Some aspects of Vorticity fields in Relativistic and Quantum Plasmas

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- ***Part I: Non-relativistic and Special relativistic Plasmas***
- ***Part II: General relativistic Plasmas***
- ***Part III: Quantum and Quantum Relativistic Plasmas***

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Part II:
VORTICITY IN GENERAL RELATIVISTIC PLASMAS

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Today...

- We explore the concept of vorticity fields in general relativistic plasmas
- We study the generation of vorticity
- We introduce the concept of Generalized helicity in general relativistic plasmas

Motivation

- We saw previously that the motion of a charged fluid in space-time generates a magnetic field, it stands to reason that if spacetime were distorted in the region occupied by a charged fluid, a magnetic field would emerge.
- According to this idea, we can explore if the properties of the plasmas can generate vorticities and magnetic fields in general relativity.
- The General Relativistic effects can open exciting possibility of spontaneous generation of magnetic fields near gravitating sources.

special relativity

It was recently demonstrated[2] that the generalized vorticity

$$\Omega = \mathbf{B} + \frac{m}{q}\nabla \times (f\gamma\mathbf{v})$$

The dynamics of Ω is given by

$$\frac{\partial\Omega}{\partial t} - \nabla \times (\mathbf{v} \times \blacksquare) = \chi_B + \chi_R$$

$$\chi_B = -\frac{c}{q\gamma}\nabla T \times \nabla\sigma\,, \qquad \chi_R = \frac{cT}{q\gamma^2}\nabla\gamma \times \nabla\sigma\,,$$

[2] Mahajan and Yoshida, PRL **105**, 095005 (2010); PoP **18**, 055701 (2011).

Plasma dynamics in General Relativity

general relativity (I): plasma dynamics of unified fields

The dynamics of an ideal plasma is obtained through $T^{\mu\nu}{}_{;\nu} = qnF^{\mu\nu}U_\nu$ (using the usual symbol ; for covariant derivatives) for the energy-momentum tensor $T^{\mu\nu} = hU^\mu U^\nu + pg^{\mu\nu}$ where $F^{\mu\nu}$ is the electromagnetic field tensor and U^μ is the normalized four-velocity ($U^\mu U_\mu = -1$ with $c = 1$), n is the density, h is the enthalpy density and p is the pressure. The charge q and the mass m of the fluid particles are invariants.

The plasma fluid fulfill the continuity equation $(nU^\mu)_{;\mu} = 0$.

The equation of motion could be written in terms of unified fields[3]

$$q\, U_\nu M^{\mu\nu} = T\sigma^{,\mu}\,, \tag{1}$$

where $M^{\mu\nu} = F^{\mu\nu} + (m/q)S^{\mu\nu}$ in terms of $S^{\mu\nu} = (fU^\nu)^{;\mu} - (fU^\mu)^{;\nu}$ and $f = h/mn$. All kinematic and thermal aspects of the fluid are now given by $S^{\mu\nu}$. The function σ is the entropy density of the fluid (where T is the temperature)

$$\sigma^{,\mu} = \frac{p^{,\mu} - mnf^{,\mu}}{nT}\,. \tag{2}$$

The antisymmetry of $M_{\mu\nu}$ guarantees that the fluid is isentropic $U_\mu\sigma^{,\mu} = 0$. Inclusion of the Maxwell equations completes the system description

$$F^{\mu\nu}{}_{;\nu} = 4\pi qnU^\mu\,. \tag{3}$$

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[3] Mahajan, Phys. Rev. Lett. **90**, 035001 (2003).

general relativity (II): 3+1 decomposition

The $3+1$ formalism allows us to obtain a set of equations that is similar to those found in special relativity.
The interval is (the shift vector is zero)

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij} dx^i dx^j \,, \quad (i,j = 1,2,3) \tag{4}$$

α is the lapse function and γ_{ij} is the 3-metric of the spacelike hypersurfaces of metric $g_{\mu\nu}$.

$$\alpha = \sqrt{-g_{00}}$$

it corresponds to the gravitational potential.
The timelike vector field $n^\mu = (-1/\alpha, 0, 0, 0)$ and $n_\mu = (\alpha, 0, 0, 0)$ [$n^\mu n_\mu = -1$ and $n^\mu \gamma_{\mu\nu} = 0$]

$$g_{\mu\nu} = \gamma_{\mu\nu} - n_\mu n_\nu$$

Thus, the $3+1$ decomposition is achieved by projecting every tensor onto n^μ in timelike hypersurfaces and onto $\gamma_{\mu\nu}$ in spacelike hypersurfaces. For example, the four-velocity $U^\mu = (\Gamma, \Gamma v^i)$, such that $n_\mu U^\mu = \alpha\Gamma$, the decomposition

$$U^\mu = -\alpha\Gamma n^\mu + \Gamma \gamma^\mu{}_\nu v^\nu \,, \tag{5}$$

allows us to write the Lorentz factor as

$$\Gamma = \left(\alpha^2 - \gamma_{\mu\nu} v^\mu v^\nu\right)^{-1/2} . \tag{6}$$

