Some aspects of Vorticity fields in Relativistic and Quantum Plasmas

Felipe A. Asenjo[1]
Universidad Adolfo Ibáñez,
Chile

- ***Part I: Non-relativistic and Special relativistic Plasmas***
- ***Part II: General relativistic Plasmas***
- ***Part III: Quantum and Quantum Relativistic Plasmas***

[1] felipe.asenjo@uai.cl

Part III:
VORTICITY IN QUANTUM PLASMAS

Felipe A. Asenjo

Today...

- We explore the concept of vorticity fields in quantum plasmas
- We introduce the concept of helicity in these plasmas

Motivation

Is there something similar to what we have been studying before?

Quantum plasma fluid

Fluidization of quantum systems

- Early Era - Madelung, Bohm, Takabayasi. They tried to understand and interpret quantum mechanics in terms of familiar classical concepts
- Content to devise appropriate fluid-like variables obeying the "expected" fluid like equations of motion: Continuity - Force balance etc.
- Quantum phenomena entered the latter through the so called "quantum forces" proportional to powers of $\hbar$
- The macroscopic formulations (for studying collective motions of quantum fluids) have invoked methodologies similar to those employed in classical plasmas
- Both the fluid and kinetic theories have been constructed:
  - simple quantum (Feix, Anderson, Haas, Kuzmenkov, etc.)
  - spin quantum (Marklund, Brodin, Andreev, Kuzmenkov, Zamanian, etc.)
  - relativistic quantum plasmas (Mahajan, Asenjo, Shukla, Hakim, Sivak, Mendonça, Biali, etc.)

- When the de Broglie wavelegth of the charged contituents of the plasma is comparable to the dimensions of the system, the quantum effects must be considered

$$\lambda_B n^{1/3} \sim 1$$

$$\lambda_B = \frac{\hbar}{mv}$$

- Quantum effects play an important role in very dense scenarios, as astrophysical ones (neutron stars, accretion disks) with strong magnetic fields, nano-scale physics (applications to condense matter), microplasmas and high-energy lasers.
- New effects in propagation modes, shock waves, solitons, inestabilities, etc.

Schrödinger equation

The fluid approach for quantum plasmas start with the Schrödinger equation

$$i\hbar\frac{\partial\psi_{(\alpha)}}{\partial t} = \left[-\frac{\hbar^2}{2m}\left(\nabla + \frac{ie}{\hbar c}\mathbf{A}\right)^2 - e\phi\right]\psi_{(\alpha)}$$

with the subindex (α) representing the particle quantum state. Using the Madelung decomposition

$$\psi_{(\alpha)} = \sqrt{n_{(\alpha)}}\exp(iZ_{(\alpha)}/\hbar)$$

where $n_{(\alpha)}$ is identified with number density and $Z_{(\alpha)}$ is the phase. The velocity is defined as

$$\mathbf{v}_{(\alpha)} = \frac{1}{m}\nabla Z_{(\alpha)} + \frac{e}{mc}\mathbf{A}$$

We obtain the fluid equations

$$\frac{\partial n_{(\alpha)}}{\partial t} + \nabla \cdot (n_{(\alpha)} \mathbf{v}_{(\alpha)}) = 0$$

$$\frac{\partial \mathbf{v}_{(\alpha)}}{\partial t} + (\mathbf{v}_{(\alpha)} \cdot \nabla)\mathbf{v}_{(\alpha)} = \frac{e}{m}(\mathbf{E} + \mathbf{v}_{(\alpha)} \times \mathbf{B}) + \frac{\hbar^2}{2m^2}\nabla \left(\frac{\nabla^2 \sqrt{n_{(\alpha)}}}{\sqrt{n_{(\alpha)}}} \right)$$

The quantum correction term is called Bohm potential.

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The quantum correction term is called Bohm potential.

