Connections in Plasmas

Felipe A. Asenjo[1]
Universidad Adolfo Ibáñez,
Chile

[1]felipe.asenjo@uai.cl

Today...

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Generalized Magnetofluid Connections in Relativistic Magnetohydrodynamics

Felipe A. Asenjo[1,*] and Luca Comisso[2,3,†]
[1]*Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibáñez, Santiago 7941169, Chile*
[2]*Dipartimento Energia, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy*
[3]*Istituto dei Sistemi Complessi-CNR, Via dei Taurini 19, 00185 Roma, Italy*
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Generalized magnetofluid connections in pair plasmas

Felipe A. Asenjo,[1,a)] Luca Comisso,[2,3,b)] and Swadesh M. Mahajan[4,c)]
[1]*Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibáñez, Santiago 7941169, Chile*
[2]*Department of Astrophysical Sciences and Princeton Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08544, USA*
[3]*Dipartimento Energia, Politecnico di Torino, Torino 10129, Italy and Istituto dei Sistemi Complessi-CNR, Roma 00185, Italy*
[4]*Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712, USA*

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Newcomb’s Theorem

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Motion of Magnetic Lines of Force*

William A. Newcomb†

Project Matterhorn, Princeton University, Princeton, New Jersey

In 1958, Newcomb showed that in a plasma that satisfies the ideal Ohms law, two plasma elements connected by a magnetic field line at a given time will remain connected by a field line for all subsequent times. This occurs because the plasma moves with a transport velocity that preserves the magnetic connections between plasma elements. This is one of the most fundamental and relevant ideas in plasma physics.

Proof: d/dt is the convective derivative

Ohm's law $\vec{E} + \vec{v} \times \vec{B} = 0$ implies

$$\partial_t \vec{B} = \nabla \times (\vec{v} \times \vec{B}) = \frac{d\vec{B}}{dt} - (\vec{v} \cdot \nabla)\vec{B}$$

Proof: d/dt is the convective derivative

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Be $d\vec{l} = \vec{x}' - \vec{x}$ the 3D vector connecting two infinitesimally close fluid elements.

$$\frac{d}{dt} d\vec{l} = \vec{v}(\vec{x}') - \vec{v}(\vec{x}) = \vec{v}(\vec{x} + d\vec{l}) - \vec{v}(\vec{x}) = (d\vec{l} \cdot \nabla)\vec{v}$$

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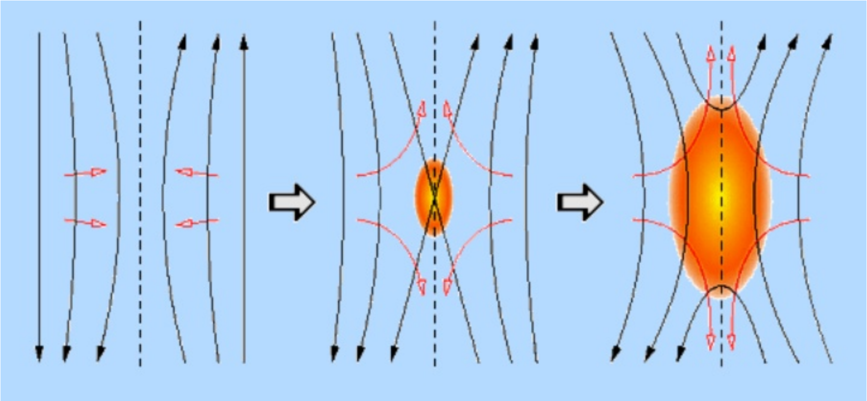
Then

$$\frac{d}{dt}(d\vec{l} \times \vec{B}) = -(d\vec{l} \times \vec{B})(\nabla \cdot \vec{v}) - \left[(d\vec{l} \times \vec{B}) \times \nabla\right] \times \vec{v}$$

Wich means that if $d\vec{l} \times \vec{B} = 0$, it always remains null



Pegoraro's generalization

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Covariant form of the ideal magnetohydrodynamic "connection theorem" in a relativistic plasma

F. PEGORARO[(a)]

Dipartimento di Fisica "Enrico Fermi", Università di Pisa - Largo B. Pontecorvo 3, I-56127 Pisa, Italy, EU

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Abstract – The magnetic connection theorem of ideal magnetohydrodynamics by Newcomb (NEWCOMB W. A., *Ann. Phys. (N.Y.)*, **3** (1958) 347) and its covariant formulation are rederived and reinterpreted in terms of a "time resetting" projection that accounts for the loss of simultaneity in different reference frames between spatially separated events.

Ohm's law

$$u^\mu = \frac{dx^\mu}{d\tau}$$

$$F^{\mu\nu} u_\nu = 0$$

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$$d/d\tau = u^\mu \partial_\mu$$

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where dl^μ is the 4D displacement of a plasma fluid element.

