



*Equilibrium States are determined by their
unstable conditionals.*

Pablo D. Carrasco

(joint w/ Federico Rodriguez-Hertz)

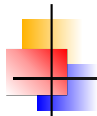
ICMC-USP visiting ICTP

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ICTP



General Thermodynamic Formalism.



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$$P_{\text{top}}(\varphi) = \sup_{\nu \in \mathcal{P}_f(M)} \{h_\nu(f) + \int \varphi d\nu\}.$$



Equilibrium States.

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- ▶ Uniqueness of equilibrium states. General (but more restrictive) methods (Bowen, or more recently, Climenhaga-Thompson).
- ▶ Properties/Description of equilibrium states. (Mixing. Bernoulli. Decay of correlations.)



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- ▶ $\varphi(x) = -\log |\det df|_{E_x^u}| \Rightarrow \mu$ is the SRB measure (need to assume f is $C^{1+\theta}$)



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We discuss now one important example where the available methods fail.



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such that (for some metric) $df|E^u$ expansion, $df|E^s$ contraction and E^c is tangent to the orbits of the action.



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For a center isometry all bundles

$E^s, E^u, E^c, E^{cs} = E^c \oplus E^s, E^{cu} = E^c \oplus E^u$ are integrable to f -invariant foliations W^* .



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Theorem A [P.C., F. Rodriguez-Hertz]

There exist $\mu_\varphi \in \mathcal{P}_f(M)$ and families of measures

$\mu^u = \{\mu_x^u\}_{x \in M}, \mu^s = \{\mu_x^s\}_{x \in M}, \mu^{cu} = \{\mu_x^{cu}\}_{x \in M}, \mu^{cs} = \{\mu_x^{cs}\}_{x \in M}$
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satisfying the following.

1. The probability μ_φ is an equilibrium state for the potential φ .
2. For every $x \in M$ the measure $\mu^\sigma, \sigma \in \{u, s, cu, cs\}$ is a Radon measure on $W^\sigma(x)$ which is positive on relatively open sets, and $y \in W^\sigma(x)$ implies $\mu_x^\sigma = \mu_y^\sigma$.



Cont.

3. If ξ is a measurable partition that refines the partition by unstable (stable) leaves then the conditionals $(\mu_\varphi)_x^\xi$ of μ_φ are equivalent to μ_x^u (resp. μ_x^s) for $\mu_\varphi - a.e.(x)$.

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4. For every $\epsilon > 0$ sufficiently small, for every $x \in M$ the measure $\mu_\varphi|D(x; \epsilon)$ has product structure with respect to the pair μ_x^u, μ_x^{cs} , i.e. its equivalent to $\mu_x^u \times \mu_x^{cs}$.

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- Given $\epsilon > 0$ there exist $a(\epsilon), b(\epsilon) > 0$ such that if

$$U(x, \epsilon, n) = \{y \in W^u(x, \epsilon) : d(f^j x, f^j y) < \epsilon, j = 0, \dots, n-1\}$$

then

$$a(\epsilon) \leq \frac{\mu_x^u(U(x, \epsilon, n))}{e^{\mathcal{S}_n \varphi(x) - nP_{\text{top}}(\varphi)}} \leq b(\epsilon).$$



Bernoulli property and uniqueness.

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Work in progress: Uniqueness also holds in the homogeneous examples (Weyl Chambers' flow).



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Implicit: Unstable manifolds coincide with Pesin's unstable manifolds.



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The families μ^u, μ^s provide the reference measures to which one can compare.



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denote $m_x =$ unstable conditional of m on $\xi(x)$.



'Change of variables' for μ^u

Theorem

For every $x \in M$

1. $\mu_{fx}^\sigma = e^{P_{\text{top}}(\varphi) - \varphi} f_* \mu_x^\sigma \quad \sigma \in \{u, cu\}.$
2. $\mu_{fx}^\sigma = e^{\varphi - P_{\text{top}}(\varphi)} f_* \mu_x^\sigma \quad \sigma \in \{s, cs\}.$



Cont. proof

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is constant on the atoms of ξ . From here one deduces that

$$y \in \xi(x) \Rightarrow \frac{\rho(y)}{\rho(x)} = \prod_{k=1}^{\infty} \frac{e^{\varphi \circ f^{-k}(y)}}{e^{\varphi \circ f^{-k}(x)}} = \Delta_x(y).$$



Cont. proof

$\Rightarrow \rho$ (provided is defined) should have the form

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It is enough to show $m = \nu$ on every $\mathcal{B}_{f^{-n}\xi}$, $n \geq 0$. An induction argument shows that after proving $m = \nu$ on $\mathcal{B}_{f^{-1}\xi}$ we are done.

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and afterwards, $m_x = \nu_x$ on $\xi(x)$ for ALL atoms, as we wanted to show.



Grazie!!!