \frac{n}{\varepsilon}(\vec{U} - \vec{U}_e)$$

- We define the Hall parameter $\varepsilon = \dfrac{c}{\omega_{pi}L_0}$

  as well as the plasma *beta* $\beta = \dfrac{p_0}{m_i n_0 v_A^2}$ and the electric resistivity $\eta = \dfrac{c^2\nu_{ie}}{\omega_{pi}^2 L_0 v_A}$

- Adding these two equations yields:

$$\boxed{n\frac{d\vec{U}}{dt} = (\vec{\nabla}\times\vec{B})\times\vec{B} - \beta\vec{\nabla}(p_i + p_e) + \nu\nabla^2\vec{U}}$$

Hall-MHD equations

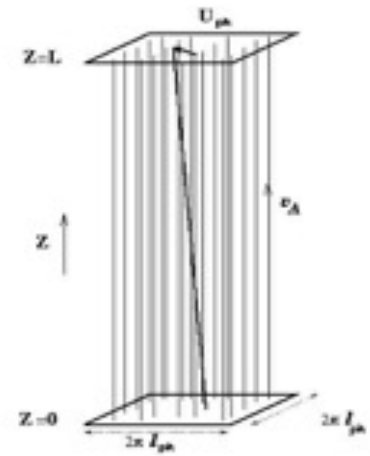
- On the other hand, using

$$\left.\begin{aligned}\vec{E} &= -\frac{1}{c}\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi \\ \vec{B} &= \vec{\nabla}\times\vec{A}\end{aligned}\right\} \rightarrow \boxed{\frac{\partial\vec{A}}{\partial t} = (\vec{U} - \frac{\varepsilon}{n}\vec{\nabla}\times\vec{B})\times\vec{B} - \vec{\nabla}\phi + \frac{\varepsilon\beta}{n}\vec{\nabla}p_e - \frac{\eta}{n}\vec{\nabla}\times\vec{B}}$$

Hall MHD in a strong field

- Let us assume a strong magnetic field along $\hat{z}$ so that

$$\vec{B} = \hat{z} + \delta\vec{B} \qquad , \qquad |\delta\vec{B}| \approx \alpha << 1$$



where α represents the typical tilt of field lines with respect to $\hat{z}$. We assume

$$\nabla_{\perp} \approx 1 \qquad , \qquad \partial_z \approx \alpha << 1$$

- The magnetic and velocity fields can be expanded in terms of potentials of order : α

$$\vec{B} = \hat{z} + \vec{\nabla} \times (a\hat{z} + g\hat{x}) = [a_y, -a_x, 1+b] \qquad , \qquad b = -g_y$$

$$\vec{U} = \vec{\nabla}\psi + \vec{\nabla} \times (\varphi\,\hat{z} + f\,\hat{x}) = [\varphi_y + \psi_x, -\varphi_x + \psi_y, u + \psi_z] \qquad , \qquad u = -f_y$$

- We want to eliminate the fast scale dynamics, characterized by $\tau_{A\perp} \approx L_{\perp} / v_A$ i.e. $\partial_t \approx 1$

- We obtain the following conditions

$$\boxed{\begin{aligned} \nabla_{\perp}^2\psi &= 0 \\ \vec{\nabla}_{\perp}[b + \beta(p_i + p_e)] &= 0 \\ \vec{\nabla}_{\perp}[\phi + \varphi - \varepsilon(b + \beta p_e)] &= 0 \end{aligned}}$$

Hall MHD in a strong field

- The relatively slower dynamics, characterized by $\tau_{A//} \approx L_{//} / v_A$ i.e. $\partial_t \approx \alpha$

  is given by the following equations (Gomez, Dmitruk & Mahajan 2008):

$$\boxed{\begin{aligned}
\partial_t a &= \partial_z(\varphi - \varepsilon b) + [\varphi - \varepsilon b, a] && + \eta \nabla_\perp^2 a \\
\partial_t \omega &= \partial_z j + [\varphi, \omega] - [a, j] && + \nu \nabla_\perp^2 \omega \\
\partial_t b &= \partial_z(u - \varepsilon j) + [\varphi, b] + [u - \varepsilon j, a] && + \eta \nabla_\perp^2 b \\
\partial_t u &= \partial_z b + [\varphi, u] - [a, b] && + \nu \nabla_\perp^2 u
\end{aligned}}$$

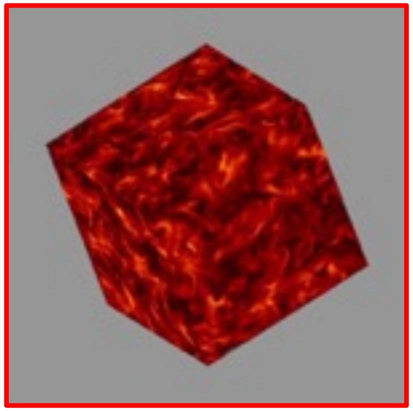
where $j = -\nabla_\perp^2 a$ and $\omega = -\nabla_\perp^2 \varphi$

- These are the RHMHD equations. Their ideal invariants (just as for 3D HMHD) are:

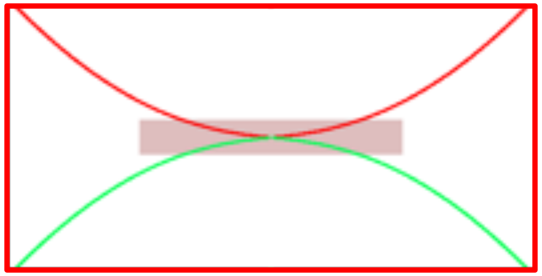
$$\boxed{\begin{aligned}
E &= \frac{1}{2}\int d^3r\,(|\vec{U}|^2 + |\vec{B}|^2) = \frac{1}{2}\int d^3r\,(|\vec{\nabla}_\perp \varphi|^2 + |\vec{\nabla}_\perp a|^2 + u^2 + b^2) && \text{energy} \\
H_m &= \frac{1}{2}\int d^3r\,(\vec{A}\cdot\vec{B}) = \int d^3r\, ab && \text{magnetic helicity} \\
H_h &= \frac{1}{2}\int d^3r\,(\vec{A} + \varepsilon \vec{U})\cdot(\vec{B} + \varepsilon\vec{\Omega}) = \int d^3r\,(ab + \varepsilon(a\omega + ub) + \varepsilon^2 u\omega) && \text{hybrid helicity}
\end{aligned}}$$