But this is a fluid description for one-particle

We have to define the total density and the total fluid velocity as

$$n = \sum_{\alpha} p_{(\alpha)} n_{(\alpha)}, \qquad \mathbf{v} = \langle \mathbf{v}_{(\alpha)} \rangle = \frac{1}{n} \sum_{\alpha} p_{(\alpha)} n_{(\alpha)} \mathbf{v}_{(\alpha)}$$

$$z_{(\alpha)} = \mathbf{v}_{(\alpha)} - \mathbf{v}, \qquad \langle z_{(\alpha)} \rangle = 0$$

where $p_{(\alpha)}$ is the probability associated to each state. This is called ensemble average.

Fluid description for quantum plasma

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{q}{m}(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - \frac{1}{mn}\nabla\cdot\Pi + \frac{\hbar^2}{2m^2}\left\langle \nabla\left(\frac{\nabla^2\sqrt{n}_\alpha}{\sqrt{n}_\alpha}\right)\right\rangle$$

where $\Pi^{ij} = mn\langle z^i_{(\alpha)} z^j_{(\alpha)}\rangle$ is the pressure tensor.

Usually is assumed

$$\frac{\hbar^2}{2m^2}\left\langle \nabla\left(\frac{\nabla^2\sqrt{n}_\alpha}{\sqrt{n}_\alpha}\right)\right\rangle \sim \frac{\hbar^2}{2m^2}\nabla\left(\frac{\nabla^2\sqrt{n}}{\sqrt{n}}\right)$$

Spin quantum plasma fluid [2]

[2] Mahajan & Asenjo, PRL **107**, 195003 (2011)

Pauli equation

$$i\hbar\frac{\partial\Psi_{(\alpha)}}{\partial t} = \left[-\frac{\hbar^2}{2m}\left(\nabla + \frac{ie}{\hbar c}\mathbf{A}\right)^2 - \frac{e\hbar}{2mc}\mathbf{B}\cdot\sigma - e\phi\right]\Psi_{(\alpha)}$$

The spinor is decomposed in a similar form as a Madelung decomposition [3]

$$\Psi_{(\alpha)} = \sqrt{n_{(\alpha)}}\exp(iZ_{(\alpha)}/\hbar)\psi_{(\alpha)}$$

with a normalized two-spinor $\psi_{(\alpha)}$.
The velocity and the spin density vector are defined as

$$\mathbf{v}_{(\alpha)} = \frac{1}{m}\left(\nabla Z_{(\alpha)} - i\hbar\psi^{\dagger}_{(\alpha)}\nabla\psi_{(\alpha)}\right) + \frac{e}{mc}\mathbf{A}$$

$$\mathbf{s}_{(\alpha)} = \frac{\hbar}{2}\psi^{\dagger}_{(\alpha)}\sigma\psi_{(\alpha)}$$

[3] Takabayasi, PTP **14**, 283 (1955); Marklund & Brodin PRL **98**, 025001 (2007).

And the fluid equations are

$$\frac{\partial n_{(\alpha)}}{\partial t} + \nabla \cdot (n_{(\alpha)} \mathbf{v}_{(\alpha)}) = 0$$

$$\begin{aligned} m\frac{\partial \mathbf{v}_{(\alpha)}}{\partial t} + m(\mathbf{v}_{(\alpha)} \cdot \nabla)\mathbf{v}_{(\alpha)} &= e(\mathbf{E} + \mathbf{v}_{(\alpha)} \times \mathbf{B}) + e\mathbf{s}_{(\alpha)_k} \nabla \mathbf{B}_k \\ &\quad - \frac{1}{n_{(\alpha)}} \partial_k \left(n_{(\alpha)} \nabla \mathbf{s}_{(\alpha)_j} \partial_k \mathbf{s}_{(\alpha)_j} \right) \\ &\quad + \frac{\hbar^2}{2} \nabla \left(\frac{\nabla^2 \sqrt{n_{(\alpha)}}}{\sqrt{n_{(\alpha)}}} \right) \end{aligned}$$

$$\frac{\partial \mathbf{s}_{(\alpha)}}{\partial t} + (\mathbf{v}_{(\alpha)} \cdot \nabla)\mathbf{s}_{(\alpha)} = \frac{e}{m}\mathbf{s}_{(\alpha)} \times \mathbf{B} + \frac{1}{mn_{(\alpha)}}\mathbf{s}_{(\alpha)} \times \partial_k \left(n_{(\alpha)} \partial_k \mathbf{s}_{(\alpha)} \right)$$

But, again, these equations are for one-particle!

