Equilibrium States are determined by their unstable conditionals.

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(joint w/ Federico Rodriguez-Hertz)

ICMC-USP visiting ICTP

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ICTP

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- Properties/Description of equilibrium states. (Mixing. Bernoulli. Decay of correlations.)

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- $\varphi(x) = -\log|det\ df|E^u_x| \Rightarrow \mu$ is the SRB measure (need to assume f is $C^{1+\theta}$)

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We discuss now one important example where the available methods fail.

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such that (for some metric) $df|E^u$ expansion, $df|E^s$ contraction and E^c is tangent to the orbits of the action.

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For a center isometry all bundles $E^s, E^u, E^c, E^{cs} = E^c \oplus E^s, E^{cu} = E^c \oplus E^u$ are integrable to f-invariant foliations W^*.

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Theorem A [P.C., F. Rodriguez-Hertz]

There exist $\mu_\varphi \in \mathcal{P}_f(M)$ and families of measures
$\mu^u = \{\mu^u_x\}_{x\in M}, \mu^s = \{\mu^s_x\}_{x\in M}, \mu^{cu} = \{\mu^{cu}_x\}_{x\in M}, \mu^{cs} = \{\mu^{cs}_x\}_{x\in M}$
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satisfying the following.

1. *The probability μ_φ is an equilibrium state for the potential φ.*
2. *For every $x \in M$ the measure $\mu^\sigma, \sigma \in \{u, s, cu, cs\}$ is a Radon measure on $W^\sigma(x)$ which is positive on relatively open sets, and $y \in W^\sigma(x)$ implies $\mu^\sigma_x = \mu^\sigma_y$.*

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3. If ξ is a measurable partition that refines the partition by unstable (stable) leaves then the conditionals $(\mu_\varphi)_x^\xi$ of μ_φ are equivalent to μ_x^u (resp. μ_x^s) for $\mu_\varphi - a.e.(x)$.

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4. For every $\epsilon > 0$ sufficiently small, for every $x \in M$ the measure $\mu_\varphi | D(x;\epsilon)$ has product structure with respect to the pair μ^u_x, μ^{cs}_x, i.e. its equivalent to $\mu^u_x \times \mu^{cs}_x$.

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5. Given $\epsilon > 0$ there exist $a(\epsilon), b(\epsilon) > 0$ such that if

$$U(x,\epsilon,n) = \{y \in W^u(x,\epsilon) : d(f^j x, f^j y) < \epsilon, j = 0,\dots,n-1\}$$

then

$$a(\epsilon) \leq \frac{\mu_x^u(U(x,\epsilon,n)))}{e^{S_n\varphi(x) - nP_{top}(\varphi)}} \leq b(\epsilon).$$

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Work in progress: Uniqueness also holds in the homogeneous examples (Weyl Chambers' flow).

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Implicit: Unstable manifolds coincide with Pesin's unstable manifolds.

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The families μ^u, μ^s provide the reference measures to which one can compare.

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'Change of variables' for μ^u

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Theorem

For every $x \in M$

1. $\mu^\sigma_{fx} = e^{P_{top}(\varphi)-\varphi} f_* \mu^\sigma_x \quad \sigma \in \{u, cu\}.$
2. $\mu^\sigma_{fx} = e^{\varphi - P_{top}(\varphi)} f_* \mu^\sigma_x \quad \sigma \in \{s, cs\}.$

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is constant on the atoms of ξ. From here one deduces that

$$y \in \xi(x) \Rightarrow \frac{\rho(y)}{\rho(x)} = \prod_{k=1}^{\infty} \frac{e^{\varphi \circ f^{-k}(y)}}{e^{\varphi \circ f^{-k}(x)}} = \Delta_x(y).$$

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It is enough to show $m = \nu$ on every $\mathcal{B}_{f^{-n}\xi}, n \geq 0$. An induction argument shows that after proving $m = \nu$ on $\mathcal{B}_{f^{-1}\xi}$ we are done.

Cont. proof

$$q(x) = \nu_x(f^{-1}\xi(x)) \Rightarrow$$

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and afterwards, $m_x = \nu_x$ on $\xi(x)$ for ALL atoms, as we wanted to show.

Grazie!!!