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$$\frac{d}{d\tau}(dl^\mu F_{\mu\nu}) = -(\partial_\nu u^\beta) dl^\alpha F_{\alpha\beta}$$

This means that if $dl^\mu F_{\mu\nu} = 0$, it always remains null

Relativistic Plasma

We extend the connection concept beyond

A plasma governed by generalized relativistic MHD equations. Effects such as thermal-inertial effects, thermal electromotive effects, current inertia effects and Hall effects.

Minkowski metric tensor $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, and an electron-ion plasma with density n, charge density $q = ne$, normalized four-velocity U^μ ($U_\mu U^\mu = -1$) and four-current density J^μ

continuity

$$\partial_\mu (q U^\mu) = 0$$

generalized momentum equation

$$\partial_\nu \left(h U^\mu U^\nu + \frac{\mu h}{q^2} J^\mu J^\nu \right) = -\partial^\mu p + J_\nu F^{\mu\nu} \, ,$$

generalized Ohm's law

$$\partial_\nu \left[\frac{\mu h}{q} (U^\mu J^\nu + J^\mu U^\nu) - \frac{\mu \Delta\mu h}{q^2} J^\mu J^\nu \right] = \frac{1}{2} \partial^\mu \Pi + q U_\nu F^{\mu\nu} - \Delta\mu J_\nu F^{\mu\nu} + q R^\mu \, .$$

h denotes the MHD enthalpy density, $\Pi = p\Delta\mu - \Delta p$, $p = p_+ + p_-$ and $\Delta p = p_+ - p_-$, $\mu = m_+ m_- / m^2$, $m = m_+ + m_-$, $\Delta\mu = (m_+ - m_-)/m$. The frictional four-force density between the fluids is

$$R^\mu = -\eta \left[J^\mu + Q(1 + \Theta) U^\mu \right] \, ,$$

where Θ is the thermal energy exchange rate from the negatively to the positively charged fluid, η is the plasma resistivity, and $Q = U_\mu J^\mu$.

As usual, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field tensor (A^μ is the four-vector potential), which obeys Maxwell's equations

$$\partial_\nu F^{\mu\nu} = 4\pi J^\mu \,, \qquad \partial_\nu F^{*\mu\nu} = 0 \,.$$

Of course, $F^{*\mu\nu} = (1/2)\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ is the dual of $F^{\mu\nu}$, and $\epsilon^{\mu\nu\alpha\beta}$ indicates the Levi-Civita symbol.

Ohm's law

$$\Sigma^\mu = \mathcal{U}_\nu \mathcal{M}^{\mu\nu} + R^\mu \,,$$

$$\mathcal{U}^\mu = U^\mu - \frac{\Delta\mu}{q} J^\mu \,, \qquad \mathcal{M}^{\mu\nu} = F^{\mu\nu} - \frac{\mu}{\Delta\mu} W^{\mu\nu} \,,$$

$$W^{\mu\nu} = S^{\mu\nu} - \Delta\mu \Lambda^{\mu\nu} = \partial^\mu \left(\frac{h}{q} \mathcal{U}^\nu \right) - \partial^\nu \left(\frac{h}{q} \mathcal{U}^\mu \right) \,,$$

$$S^{\mu\nu} \;\; = \;\; \partial^\mu \left(\frac{h}{q} U^\nu \right) - \partial^\nu \left(\frac{h}{q} U^\mu \right) \,,$$

$$\Lambda^{\mu\nu} \;\; = \;\; \partial^\mu \left(\frac{h}{q^2} J^\nu \right) - \partial^\nu \left(\frac{h}{q^2} J^\mu \right) \,.$$

and $\Sigma^\mu = \partial^\mu \left[\mu h Q / q^2 + \mu h / (q \Delta\mu) \right] + (\mu / \Delta\mu) \chi^\mu$, with

$$\chi^\mu = U_\nu \partial^\nu \left(\frac{h}{q} U^\mu \right) + \frac{\Delta\mu Q}{q} \partial^\mu \left(\frac{h}{q} \right) - \frac{\Delta\mu}{2\mu q} \partial^\mu \Pi \,.$$

Curl of the Ohm's law

$$\frac{d\mathcal{M}^{\lambda\phi}}{d\tau} = \partial^{\lambda}\mathcal{U}_{\nu}\mathcal{M}^{\phi\nu} - \partial^{\phi}\mathcal{U}_{\nu}\mathcal{M}^{\lambda\nu} - \frac{\mu}{\Delta\mu}\mathcal{Z}^{\lambda\phi} + \partial^{\lambda}R^{\phi} - \partial^{\phi}R^{\lambda} \,,$$

with $d/d\tau = \mathcal{U}_{\nu}\partial^{\nu}$, and

$$\mathcal{Z}^{\lambda\phi} = \mathcal{Z}_h^{\lambda\phi} + \mathcal{Z}_p^{\lambda\phi} + \mathcal{Z}_H^{\lambda\phi} + \mathcal{Z}_c^{\lambda\phi} \,,$$

