Brownian motion, evolving geometries and entropy formulas

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Manifolds evolving under a geometric flow are currently a very active field of research. A prominent example is the so-called Ricci flow, which is a kind of nonlinear heat equation for Riemannian metrics on a differentiable manifold. It smoothens out the geometry of the manifold towards the best possible one decided by the topology of the manifold. We develop basic notions of stochastic differential geometry in the framework of evolving manifolds. This includes in particular the notion of a canonical Brownian motion in this setting.

A basic problem in geometry is to characterize Ricci curvature in terms of functional inequalities. It is well-known that this can be done in various ways, for instance, via gradient estimates for the heat equation, via log-Sobolev or Poincaré inequalities, via Harnack inequalities, via transportation cost inequalities, etc. Manifolds evolving under Ricci flow for instance share many properties of Ricci flat manifolds. Hence it is not surprising that also super-Ricci flow (and even Ricci flow) can be characterized by functional inequalities. We shall discuss these topics in the first part of the lectures.

Of particular interest in the theory are entropy formulas for positive solutions of the heat equation under a geometric flow. We explain how tools from Stochastic Analysis can be used to study the evolution of entropies, analogous to Perelman's entropy functionals. Motivated by the work of Perelman, several entropy formulas are introduced; their analytic and geometric significance is discussed in various cases. Geometric conditions are given which guarantee convexity of the entropy functionals.