Some applications

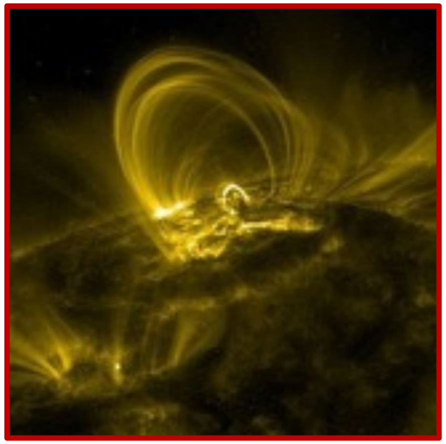
- We studied a number of astrophysical problems, within the general framework of MHD:
- 3D Hall-MHD turbulent dynamos.
(Mininni, Gomez & Mahajan 2003, 2005;
Gomez, Dmitruk & Mininni 2010)



- 2.5 D Hall-MHD magnetic reconnection
in the Earth magnetosphere
(Morales, Dasso & Gomez 2005, 2006)



- 3D HD helical fluid turbulence
(Gomez & Mininni 2004)
- RMHD heating of solar coronal loops
(Dmitruk & Gomez 1997, 1999)



- RHMHD turbulence in the solar wind
(Martin, Dmitruk & Gomez 2010, 2012)
- Hall magneto-rotational instability in accretion disks
(Bejarano, Gomez & Brandenburg 2011)

RMHD applied to coronal loop heating

- The solar corona is a topologically complex array of loops (TRACE movie 171 A)
- Coronal loops are magnetic flux tubes with their footpoints anchored deep in the convective region.
- They confine a tenuous and hot plasma. Typical densities are n = 10^9 cm^{-3} and temperatures are T = 2-3.10^6 K.

- The magnetic field provides not just the confinement of the plasma, but also the energy to heat it up to coronal temperatures (Parker 1972, 1988; van Ballegooijen 1986; Einaudi et al. 1996).
- One of the key ingredients is the free energy available in the sub-photospheric convective region. Convective motions move the footpoints of fieldlines, thus building up magnetic stresses. See **Mandrini, Demoulin & Klimchuk 2000** for a comprehensive comparison between theoretical models and observations for a large number of active regions.
- However, the typical length scale of these magnetic stresses is way too large for the Ohmic dissipation to do the job, since

$$\boxed{\tau_{diss} \approx \ell^2 / \eta}$$

RMHD Equations

- Reduced MHD is a self-consistent approximation of the full MHD equations whenever:
  (a) one component of the magnetic field is much stronger than the others and,
  (b) spatial variations are smoother along than across (Strauss 1976).

$$\partial_t a = v_A \partial_z \varphi + [\varphi, a] + \eta \nabla_\perp^2 a$$

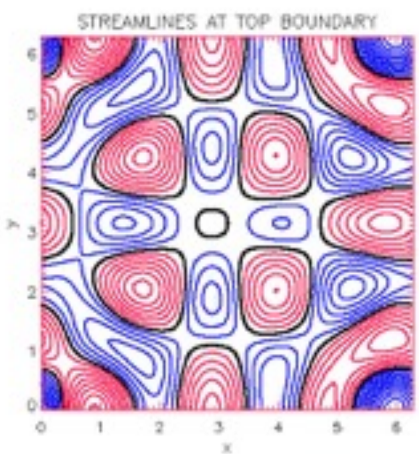
$$\partial_t \omega = v_A \partial_z j + [\varphi, \omega] - [a, j] + \eta \nabla_\perp^2 \omega$$

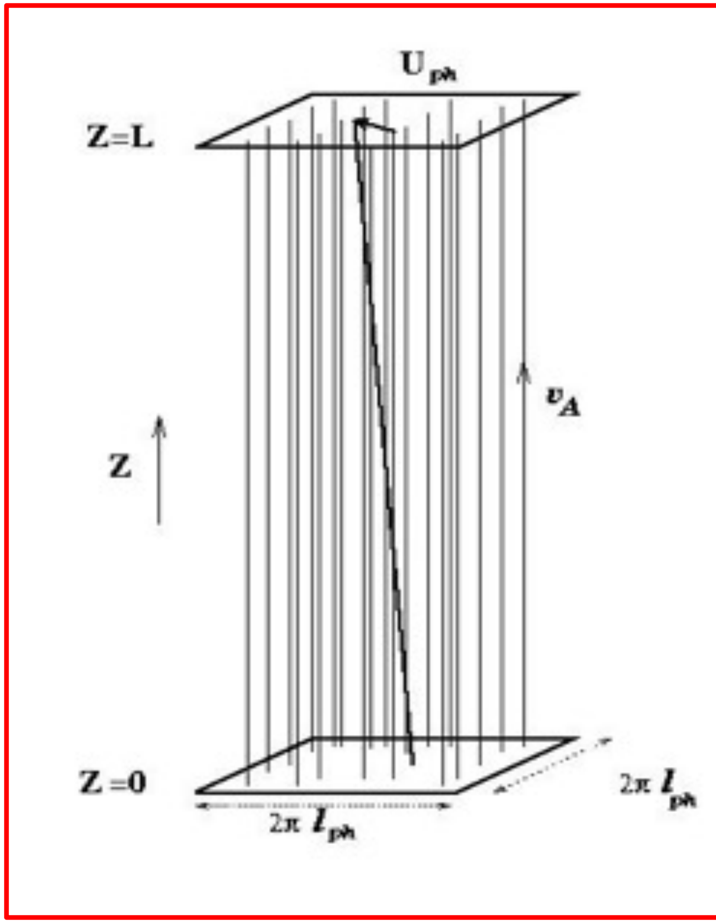
$$\vec{b} = v_A \hat{z} + \vec{\nabla}_\perp \times (a\,\hat{z}) \quad , \quad \vec{u} = \vec{\nabla}_\perp \times (\varphi\,\hat{z})$$

$$\omega = -\nabla_\perp^2 \varphi \quad , \quad j = -\nabla_\perp^2 a$$

- These equations describe the evolution of the velocity (u) and magnetic field (b) inside the box, assuming periodic boundary conditions at the sides.