Plasma equations - momentum

emsemble average

$$n = \sum_\alpha p_{(\alpha)} n_{(\alpha)}, \quad \mathbf{v} = \langle \mathbf{v}_{(\alpha)} \rangle, \quad \mathbf{s} = \langle \mathbf{s}_{(\alpha)} \rangle$$

$$z_{(\alpha)} = \mathbf{v}_{(\alpha)} - \mathbf{v}, \quad w_{(\alpha)} = \mathbf{s}_{(\alpha)} - \mathbf{s}, \quad \langle z_{(\alpha)} \rangle = \langle w_{(\alpha)} \rangle = 0$$

we obtain the continuity equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0$$

The equation for evolution for velocity

$$mn \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = ne \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right) - \nabla \cdot \Pi + \mathbf{F}_Q$$

where $\Pi^{ij} = mn \langle z^i_{(\alpha)} z^j_{(\alpha)} \rangle$ is the pressure tensor, and

$$\begin{aligned} \mathbf{F}_Q &= en\mathbf{s}_k \nabla \mathbf{B}_k + \frac{n\hbar^2}{2} \left\langle \nabla \left(\frac{\nabla^2 \sqrt{n_{(\alpha)}}}{\sqrt{n_{(\alpha)}}} \right) \right\rangle \\ & \quad -\partial_k \left(n \nabla \mathbf{s}_j \partial_k \mathbf{s}_j + n \langle \nabla w_{(\alpha)j} \rangle \partial_k \mathbf{s}_j + n \langle \nabla \mathbf{s}_{(\alpha)j} \partial_k w_{(\alpha)j} \rangle \right) \end{aligned}$$

Plasma equations - spin

$$n\frac{\partial \mathbf{s}}{\partial t} + n(\mathbf{v}\cdot\nabla)\mathbf{s} = \frac{en}{m}\mathbf{s}\times\mathbf{B} + \nabla\cdot K + \Omega_Q$$

where $K^{ij} = n\langle z^i_{(\alpha)} w^j_{(\alpha)}\rangle$ is the thermal spin coupling tensor, and Ω_Q is a quantum correction

$$\begin{aligned}\Omega_Q &= \frac{1}{m}\mathbf{s}\times\partial_k\left(n\partial_k\mathbf{s}\right) + \frac{1}{m}\mathbf{s}\times\partial_k\left(n\langle\partial_k w_{(\alpha)}\rangle\right) \\ &\quad + \frac{n}{m}\left\langle \frac{1}{n_{(\alpha)}} w_{(\alpha)}\times\partial_k\left(n_{(\alpha)}\partial_k\mathbf{s}_{(\alpha)}\right)\right\rangle\end{aligned}$$

The Spin Quantum Plasma System

The macroscopic continuity, force balance and spin evoution equations are (n as density, $\mathbf{v}$ as the fluid velocity and $\mathbf{S}$ as the spin vector, $\mu = q\hbar/2mc$ as the magnetic moment, and $\mathbf{S}\cdot\mathbf{S} = 1$):

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0 \tag{1}$$

$$m\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{v} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right) + \mu S_j \nabla \widehat{B}_j + \Xi \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{S} = \frac{2\mu}{\hbar}\left(\mathbf{S}\times\widehat{\mathbf{B}}\right) \tag{3}$$

Neglection of effects like the spin-spin and the thermal-spin couplings.