Vorticity generation and helicity in General Relativity

Magnetofluid unification

The generalized electric (ξ^μ) and magnetic (Ω^μ) fields in terms of $M^{\mu\nu}$ are

$$\xi^\mu = n_\nu M^{\nu\mu}\,, \qquad \Omega^\mu = \frac{1}{2} n_\rho \epsilon^{\rho\mu\sigma\tau} M_{\sigma\tau}\,, \tag{7}$$

both spacelike ($n_\mu \xi^\mu = 0$ and $n_\mu \Omega^\mu = 0$). The magnetofluid tensor reads

$$M^{\mu\nu} = \xi^\mu n^\nu - \xi^\nu n^\mu - \epsilon^{\mu\nu\rho\sigma} \Omega_\rho n_\sigma$$

$$\xi = \mathbf{E} - \frac{m}{\alpha q} \nabla \left(f \alpha^2 \Gamma \right) - \frac{m}{\alpha q} \frac{\partial}{\partial t} \left(f \Gamma \mathbf{v} \right), \tag{8}$$

$$\Omega = \mathbf{B} + \frac{m}{q} \nabla \times \left(f \Gamma \mathbf{v} \right). \tag{9}$$

The generalized magnetic field Ω is the generalized vorticity, GV. General relativity enters the definition of GV through Γ and ∇ (calculated with γ_{ij}).
The plasma equations are

$$q \alpha \Gamma \mathbf{v} \cdot \xi = -T \frac{\partial \sigma}{\partial t}\,, \tag{10}$$

while the plasma momentum evolution equation is

$$\alpha \Gamma \xi + \Gamma \mathbf{v} \times \Omega = \frac{T}{q} \nabla \sigma\,. \tag{11}$$

Generalized Vorticity generation [4]

The antisymmetry of $M^{\mu\nu}$ implies that its dual must obey $M^{*\mu\nu}{}_{;\nu} = 0$. The $3+1$ decomposition of this equation leads to $\partial\Omega/\partial t = -\nabla \times (\alpha\xi)$. Using this

$$\frac{\partial \Omega}{\partial t} - \nabla \times (\mathbf{v} \times \Omega) = \Xi_B + \Xi_R \,, \tag{12}$$

Ξ_B and Ξ_R are the sources of the vorticity Ω. These drives are nonzero only for inhomogeneous thermodynamics

$$\Xi_B = -\left(\frac{1}{q\Gamma}\right) \nabla T \times \nabla \sigma \,, \qquad \Xi_R = \frac{T\,\Gamma}{2q} \left[-\nabla \alpha^2 + \nabla \left(\gamma_{ij} v^i v^j \right) \right] \times \nabla \sigma \,, \tag{13}$$

Ξ_B is the traditional Biermann battery corrected by curvature. Ξ_R is the general relativistic drive and it is the principal object of this search.

- The relativistic drive Ξ_R is radically transformed from its flat space antecedent. The striking result is that the gravitational potential α, can produce a magnetic field in any region populated by charged particles even if their local velocities are negligible.
- Ξ_B and Ξ_R are non-magnetic thermodynamic source terms that create the conditions for the linear growth of the magnetic fields from zero initial value (batteries).

[4] Asenjo, Mahajan & Qadir, PoP**20**, 022901 (2013);

Generalized relativistic helicity

Again we define

$$K^{\mu} = \frac{1}{2\sqrt{-g}} \varepsilon^{\mu\nu\alpha\beta} P_{\nu} M_{\alpha\beta}$$

where

$$P_{\mu} = A_{\mu} + \frac{m}{q} f U_{\mu}$$

$$\begin{aligned} K^{\mu}{}_{;\mu} &= \frac{1}{\sqrt{-g}} \partial_{\mu} \left(\sqrt{-g} K^{\mu} \right) = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} P_{\nu;\mu} M_{\alpha\beta} + \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} P_{\nu} M_{\alpha\beta;\mu} \\ &= \varepsilon^{\mu\nu\alpha\beta} P_{\nu;\mu} M_{\alpha\beta} \end{aligned}$$