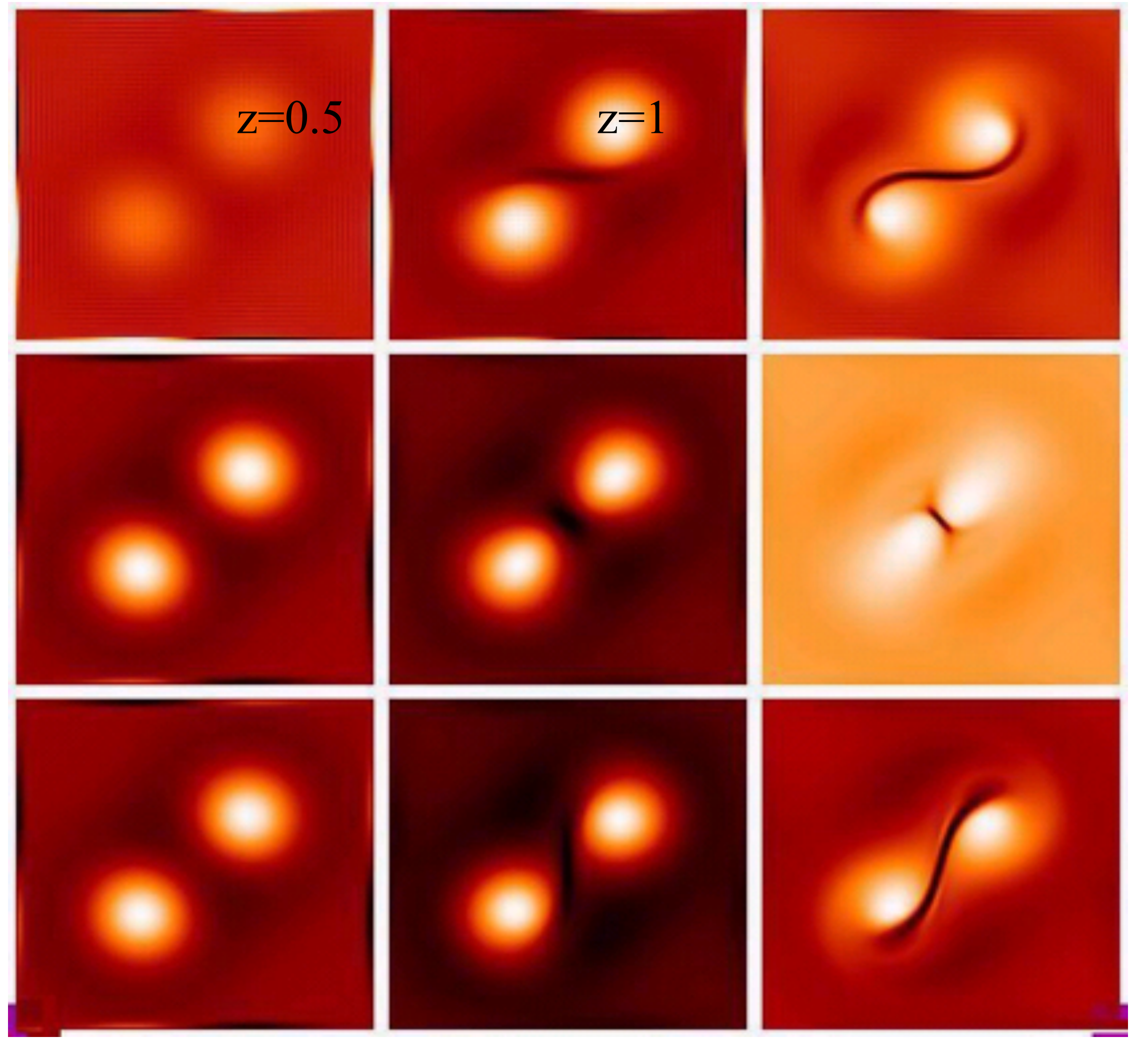
- We enforce stationary velocity field (U_{ph}) at the top plate.





Current density distribution

Current density



time →

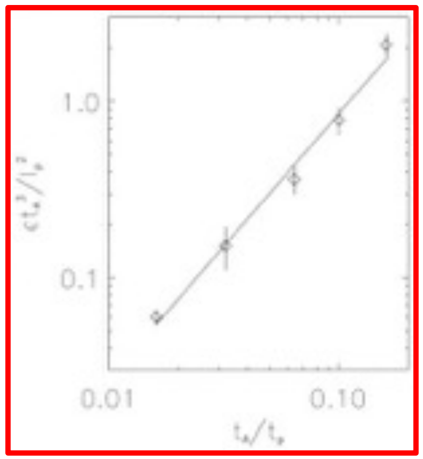
RMHD simulations

- We perform long time integrations of the RMHD equations. Lengths are in units of the photospheric motions (ℓ_{ph}) and times are in units of the Alfven time (t_A) along the loop.

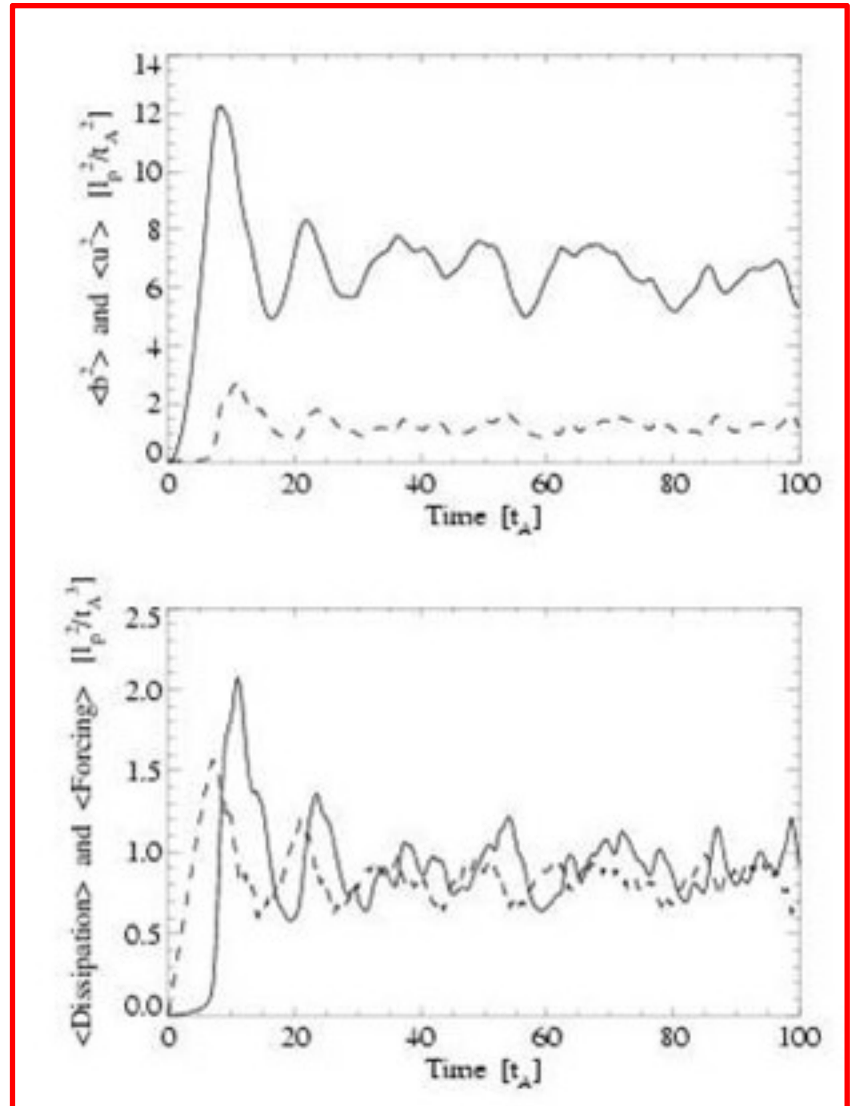
- Spatial resolution is 256x256x48 and the integration time is 4000 t_A. We use a spectral scheme in the xy-plane and finite differences along z.

