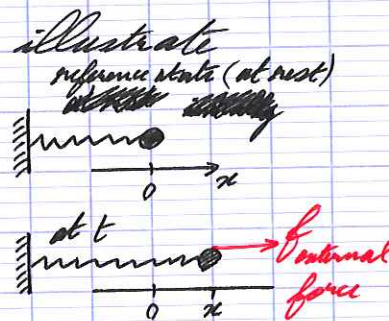


Generalized hydrodynamics of the cytoskeleton

We have seen that motors embedded in an elastic matrix can effect macroscopic active stresses. How will these stresses drive large-scale flows in the cytoskeleton viewed as an active material? Here we derive the flow equations for a weakly out-of-equilibrium ~~fluid~~ ^{simplified} fluid cytoskeleton from general thermodynamic and symmetry considerations. We will see that the active stress discussed in the last chapter emerges naturally as a driver of these flows.

I) General strategy

We tackle a very elementary system to illustrate the approach of this chapter. Think of a spring attached to a wall connected with a wall. Inertia is irrelevant (overdamped), and the only variable required to describe the thermodynamics of the system is x .



We want a generic, linear-response description of the dynamics of the ball when subjected to an external force, assuming it remains close to the reference state.

Step 1: Choosing the variables

The variable is x , and the free energy is a function only of this state variable: $F(x)$.

Step 2: Response under a reversible transformation

Assume an increment x by dx : $x \rightarrow x + dx$. The work received by the system from the external force is $\delta W = f \cdot dx$. For a reversible transformation, the change in free energy is thus

$$\delta F_{rev} = \delta W = f \cdot dx$$

Step 3: Irreversible transformation - dissipated free energy

In general $dF = \delta F_{reversible} + \delta F_{irreversible}$
with $\delta F_{irreversible} \leq 0$ according to the 2nd ppl of thermodynamics
 \uparrow equality at equilibrium

Besides, F is a state variable, so

$$dF = f^{eq} dx \quad \text{with } f^{eq} = \frac{\partial F}{\partial x}$$

Therefore:

$$\delta F_{im} = dF - \delta F_{rev} \\ = -(f - f^{eq}) \cdot dx$$

Assuming the change takes place over a time interval dt :

$$\frac{dF_i}{dt} = -f^{diss} \cdot \dot{x} \quad \text{with } f^{diss} = f - f^{eq}$$

"force" \rightarrow "flux"

Step 4: Flux-force relationships.

Generically the flux depends on the applied force; both vanish in the reference state. Thus to linear order

$$\dot{x} = A \cdot f^{diss}$$

\uparrow "phenomenological coefficient"

And $\frac{dF_i}{dt} = -A (f^{diss})^2 \leq 0$ imposes $A > 0$.

Step 5: Specifying the equilibrium forces.

On the simplest case we pick $F = \frac{kx^2}{2} \Rightarrow f' = kx$

This fully specifies the dynamics:

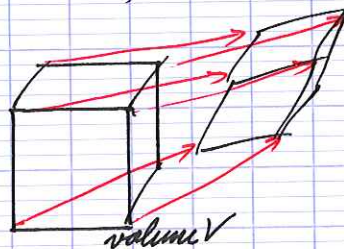
$$\dot{x} = A \cdot (f - kx)$$

II) Choosing the variable

We look at a fluid (simpler than the elastic response, valid on long time scales) with a local filament alignment characterized by a local polarization density $\vec{p}_i(\vec{r})$; this is similar to the magnetization field of a Heisenberg magnet. We are below the alignment transition, so the free energy favors a finite polarization $\vec{p}_i \cdot \vec{p}_i = p_0^2$; the orientation of the polarization is a soft (Nambu - Goldstone) mode.



We follow a Lagrangian fluid particle that undergoes a deformation $u_i(\vec{r})$; we keep track of its gradient $\partial_i u_j(\vec{r})$. Since the medium is a fluid the free energy depends only on the volume of the box, not its shape.