The spin interact with the effective magnetic field

$$\widehat{\mathbf{B}} = \mathbf{B} + \frac{\hbar c}{2qn}\partial_i\left(n\partial_i\mathbf{S}\right) \,, \tag{4}$$

composed of the two parts; there is nonlinear spin-spin force.
The pressure gets contributions from

$$\Xi = -\frac{1}{n}\nabla p + \frac{\hbar^2}{2m}\nabla\left(\frac{\nabla^2\sqrt{n}}{\sqrt{n}}\right) + \frac{\hbar^2}{8m}\nabla\left(\partial_j S_i \partial_j S_i\right) , \tag{5}$$

the classical pressure p, the Bohm potential, and the effective spin pressure.

- The dynamics of an ideal classical fluid (blue) is extended to include the quantum/spin (red) effects.

Quantum force-destruction of the standard ideal vortex

Let us now revisit the force balance equation for a spin quantum plasma in vortical language. For a barotropic fluid, the equations of motion are

$$\frac{\partial \mathbf{P}_c}{\partial t} = \mathbf{v} \times \Omega_c + \frac{\hbar}{2m} S_j \nabla \widehat{B}_j + \frac{c}{q} \widehat{\Xi}\,, \tag{6}$$

with $\widehat{\Xi} = \Xi - \nabla(q\phi + m\mathbf{v}^2/2)$ and

$$\mathbf{P}_c = \mathbf{A} + \frac{mc}{q}\mathbf{v} \qquad\qquad \Omega_c = \nabla \times \mathbf{P}_c$$

And its curl

$$\frac{\partial \Omega_c}{\partial t} = \nabla \times (\mathbf{v} \times \Omega_c) + \frac{\hbar}{2m} \nabla S_j \times \nabla \widehat{B}_j\,, \tag{7}$$

Spin quantum forces “destroy” the canonical vortical structure for Ω_c!

vortex Dynamics - Helicity:

Ideal vortex dynamics insures conservation of field helicity.

For the ideal classical vortex dynamics, the conserved classical generalized helicity takes the form $[\langle\,\rangle = \int d^3x]$

$$h_c = \langle \Omega_c \cdot \mathbf{P}_c \rangle \tag{8}$$

Helicity conservation is a topological constraint and is the primary determinant for the formation of non trivial self-organizing equilibrium configurations in plasmas.

The spin forces act as a quantum source for classical helicity

$$\frac{dh_c}{dt} = \frac{\hbar}{m} \left\langle \Omega_{ci} S_j \partial^i \widehat{B}^j \right\rangle \,, \tag{9}$$

and it could cause transitions to a different helicity state.

Potential forces- Bohm potential etc-do not contribute to vorticity evolution.

Spinning Quantum fluid again

Let us go back to the dynamics of a spinning fluid:

$$\frac{\partial \Omega_c}{\partial t} = \nabla \times (\mathbf{v} \times \Omega_c) + \frac{\hbar}{2m} \nabla S_j \times \nabla \widehat{B}_j \,, \tag{10}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right) \mathbf{S} = \frac{2\mu}{\hbar} \left(\mathbf{S} \times \widehat{\mathbf{B}}\right) \tag{11}$$

The spin field, in addition, satisfies $\mathbf{S} \cdot \mathbf{S} = 1$.

Question: Does the system allow a grand generalized vorticity?

If so what would the spin vorticity look like and what may it mean?

One must manipulate (17) in some creative way.
The aim, clearly, is to eliminate the “force” term in Eq. (16).

Looking for Spin Vorticity:

If we were able to convert Eq.17 into the form,

$$\frac{\partial \Omega_s}{\partial t} = \nabla \times (\mathbf{v} \times \Omega_s) + \frac{\hbar}{2m} \nabla S_j \times \nabla \widehat{B}_j \,, \tag{12}$$

then

$$\Omega_- = \Omega_c - \Omega_s$$

would, indeed, obey the standard vortex dynamics

$$\frac{\partial \Omega_-}{\partial t} = \nabla \times (\mathbf{v} \times \Omega_-) \tag{13}$$

Is there such an Ω_s?