- The time averaged dissipation rate is found to scale like (Dmitruk & Gómez 1999)

$$\varepsilon \approx \frac{\rho\, \ell_{ph}^2}{t_A^3} \left(\frac{t_A}{t_{ph}}\right)^{3/2}$$

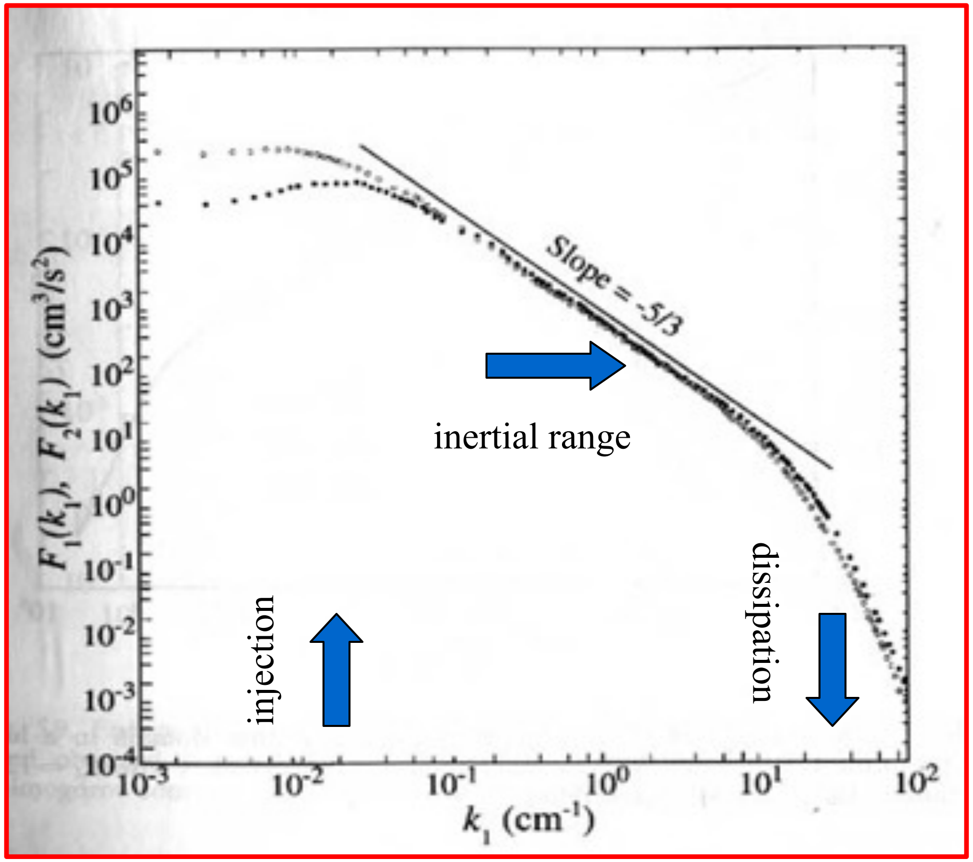


- It is essentially independent of the Reynolds number, as expected for stationary turbulence.



Stationary turbulence

- Energy cascade
  - energy flux toward high k
  - vortex breakdown



- Scale invariance
  - energy flux in k: $\longrightarrow$ $\varepsilon_k \approx \frac{u_k^2}{\tau_k}$
  - energy power spectrum: $\longrightarrow$ $E_k \approx \frac{u_k^2}{k}$

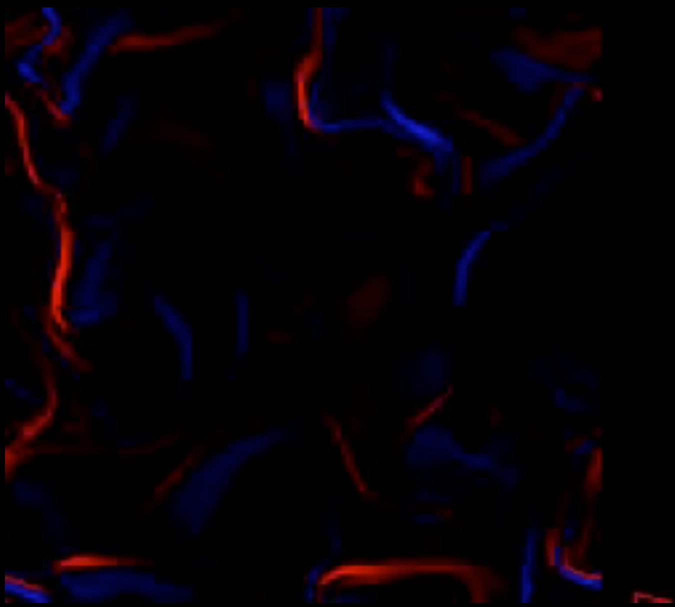
$$\tau_k \approx \frac{1}{ku_k}\ ,\ \varepsilon_k \approx \frac{u_k^2}{\tau_k} = const.$$