And the Generalized vorticity

$$h \equiv \int \sqrt{-g} K^{0} d^{3}x = \int \varepsilon^{0ijk} P_{i} M_{jk} d^{3}x = \int \mathcal{P} \cdot \Omega d^{3}x$$

$$\begin{aligned} \int \sqrt{-g} K^{\mu}{}_{;\mu} d^{3}x &= \int \partial_{\mu} \left(\sqrt{-g} K^{\mu} \right) d^{3}x = \int \partial_{0} \left(\sqrt{-g} K^{0} \right) d^{3}x = \partial_{t} h \\ &= \int \partial_{t} \mathcal{P} \cdot \Omega d^{3}x + \int \mathcal{P} \cdot \partial_{t} \Omega d^{3}x = \int \frac{2}{q\Gamma} \nabla \sigma \cdot \Omega d^{3}x \end{aligned}$$

Around a black hole

accreting plasma around a Schwarzschild black hole (I)

The plasma moves in an accretion disk in the equatorial plane ($\theta = \pi/2$). Its orbital velocity is $v^\phi = r\dot{\phi} = c\sqrt{r_0/2r}$ (being r_0 the Schwarzschild radius), and it is larger than the radial velocity.

At $5r_0$, the usual definition for entropy is valid. If the plasma obeys a barotropic equation of state, then $(T/c)\nabla\sigma \equiv \zeta k_B \nabla T$ where ζ is of order unity and the Biermann batery vanishes.

For the model described above, the general relativistic drive becomes

$$\Xi_R = \frac{3\zeta c k_B r_0 \alpha}{4\,e\,r^3}\left(1 - \frac{3r_0}{2r}\right)^{-1/2} \frac{\partial T}{\partial \phi}\hat{e}_z\,, \tag{14}$$

where the variations of the temperature have been taken in cylindrical geometry, and we have neglected the toroidal temperature gradients compared with the poloidal variations, $\partial_\theta T \ll \partial_\phi T$.

All the charged matter of the accretion disk contributes to Ξ_R, and acts as a source for Ω. For the stable orbit at $r = 5r_0$, the total relativistic drive is ($M_\odot$ is the solar mass)

$$\Xi_{R\text{total}} = \int_0^{2\pi} d\phi\ \Xi_R \approx 3\times 10^{-2}\zeta\left(\frac{M_\odot}{M}\right)^{9/4}\hat{e}_z\,, \tag{15}$$

where the disk radiates like a blackbody[5] $\int \partial_\phi T d\phi \approx 5\times 10^7 (M_\odot/M)^{1/4}$K.

[5]M. Vietri, *Foundations of high-energy astrophysics* (2008).

accreting plasma around a Schwarzschild black hole (II)

For a short time τ when the nonlinear terms involving Ω are negligible, Ω grows linearly with time $\Omega_{\text{total}} \approx \Xi_{R\text{total}}\tau$.
A measure of τ is provided by $|\Omega_{\text{total}}|\tau^{-1} \simeq |\nabla \times (\mathbf{v} \times \Omega_{\text{total}})|$ implying that $\tau \simeq L/|\mathbf{v}|$, where L is the length of variation of the $|\mathbf{v} \times \Omega|$ force. Taking the length L on which $|\mathbf{v}|$ varies to be of the order of $5r_0/\alpha$, the time for initial linear phase of GV seed is

$$\tau = \frac{5r_0}{|\mathbf{v}|\alpha} \approx 1.7 \times 10^{-4} \left(\frac{M}{M_\odot}\right), \tag{16}$$

in seconds, where the velocity is of order v^ϕ. The total strength of the magnetic field generated (in gauss) for the "test" plasma matter accreting at a distance $5r_0$ is

$$|\Omega_{\text{total}}| \approx 5 \times 10^{-6}\zeta \left(\frac{M_\odot}{M}\right)^{5/4}. \tag{17}$$

The initial seed is supposed to be small. It is created in a very short initial time in a state where there was no magnetic field to begin with. The existence of this seed is crucial to the startup of the standard processes of long-time magnetic field generation, like the dynamo process or the magneto-rotational instability. The dynamo process can operate only when it has some initial magnetic field to amplify; we have shown that the General Relativistic drive can provide the needful.