Quantum Spin vorticity:

The spin vorticity $(1, 2, 3$ denote components of $\mathbf{S})$

$$\Omega_s = S_1\left(\nabla S_2 \times \nabla S_3\right) + S_2\left(\nabla S_3 \times \nabla S_1\right) + S_3\left(\nabla S_1 \times \nabla S_2\right)\ , \tag{14}$$

The constraint $S_1^2 + S_2^2 + S_3^2 = 1$ $(\Rightarrow S_1 \nabla S_1 + S_2 \nabla S_2 + S_3 \nabla S_3 = 0)$ allows an alternate simpler expression(and cyclical counterparts)

$$\Omega_s = \frac{\nabla S_1 \times \nabla S_2}{S_3} \tag{15}$$

In component form $\Omega_s{}^i = \frac{1}{2}\varepsilon^{ijk}\varepsilon^{lmn} S_l \partial_j S_m \partial_k S_n$.

Ω_s is, indeed a vorticity, it is the curl of a vector field: $\Omega_s = \nabla \times \mathbf{P}_s$

$$\mathbf{P}_s = -S_3 \nabla[\arctan(S_2/S_1)]$$

.

The potential $\mathbf{P}_s$ is in the Clebsch form

The Potential and the final set

The potential $\mathbf{P}_s$ obeys

$$\frac{\partial \mathbf{P}_s}{\partial t} = \mathbf{v} \times \Omega_s + \frac{q}{mc} S_j \nabla \widehat{B}_j \,, \tag{16}$$

We have , by this time, created a whole plethora of equations. A possible complete set independent set is

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{S} = \frac{2\mu}{\hbar} \left(\mathbf{S} \times \widehat{\mathbf{B}} \right) \tag{17}$$

$$\frac{\partial \Omega_-}{\partial t} = \nabla \times (\mathbf{v} \times \Omega_-) \tag{18}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J} + 4\pi \nabla \times \mathbf{M} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \tag{19}$$

with $\mathbf{J}$ as the current density, and $\mathbf{M} = \mu n \mathbf{S}$ as the magnetization density.

The Conserved Helicity

We have a potential vector field $\mathbf{P}_- = \mathbf{P}_c - (\hbar c/2q)\mathbf{P}_s$ satisfying

$$\frac{\partial \mathbf{P}_-}{\partial t} = \mathbf{v} \times \Omega_- + \frac{c}{q}\widehat{\Xi}\,, \tag{20}$$

One defines the Grand Generalized Helicities $h_- = \langle \mathbf{P}_- \cdot \Omega_- \rangle$ which is a constant of motion

$$\frac{dh_-}{dt} = 0\,. \tag{21}$$

The existence of h_- is quite an amazing result. We have found this constant of motion that straddles the classical and quantum domains

More on Quantum Spin Vorticity

The spin vector lies on the surface of a unit sphere: $\mathbf{S} \cdot \mathbf{S} = 1$. Allows a parametric representation $S_3 = \cos\theta, S_2 = \sin\theta\cos\phi, S_1 = \sin\theta\sin\phi$

The spin vorticity ($\mathbf{P}_s = -S_3\nabla[\arctan(S_2/S_1)] = -\cos\theta\nabla\phi$)

$$\Omega_s = \frac{\nabla S_1 \times \nabla S_2}{S_3} = \nabla \times \mathbf{P}_s = \sin\theta\nabla\theta \times \nabla\phi \tag{22}$$

displays very interesting characteristics.