- Therefore $\longrightarrow$ $\boxed{E_k \approx \frac{u_k^2}{k} = \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}}$

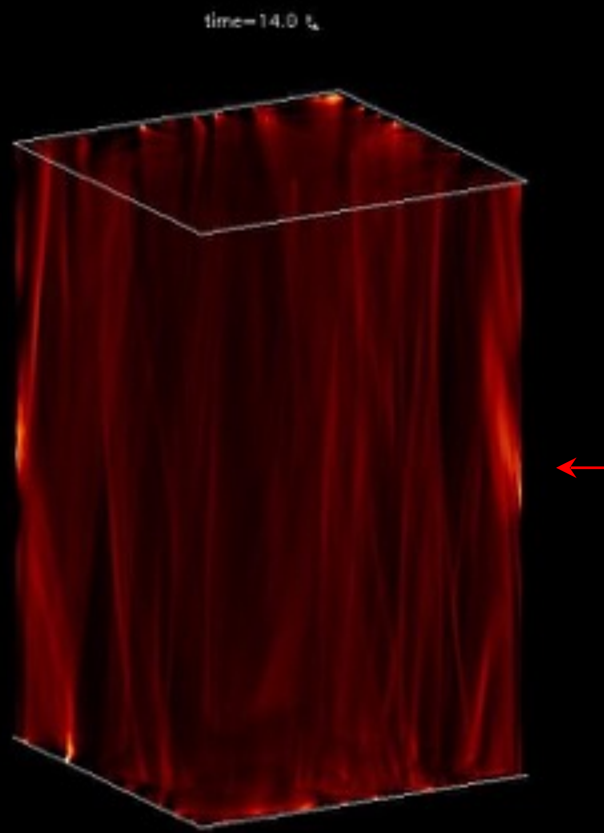
Kolmogorov spectrum (K41)

Dissipative structures: current sheets in 2D

- Most of the energy dissipation takes place in current sheets. We display the current density (upflows & downflows) along the loop in a transverse cut.



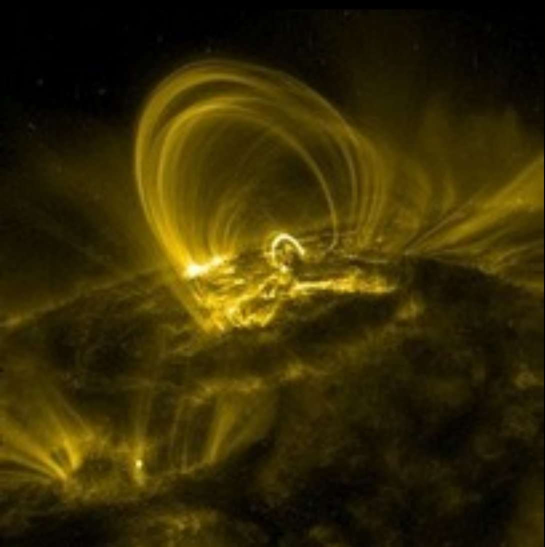
- Versus height.

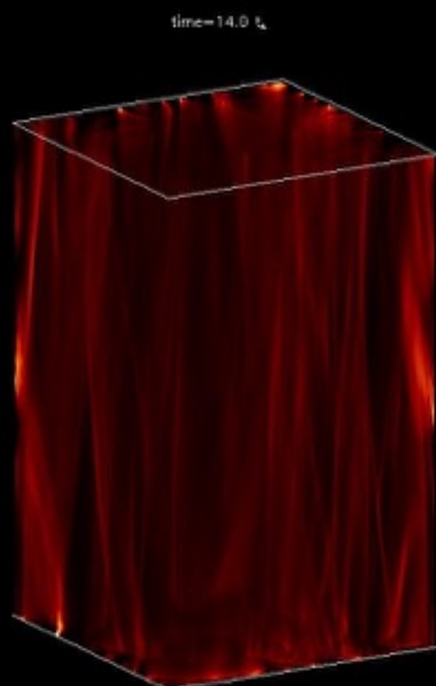


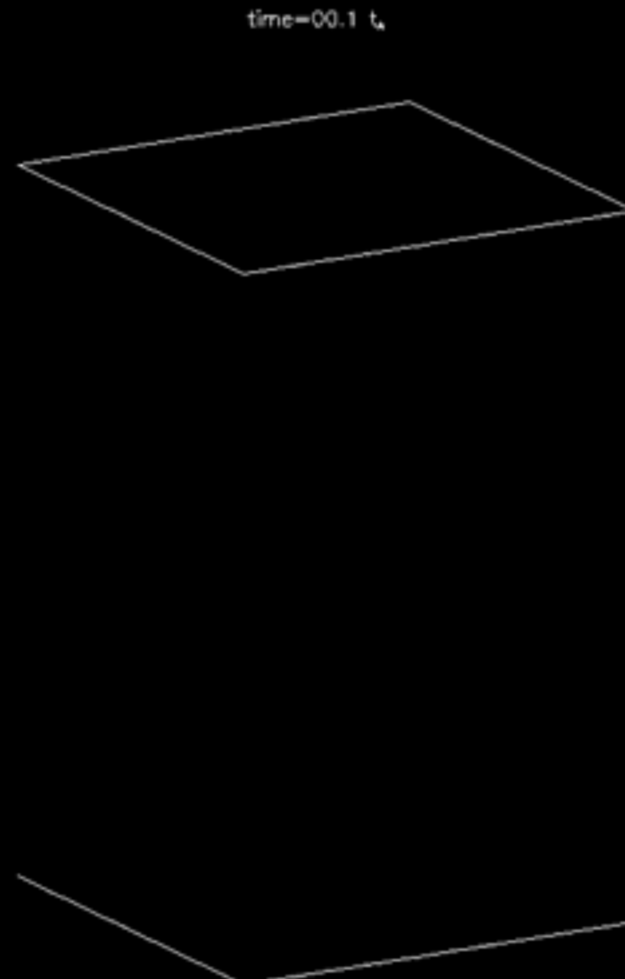
- Versus time.

Dissipative structures: current sheets in 3D

- 3D distribution of the energy dissipation rate.
- We display the dissipation rate during 20 Alfven times with a cadence of 0.1 t_A.







