L-SPACES, TAUT FOLIATIONS, AND ORDERABILITY

Abstract. Exercises for a three-hour lecture series at the Advanced School on Geometric Group Theory and Low-Dimensional Topology at ICTP, Trieste, Italy, May 25-27, 2016.

1. Exercises for Day One.

Exercises marked by a spade (♠) are more challenging.

(1) Prove that if Y is a closed, connected, simply-connected 3-manifold, then Y is an irreducible homology sphere. Your proof should invoke the names Hurewicz, Poincaré, and Papakyriakopolous.
(2) Prove that if p, q, and r are positive integers such that $qr \equiv 1 \pmod{p}$, then $L(p,q) \approx L(p,r)$.
(3) Prove that the ordinary first homology group of Y_n is cyclic of order $n-4$.
(4) Prove that $\pi_1(Y_n) \approx \langle x, y, z \,|\, x^2 = y^3 = z^{n+2}, x = zy\rangle$.
(5) Prove that $\pi_1(Y_3)$ is a non-trivial group by showing that it acts by symmetries of a regular icosahedron; prove that $\pi_1(Y_5)$ is a non-trivial group by showing that it acts by symmetries of a tesselation of $\mathbb{H}^2$ by $(\pi/2, \pi/3, \pi/7)$-triangles.
(6) (♠) Prove that, in fact, $\pi_1(Y_3)$ has order 120.
(7) Suppose that $\varphi : \Sigma_g \to \Sigma_g$ is an orientation-preserving diffeomorphism of a closed surface Σ_g. Show that the mapping torus $M(\varphi)$ admits a Heegaard decomposition of genus $2g$.

    A look ahead: given a Heegaard diagram $H = (\Sigma_g, \{\alpha_1, \dots, \alpha_g\}, \{\beta_1, \dots, \beta_g\})$, we will construct a chain complex $\widehat{CF}(H)$ that is freely generated by g-tuples of points on Σ_g, such that in each g-tuple, there is one point on each α curve and one point on each β curve.

(8) Determine the number of generators for the Heegaard diagram that presents Y_n.
(9) (♠) Prove that if H is a Heegaard diagram and $\widehat{CF}(H)$ has rank one, then H presents S^3.