Within the box, ATP molecules are hydrolyzed at a certain rate. We denote by n the number of molecules hydrolyzed. For each hydrolysis event, a free energy $\Delta\mu$ is deposited in the box.

→ per unit volume of the system

We consider the system constituted by the cytoskeleton within the box not including ATP and its hydrolysis products. For this system, $F(V, p^2)$.

III) Response under a reversible transformation.

The outside world exerts 3 types of "forces" on our box:

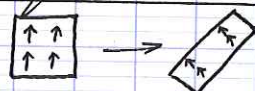
- a stress σ_{ij} exerted by the neighboring regions of the fluid
- a local field h_i , also from the neighborhood which tends to align p_i
- a released chemical free energy $\Delta\mu$ per hydrolysis event.

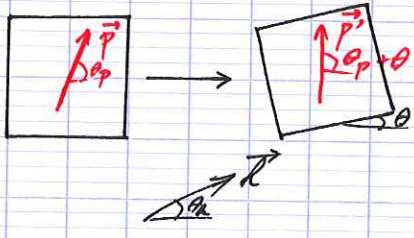
Thus the total work performed on the system can be decomposed as

$$\delta W = \delta W_1 + \delta W_2 + \delta W_3,$$

where 1, 2 and 3 are three independent transformations detailed below

1. Affine deformation connecting the molecular orientation

E.g.,  arrows rotate with the material; as a result, work results from the external stress, namely $\delta W_1 = \sigma_{ij} d_i u_j$, but also from the alignment field in case of a rotation:



$$\left. \begin{aligned} d(\partial_x u_y) &= \theta \\ d(\partial_y u_x) &= -\theta \end{aligned} \right\} \Rightarrow \theta = \frac{\partial_x u_y - \partial_y u_x}{2}$$

The work exerted by the field is $\delta W_2 = \vec{p} \cdot \vec{h} - \vec{p} \cdot \vec{h}$

$$\begin{aligned} \delta W_{2,h} &= V \cdot p \cdot h [\cos(\theta_p + \theta) - \cos(\theta_p)] \quad \text{in 2d (for simplicity)} \\ &= -V p h \sin \theta_p \cdot \theta \quad \text{for small } \theta \end{aligned}$$

and $p h \sin(\theta_p - \theta) = \vec{h} \wedge \vec{p} = h_x p_y - h_y p_x$

Thus

$$\begin{aligned} \delta W_{1,h} &= -V (h_x p_y - h_y p_x) \cdot \frac{\partial_x u_y - \partial_y u_x}{2} \\ &= -V \frac{h_x p_y - h_y p_x}{2} \cdot d(\partial_x u_y) - V \frac{h_y p_x - h_x p_y}{2} d(\partial_y u_x) \\ &= V \frac{p_i h_j - p_j h_i}{2} d(\partial_i u_j) \end{aligned}$$

$$\Rightarrow \delta W_1 = V [\sigma_{ij} + \frac{1}{2} (p_i h_j - p_j h_i)] d(\partial_i u_j)$$

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2. Rotation of the polarization without deformation

$$\delta W_2 = V \cdot h_i \cdot dp_i$$

3. ATP hydrolysis

$$\delta W_3 = V \cdot \Delta \mu \cdot dn$$

IV) Irreversible transformation

$$\delta F_{rev} = V \left[\sigma_{ij} + \frac{1}{2} (p_i h_j - p_j h_i) \right] d(\epsilon_i \epsilon_j) + V h_i dp_i + V \Delta \mu \cdot dn$$

$$dF = -\mathcal{S}^{eq} dV + V \cdot H_i^{eq} d(p_i \cdot p_i) \quad \text{w/ } \mathcal{S}^{eq} = -\left. \frac{\partial F}{\partial V} \right|_{p_i} \quad H_i^{eq} = \left. \frac{\partial F}{\partial p_i} \right|_V$$

$$= V \cdot \sigma_{ij}^{eq} \cdot d(\epsilon_i \epsilon_j) + V \cdot h_i^{eq} dp_i \quad \text{w/ } \sigma_{ij}^{eq} = -\mathcal{S}^{eq} \delta_{ij} \quad h_i^{eq} = 2 H_i^{eq} p_i$$