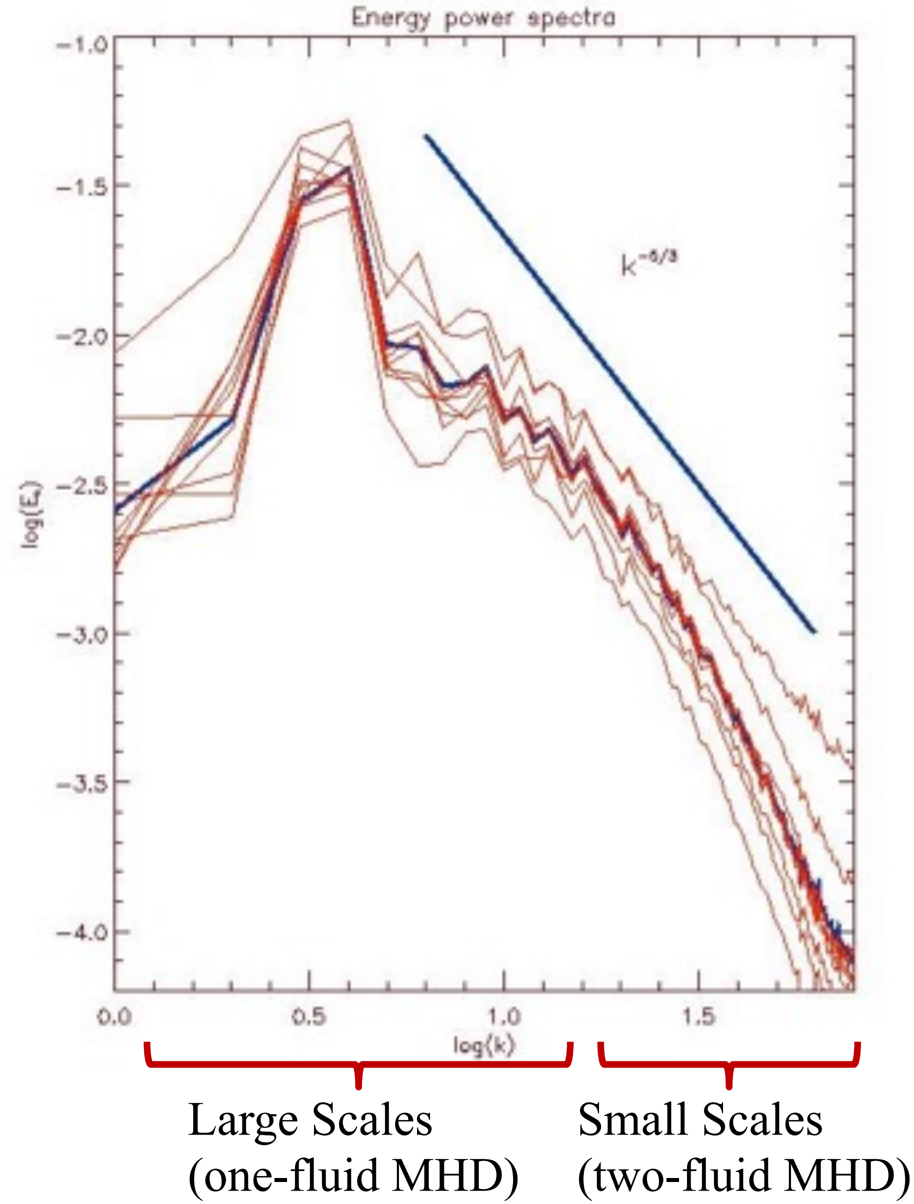
Large and small scales: Energy power spectra

- The energy spectra are shown here. The **red lines** correspond to ten spectra taken at different times (separated by 10 t_A). The **blue trace** is the time averaged version.

- The Kolmogorov slope is displayed for reference, but the moderate spatial resolution of these runs is insufficient for a serious spectral analysis.

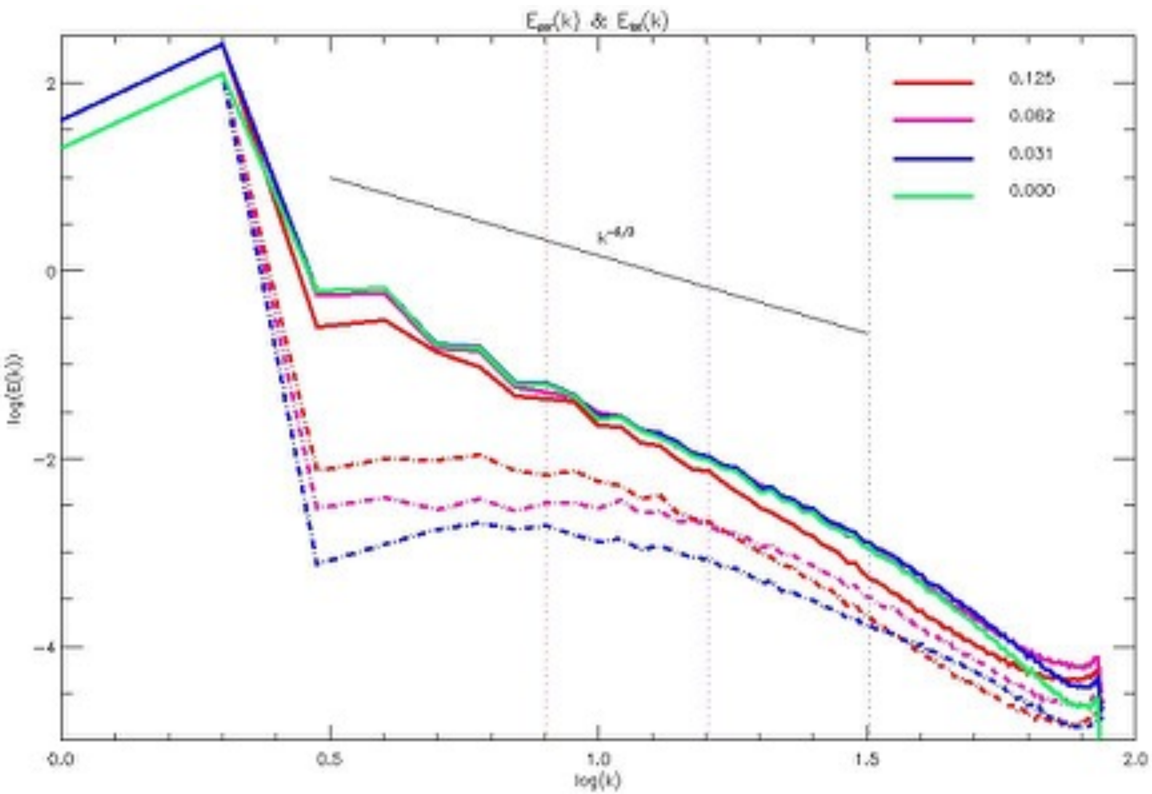
- Viscosity and resistivity are large enough to spatially resolve the dissipative structures properly.