In cosmology

Friedmann–Robertson–Walker

The spatially flat universe

$$ds^2 = -dt^2 + a^2 \gamma_{ij} dx^i dx^j \,, \quad (i,j = 1,2,3) \tag{18}$$

$a = a(t)$ is the time-dependent scale factor of the Universe, and $\gamma_{ij} = (1,1,1)$ is the 3-metric of the spacelike hypersurfaces of the flat spacetime.
The four-velocity $U^\mu = (\Gamma, \Gamma v^i)$,

$$U_\mu = -\Gamma n_\mu + a^2 \Gamma \gamma_{\mu\nu} v^\nu \,, \tag{19}$$

such that $n_\mu U^\mu = \Gamma$, and the Lorentz factor is given by

$$\Gamma = \left(1 - a^2 v^2\right)^{-1/2} \,, \tag{20}$$

Generalized magnetofluid fields

$$\xi^\mu = n_\nu M^{\nu\mu} \,, \qquad \Omega^\mu = \frac{1}{2} n_\rho \epsilon^{\rho\mu\sigma\tau} M_{\sigma\tau} \,, \tag{21}$$

$$M^{\mu\nu} = \xi^\mu n^\nu - \xi^\nu n^\mu - \epsilon^{\mu\nu\rho\sigma} \Omega_\rho n_\sigma \,. \tag{22}$$

$$\xi = \mathbf{E} - \frac{m}{qa^2}\left[\nabla\left(f\Gamma\right) + \frac{\partial}{\partial t}\left(fa^2\Gamma\mathbf{v}\right)\right] \,, \tag{23}$$

$$\Omega = \mathbf{B} + \frac{ma^2}{q} \nabla \times (f\Gamma\mathbf{v}) = \nabla \times \left(\mathbf{A} + \frac{a^2 mf\Gamma}{q}\mathbf{v}\right) \,, \tag{24}$$

Plasma dynamics

$$a^2 \mathbf{v} \cdot \xi = \frac{T}{q\Gamma} \frac{\partial \sigma}{\partial t} \,, \tag{25}$$

$$\xi + \mathbf{v} \times \Omega = -\frac{T}{q\Gamma} \nabla \sigma \,. \tag{26}$$

$$\frac{\partial}{\partial t} \left(a^3 \Omega \right) + a^3 \nabla \times \xi = 0 \,, \tag{27}$$

$$\frac{1}{a^3} \frac{\partial}{\partial t} \left(a^3 \Omega \right) - \nabla \times (\mathbf{v} \times \Omega) = \Xi_B + \Xi_R \,. \tag{28}$$

The vector fields Ξ_B and Ξ_R are the sources of the vorticity Ω. The cosmological Biermann battery

$$\Xi_B = \frac{1}{q\Gamma} \nabla T \times \nabla \sigma \,, \tag{29}$$

and the general relativistic drive for a cosmological background

$$\Xi_R = \frac{T}{q} \nabla \left(\frac{1}{\Gamma} \right) \times \nabla \sigma = -\frac{a^2 T \,\Gamma}{2q} \nabla v^2 \times \nabla \sigma \,. \tag{30}$$

Relativistic cosmological drive[6]

Universe in the radiation-dominated era $a \propto t^{1/2}$ and $T \propto a^{-1} \propto t^{-1/2}$. In this case $\sigma = \sigma(T)$, and the Biermann battery drive vanishes.
The plasma velocity goes as $|\mathbf{v}| \propto a^{-1}$. If the temperature $T(t, r, \theta) = T_s(r, \theta)/a$, and the velocity $\mathbf{v}(t, r, \theta) = \mathbf{v}_s(r, \theta)/a$, then

$$|\Xi_R| \propto \frac{\Gamma_s}{q\ t^{1/2}} |\nabla v_s^2 \times \nabla T_s| \,, \tag{31}$$

where $v_s^2 = \mathbf{v}_s \cdot \mathbf{v}_s$, and the Lorentz factor $\Gamma_s = \left(1 - v_s^2\right)^{-1/2}$
For some short enough time (the initial seed generation phase) we can neglect the nonlinear terms involving Ω. The vorticity seed is found to grow as

$$|\Omega| \propto \frac{\Gamma_s}{q} |\nabla v_s^2 \times \nabla T_s| t^{1/2} \,, \tag{32}$$

being proportional to the scale factor of the Universe expansion ($\propto a$).

The vorticity generated (and therefore the magnetic field) grows as the early-Universe.

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[6] Asenjo & Mahajan, submitted to PoP

More general spacetimes

- Rotating black holes (Kerr metric)
  [Qadir, Asenjo & Mahajan, Phys. Scr. **89**, 084002 (2014);
  Bhattacharjee, Das & Mahajan, PRD **91**, 123005 (2015)]
- Non-minimal gravity coupling
  [Bhattacharjee, Das & Mahajan, PRD **91**, 064055 (2015)]

That’s all (for now).
Thanks!