All components of $\mathbf{S}$ must be nonzero and inhomogeneous for a nonzero Ω_s

The Clebsch form forces the Helicity density of the pure spin field to be zero

$$\Omega_s \cdot \mathbf{P}_s = 0 \tag{23}$$

Spin vorticity, however, contributes in a fundamental way to the conserved helicity $h_- = \langle \mathbf{P}_- \cdot \Omega_- \rangle$

Conserved Helicity- Some details

Let us spell out the conserved helicity

$$h_- = \langle \mathbf{P}_- \cdot \Omega_- \rangle = \langle (\mathbf{P}_c + \mathbf{P}_s) \cdot (\Omega_c + \Omega_s) \rangle = \langle \mathbf{P}_c \cdot \Omega_c + 2\mathbf{P}_c \cdot \Omega_s \rangle \quad (24)$$

It is thus, through the cross term $h_{cross} = \langle \mathbf{P}_c \cdot \Omega_s \rangle$ that the spin vorticity affects the overall dynamics.

- Helicity is an invariant measure of the "complexity" of a vector field-the connectedness or knottedness of the flow lines.
- The spin field , by itself, is simple (zero helicity)
- But the total relevant field- canonical plus spin-does , indeed, support complexity and structure formation.

Applications

- Superconductor-like equilibrium spin plasma states; Mahajan & Asenjo, Phys. Rev. Lett. **107**, 195003 (2011)
- Electromagnetic instabilities due to spin; Braun, Asenjo & Mahajan, Phys. Rev. Lett. **109**, 175003 (2012)
- Relaxed states for spin neutron plasmas; Mahajan & Asenjo, Phys. Lett. A **377**, 1430 (2013)
- Effect of spin in radiation reaction, Mahajan, Asenjo & Hazeltine, MNRAS 446, 4112 (2015)

Relativistic quantum plasma fluid[4]

[4] Mahajan & Asenjo, International Journal of Theoretical Physics **54** 1435 (2015)

Hot Relativistic Perfect Fluid

$$T^{\mu\nu} = \frac{n}{w} f \, p^\mu p^\nu + \Pi \, \eta^{\mu\nu} \,, \tag{25}$$

where p^μ is the fluid kinematic momentum, Π is the pressure, f is the enthalpy, n is the invariant number density, and w is a constant with dimensions of mass.

$$\partial_\mu (n p^\mu) = 0 \,. \tag{26}$$

$$p^\mu{}_{eff} \equiv P^\mu = f \, p^\mu \,, \tag{27}$$

$$T^{\mu\nu} = g \, P^\mu P^\nu + \Pi \, \eta^{\mu\nu} \,, \tag{28}$$

$$\partial_\mu (g P^\mu) = 0 \,. \tag{29}$$

where

$$g = \frac{n}{fw}$$

Quantization for the effective momentum

$$g P^\mu P^\nu \Longrightarrow g \left(P^\mu P^\nu + \frac{\hbar}{2i} \partial^\mu \frac{\hbar}{2i} \partial^\nu \ln g \right) . \tag{30}$$

This prescription looks both similar to, and distinct from the conventional prescription $p^\mu = -i\hbar\partial^\mu$ invoked in particle quantum mechanics.

$$T_q^{\mu\nu} = g\, P^\mu P^\nu + \Pi\, \eta^{\mu\nu} - \frac{\hbar^2}{4} g \partial^\mu \partial^\nu \ln g\, , \tag{31}$$

Then, the equation of motion $0 = \partial_\mu T_q^{\mu\nu}$ gives

$$0 = \frac{1}{g} \partial^\nu \Pi + P^\mu \partial_\mu P^\nu - \frac{\hbar^2}{4} \partial^\nu \left(\frac{1}{2} \partial_\mu \zeta \partial^\mu \zeta + \Box \zeta \right) \tag{32}$$

with

$$\zeta = \ln(n/f)$$

We are going Quantum

$$P^{\mu} = \partial^{\mu} S\,, \tag{33}$$

$$\Pi = \Pi(g) \tag{34}$$

When both these conditions are satisfied, the equation of motion may be integrated

$$d = \bar{\Pi} + \frac{1}{2}(\partial_{\mu} S)(\partial^{\mu} S) - \frac{\hbar^2}{4}\left(\frac{1}{2}\partial_{\mu}\zeta\partial^{\mu}\zeta + \Box\zeta\right) \tag{35}$$