- In one-fluid MHD, the only kinetic effects were viscosity and resistivity. A two-fluid description would bring new physics into play.



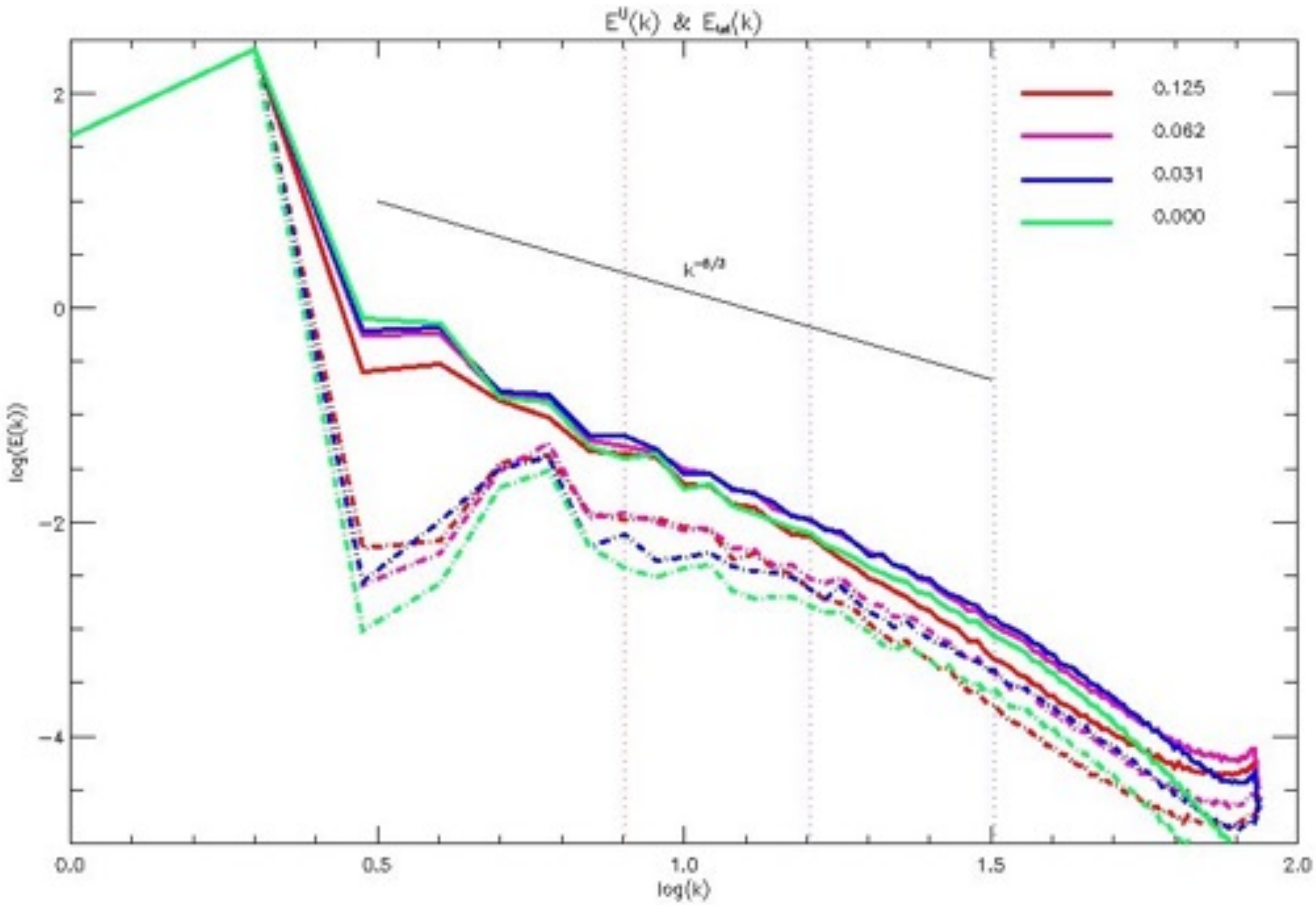
Energy spectra

- We also computed energy power spectra for different values of the Hall parameter ε .
- The Kolmogorov slope $k^{-5/3}$ is also displayed for reference.
- The dotted curves correspond to the parallel energy spectra.
- The vertical dotted lines indicate the location of the Hall scale $k_\varepsilon \cong \frac{1}{\varepsilon}$ for each run.



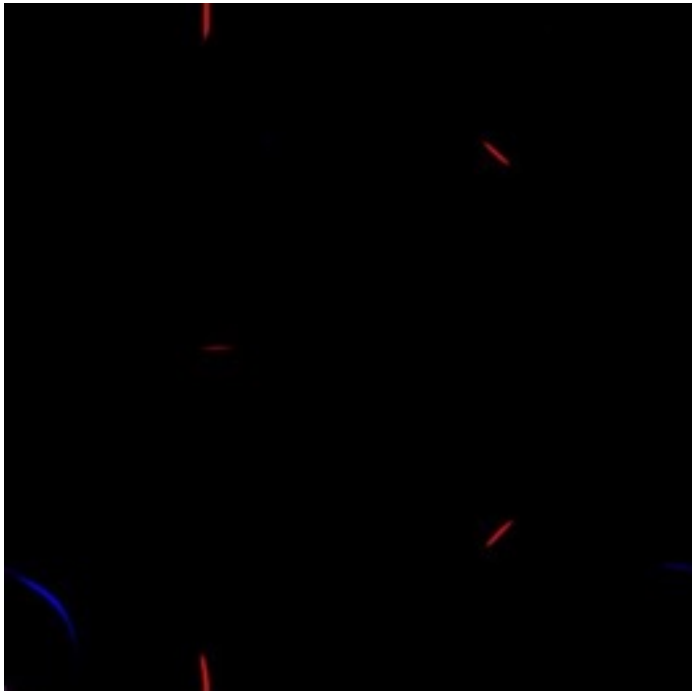
Energy spectra

- Energy power spectra for different values of ε.
- The dotted curves are the spectra for kinetic energy.