where

$$\begin{aligned}
\mathcal{Z}_h^{\lambda\phi} &= \Delta\mu\left[\partial^{\lambda}\left(\frac{Q}{q}\right)\partial^{\phi}\left(\frac{h}{q}\right) - \partial^{\phi}\left(\frac{Q}{q}\right)\partial^{\lambda}\left(\frac{h}{q}\right)\right] \,, \\
\mathcal{Z}_p^{\lambda\phi} &= \frac{\partial^{\lambda}q}{q^2}\partial^{\phi}\left(p + \frac{\Delta\mu}{2\mu}\Pi\right) - \frac{\partial^{\phi}q}{q^2}\partial^{\lambda}\left(p + \frac{\Delta\mu}{2\mu}\Pi\right) \,, \\
\mathcal{Z}_H^{\lambda\phi} &= \partial^{\lambda}\left(\frac{1}{q}J_{\nu}F^{\phi\nu}\right) - \partial^{\phi}\left(\frac{1}{q}J_{\nu}F^{\lambda\nu}\right) \,, \\
\mathcal{Z}_c^{\lambda\phi} &= -\partial^{\lambda}\left[\frac{\mu}{q}J^{\alpha}\partial_{\alpha}\left(\frac{h}{q^2}J^{\phi}\right)\right] + \partial^{\phi}\left[\frac{\mu}{q}J^{\alpha}\partial_{\alpha}\left(\frac{h}{q^2}J^{\lambda}\right)\right] \,.
\end{aligned}$$

$\mathcal{Z}_h^{\lambda\phi}$ and $\mathcal{Z}_p^{\lambda\phi}$ are due to the thermal-inertial and thermal electromotive effects. The contributions coming from the Hall effect in the generalized Ohm's law are instead retained by the tensor $\mathcal{Z}_H^{\lambda\phi}$, while $\mathcal{Z}_c^{\lambda\phi}$ appears owing to current inertia effects.

Displacement of a plasma element

Define a general displacement four-vector Δx^μ of a general element that is transported by the general four-velocity

$$\frac{\Delta x^\mu}{\Delta \tau} = \mathcal{U}^\mu + \frac{\mu}{\Delta \mu}\mathcal{D}^\mu \,,$$

where $\Delta \tau$ is the variation of the proper time and $\mathcal{D}^\mu$ is a four-vector field which satisfies the equation

$$\mathcal{M}^{\nu\phi}\partial^\lambda \mathcal{D}_\nu - \mathcal{M}^{\nu\lambda}\partial^\phi \mathcal{D}_\nu = \mathcal{Z}^{\lambda\phi} \,.$$

The four-vector $\mathcal{D}^\mu$ contains all the (inertial-thermal-current-Hall) information of $\mathcal{Z}^{\mu\nu}$.
We introduce the event-separation four-vector $dl^\mu = x'^\mu - x^\mu$ between two different elements. Then $(d/d\tau)dl^\mu = \mathcal{U}'^\mu + (\mu/\Delta\mu)\mathcal{D}'^\mu - \mathcal{U}^\mu - (\mu/\Delta\mu)\mathcal{D}^\mu = \mathcal{U}^\mu(x_\alpha + dl_\alpha) + (\mu/\Delta\mu)\mathcal{D}^\mu(x_\alpha + dl_\alpha) - \mathcal{U}^\mu(x_\alpha) - (\mu/\Delta\mu)\mathcal{D}^\mu(x_\alpha)$. Therefore, the four-vector dl^μ fulfills

$$\frac{d}{d\tau}dl^\mu = dl^\alpha \partial_\alpha \left(\mathcal{U}^\mu + \frac{\mu}{\Delta\mu}\mathcal{D}^\mu\right) \,.$$

Connections when resistivity is neglected!

Finally we find

$$\frac{d}{d\tau}\left(dl_\lambda \mathcal{M}^{\lambda\phi}\right) = -\left(dl_\lambda \mathcal{M}^{\lambda\nu}\right)\partial^\phi\left(\mathcal{U}_\nu + \frac{\mu}{\Delta\mu}\mathcal{D}_\nu\right) .$$

This equation reveals the existence of generalized magnetofluid connections that are preserved during the plasma dynamics. Indeed, from this equation it follows that if $dl_\lambda \mathcal{M}^{\lambda\phi} = 0$ initially, then $d/d\tau(dl_\lambda \mathcal{M}^{\lambda\phi}) = 0$ for every time, and so $dl_\lambda \mathcal{M}^{\lambda\phi}$ will remain null at all times.

The "magnetofluid connection equation" all previous results for a relativistic electron-ion MHD plasma with thermal-inertial, Hall, thermal electromotive and current inertia effects

$$dl_\lambda \mathcal{M}^{\lambda\phi} = dl_\lambda F^{\lambda\phi} - \frac{\mu}{\Delta\mu} dl_\lambda W^{\lambda\phi} ,$$

Awesome, right?

Thanks!