where d is an integration constant and $\bar{\Pi}(g)$ is determined by

$$\bar{\Pi} = \int d(\ln g)\frac{d\Pi}{dg}\,. \tag{36}$$

In terms of S and ζ, the continuity equation reads $0 = \partial_{\mu}\partial^{\mu} S + \partial_{\mu}\zeta\partial^{\mu} S$

$$\Psi = \sqrt{\frac{n}{f}}\; e^{iS/\hbar}, \tag{37}$$

we are able to derive

$$\left[\hbar^2\partial_{\mu}\partial^{\mu} - \bar{\Pi} + d\right]\Psi = 0, \tag{38}$$

$$\Psi^{*}\Psi = \frac{n}{f} \tag{39}$$

The Nonlinear Term

To get a feel for the nonlinear term, let us take a specific equation of state

$$\Pi = ag^{\Gamma} \longrightarrow \bar{\Pi} = \frac{a\Gamma}{\Gamma - 1} g^{\Gamma-1} \tag{40}$$

This choice converts the previous equation into

$$\left[\hbar^2 \partial_\mu \partial^\mu - \lambda (\Psi^* \Psi)^{\Gamma-1} + d\right] \Psi = 0, \tag{41}$$

where $\lambda = (a/w^{\Gamma-1})\Gamma/(\Gamma - 1)$ is a fluid specific constant.
For $\Gamma = 2$, corresponds to the highly investigated Ψ^4 theories that have been invoked as models for spontaneous symmetry breaking (when the vacuum does not have the symmetries of the Lagrangian).
The choice $d = -m^2$, leads to a nonlinear extension of the KG field, but if $d = \mu^2 > 0$, $\lambda > 0$, the field develops a finite vacuum expectation value.

Switching on the Electromagnetic Field

$$\partial_\mu T^{\mu\nu} = q\frac{n}{w}F^{\mu\nu}p_\mu = qgF^{\mu\nu}P_\mu \tag{42}$$

$$P^\mu + qA^\mu = \partial^\mu S, \tag{43}$$

One ends up deriving the equation of motion of the KG subjected to an electromagnetic field,

$$\left[-(i\hbar\partial_\mu - qA_\mu)(i\hbar\partial^\mu - qA^\mu) - \lambda\bar{\Pi}(\Psi^*\Psi) + d\right]\Psi = 0, \tag{44}$$

Non relativistic Limit- Landau Ginzburg Model

$$i\hbar\frac{\partial\Psi}{\partial t}=\left[-\frac{\hbar^2}{2\mu}\left(\partial^k-\frac{iq}{\hbar}A^k\right)^2+q\phi+\lambda|\Psi^*\Psi|^{(\Gamma-1)}\right]\Psi, \qquad (45)$$

The nonlinear Schrödinger equation that for $\Gamma=2$, is the Super-electron of the Landau-Ginzburg model. Notice that we could just as well derive (45) by directly working with the non relativistic limit of the energy momentum tensor.

Even more have been done...

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A hydrodynamical model for relativistic spin quantum plasmas

Felipe A. Asenjo,[1,a)] Víctor Muñoz,[1,2] J. Alejandro Valdivia,[1,2] and Swadesh M. Mahajan[3]
[1]*Departamento de Física, Facultad de Ciencias, Universidad de Chile, Casilla 653, 7800024 Santiago, Chile*
[2]*Centro para el Desarrollo de la Nanociencia y la Nanotecnología (CEDENNA), 9170124 Santiago, Chile*
[3]*Institute for Fusion Studies, The University of Texas at Austin, Austin, Texas 78712, USA*

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Relativistic quantum vorticity of the quadratic form of the Dirac equation

Felipe A Asenjo[1] and Swadesh M Mahajan[2]

[1] Facultad de Ingeniería y Ciencias, Universidad Adolfo Ibáñez, Santiago, Chile
[2] Institute for Fusion Studies, The University of Texas at Austin, TX 78712, USA

Now that's all.
Thanks!