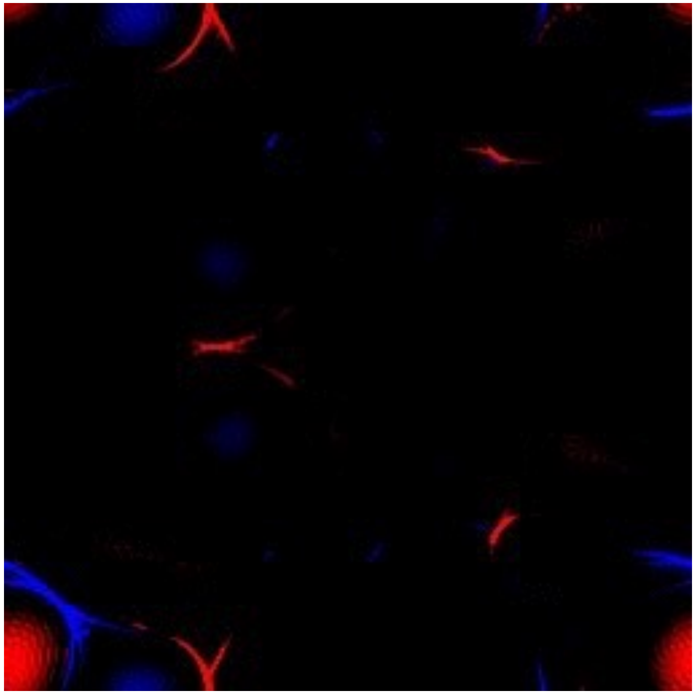


Current sheets in RHMHD

- Energy dissipation concentrates on very small structures known as current sheets, in which current density flows almost parallel to z.
- The picture shows **positive** and **negative** current density in a transverse cut at $z = \frac{1}{2}$, for pure RMHD (i.e. $\varepsilon = 0$).
- When the Hall effect is considered, current sheets display the typical Petschek-like structure.

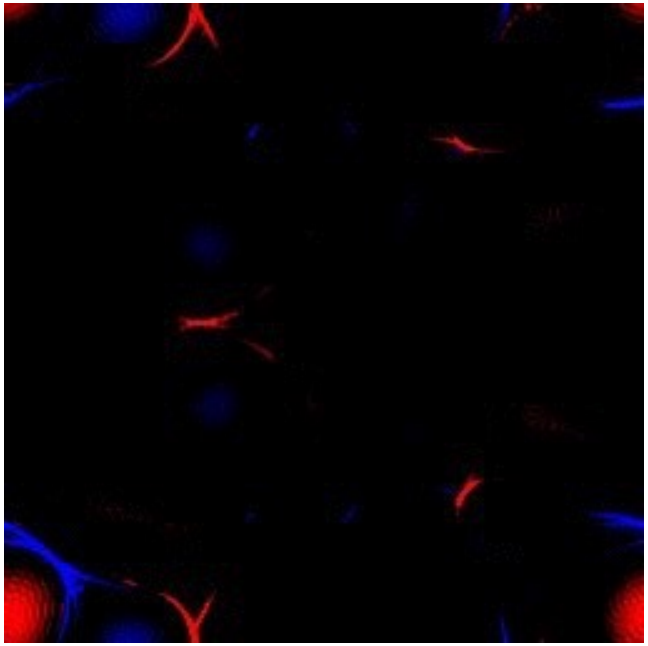


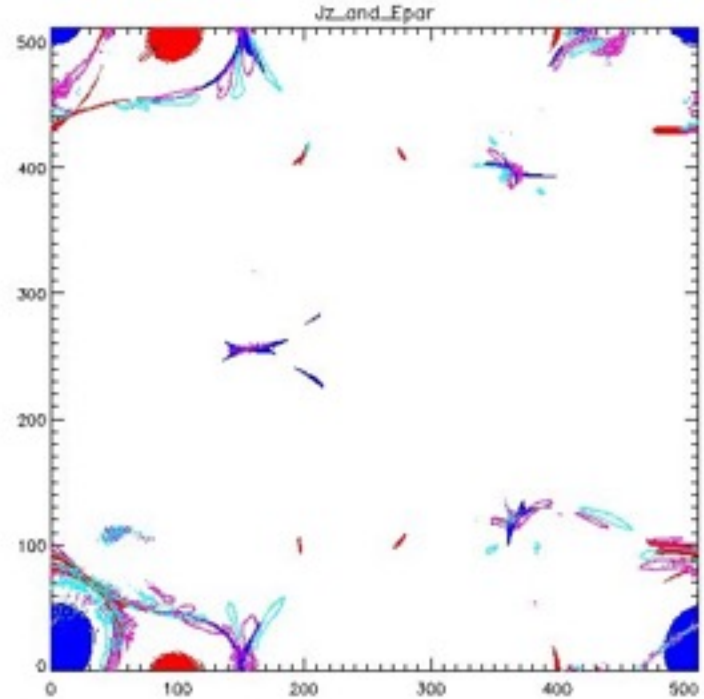
$\varepsilon = 0.0$



$\varepsilon = 0.1$

Parallel electric field





- One of the important new features of the Hall effect, is the presence of a parallel electric field, i.e. $E_{//} = \dfrac{\vec{E} \cdot \vec{B}}{|\vec{B}|}$
- To order α^2 it can be computed as $E_{//} = \varepsilon \left(\partial_z b - [a,b]\right)$

  and of course it can potentially accelerate particles along magnetic field lines.
- Current density is displayed in **red** and **blue**, while contours coloured in **light blue** and **pink** correspond to the parallel electric field.

Conclusions

- In this second lecture we introduced the Hall-MHD equations, which is an adequate theoretical framework to describe a number of astrophysical and laboratory applications.
- We also presented to so called reduced approximation, which is appropriate for plasmas embedded in relatively strong magnetic fields.
- We numerically integrated the Hall-MHD equations (spectral and Runge-Kutta) in the presence of a strong external magnetic field.
- As a first application, we showed RMHD simulations (no Hall effect yet) to study the internal dynamics of magnetic loops in the solar corona.
- We introduce the Hall effect and focused on its potential relevance in the dynamics of small scales and magnetic reconnection in the dissipative structures of turbulence.