Thus

$$\delta F_{irr} = -V \left[\sigma_{ij} + \frac{1}{2} (p_i h_j - p_j h_i) - \sigma_{ij}^{eq} \right] d(\epsilon_i \epsilon_j) - V (h_i - h_i^{eq}) dp_i - V \Delta \mu \cdot dn$$

$$\Rightarrow \frac{1}{V} \frac{dF_i}{dt} = - \underbrace{\sigma_{ij}^d}_{\text{force}} \underbrace{\frac{d(\epsilon_i \epsilon_j)}{dt}}_{\text{flux}} - \underbrace{h_i^d}_{\text{force}} \underbrace{dp_i}_{\text{flux}} - \underbrace{\Delta \mu}_{\text{force}} \underbrace{dn}_{\text{flux}}$$

velocity

V) Flux-force relations

We have to convert order $\begin{pmatrix} \sigma_{ij}^d \\ p_i^d \\ \Delta \mu \end{pmatrix} = \underline{A} \begin{pmatrix} \dot{\epsilon}_i \dot{\epsilon}_j \\ \dot{p}_i \\ \dot{n} \end{pmatrix}$

However tradition demands that we rewrite this as

$$\begin{pmatrix} \sigma_{ij}^d \\ p_i^d \\ \Delta \mu \end{pmatrix} = \underline{A}' \begin{pmatrix} \dot{\epsilon}_i \dot{\epsilon}_j \\ \dot{p}_i \\ \dot{n} \end{pmatrix}$$

Note that $\frac{dF_i}{dt} = - \begin{pmatrix} \sigma_{ij}^d \\ p_i^d \\ \Delta \mu \end{pmatrix}^T \cdot \underline{A}' \cdot \begin{pmatrix} \dot{\epsilon}_i \dot{\epsilon}_j \\ \dot{p}_i \\ \dot{n} \end{pmatrix}$

implying that \underline{A}' is positive definite.

More explicitly, we have

$$\begin{aligned} \sigma_{ij}^d &= A_{ijkl}^{(1,1)} \partial_k v_l + A_{ijk}^{(1,2)} h_k^d + A_{ij}^{(1,3)} \Delta \mu \\ p_i &= A_{ijk}^{(2,1)} \partial_j v_k + A_{ij}^{(2,2)} h_j^d + A_i^{(2,3)} \Delta \mu \\ n &= A_{ij}^{(3,1)} \partial_i v_j + A_i^{(3,2)} h_i^d + A_i^{(3,3)} \Delta \mu \end{aligned}$$

Each of the $A^{(d,P)}$ may depend on p_i . While this linear relation a priori has $(d^2 + d + 1)^2$ independent coefficients, its complexity is considerably reduced by imposing

• Spatial symmetries

- invariance under simultaneous rotations of space and p_i
- rotations around p_i (in $d \geq 3$)

• Time-reversal symmetries

The Onsager reciprocity relations impose that diagonal couplings between fluxes and forces of identical symmetries be identical symmetric; and that couplings between terms of opposite symmetries be antisymmetric (Landau, Stat Phys I § 120).

Since $\partial_k v_l$ odd
 h_k^d even
 $\Delta \mu$ even

we have

$$\begin{aligned} A_{ijkl}^{(1,1)} &= A_{llij}^{(1,1)} \\ A_{ij}^{(2,2)} &= A_{ji}^{(2,2)} \\ A_{ijk}^{(1,2)} &= -A_{kij}^{(2,1)} \\ A_{ij}^{(2,3)} &= -A_{ji}^{(3,2)} \\ A_i^{(3,3)} &= A_i^{(3,3)} \end{aligned}$$

Inequalities also hold on the coeffs due to \underline{A} 's positive definiteness.

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At the end of the day, and with some further arbitrary simplifications by the authors of (Kruuse et al. EPJE 2005) aimed at making the problem tractable, we retain the first two equations: active stress = Σ_{ij}

$$\sigma_{ij}^d = 2\eta \frac{\partial_i v_j + \partial_j v_i}{2} + \gamma \frac{p_i h_i^d + p_j h_j^d}{2} + \gamma \Delta \mu \delta_{ij} + \gamma \Delta \mu p_i p_j$$

$$\frac{D p_i}{D t} = -\underbrace{\gamma}_{\text{Onsager}} \mu_{ij} p_j + \frac{1}{\gamma} h_i^d + \lambda p_i \Delta \mu$$

forces are $\Delta \mu$ just like in ch. IV

↳ Lagrangian derivative

And we forget about the last equation, because we consider a situation where the system is in contact with a reservoir of ATP imposing a fixed $\Delta \mu$.

These are the equations that we will start from next time.

VI) Specifying the equilibrium forces

1) Within the Lagrangian particle

$F = V \cdot [f_p(\vec{p}^2), f_v(v)]$: assumed very peaked $\Rightarrow H^1$ is a Lagrange multiplier imposing $|\vec{p}| = p_0$

Similarly, we assume incompressibility $\Rightarrow \gamma^1$ Lagrange multiplier imposing $\partial_i v_i = 0$.

2) Nonlocal couplings

More rigorously, we can resort to a gradient expansion to determine how the polarization aligns in a non-constant polarization field

$$\mathcal{F} = \int dV f_p(\vec{p}^2) + f_{ij}^{(1)}(\xi p_i) \cdot \partial_i p_j + f_{ijkl}^{(1)}(\xi p_i) \partial_i p_j \partial_k p_l$$

Imposing invariance under $\vec{x} \rightarrow -\vec{x}$ kills all 1st order terms in gradients. Further requiring $\vec{p} \rightarrow -\vec{p}$ as

well as rotations of \vec{p} and \vec{n} simultaneously yields the Frank energy of a liquid crystal:

$$\mathcal{F} = \frac{1}{2} \int dV \{ K_1 (\nabla \cdot \vec{p})^2 + K_2 [\vec{p} \cdot (\nabla \times \vec{p})]^2 + K_3 [\vec{p} \wedge (\nabla \times \vec{p})]^2 \}$$



splay



twist
(only in $d \geq 3$)



bend

On 2d

$$\mathcal{F}^{(2)} = \int dx dy \left[\frac{K_1}{2} (\nabla \cdot \vec{p})^2 + \frac{K_2}{2} (\nabla \times \vec{p})^2 \right]$$

from which we deduce

~~$$h_i = - \frac{\delta \mathcal{F}^{(2)}}{\delta p_i(\vec{r})}$$~~

~~$$= - \frac{\delta}{\delta p_i(\vec{r})} \int d\vec{r} \left\{ \frac{K_1}{2} (\nabla_j p_j)^2 + \frac{K_2}{2} (\epsilon_{jlk} \nabla_j p_l)^2 \right\}$$~~

~~$$= - \int d\vec{r} \left\{ K_1 \nabla_j p_j \nabla_i p_i + K_2 \epsilon_{jlk} \nabla_j p_l \epsilon_{ikm} \nabla_m p_n \right\} \delta(p_i - \hat{p}_i)$$~~

~~$$= K_1 \nabla_j p_j \nabla_i p_i + K_2 \epsilon_{ijl} \nabla_j p_l \epsilon_{lmn} \nabla_m p_n$$~~

~~$$= K_1 \nabla_j p_j \nabla_i p_i$$~~

And for small changes of $|\vec{p}|$ from its optimal value we have

$$H^{eq} \propto p - p_0$$

We thus write

$$\vec{h}^{eq} = k(p - p_0) \hat{p} = \frac{\vec{p}}{p}$$