Slow, Phenomenological Dynamics

From Navier Stokes to Chiral Granular Gases: A practical guide and New Applications

Slow, Phenomenological Dynamics

- Phenomenological – based on symmetries conservation laws, and thermodynamics, not microscopic dynamics
- Slow – only modes with frequency $\omega\tau \ll 1$; τ=longest microscopic collision time
  - Hydrodynamic $\omega \to 0 \text{ as } q \to 0$
    - Conservations laws $\partial_t \varphi + \partial_i j_i^{\varphi} = 0$
    - Broken Symmetry: no restoring force at $q=0$
  - Other
    - Near critical point $\omega \sim \tau_c^{-1} \sim (T - T_c)^{z\nu} \ll \tau^{-1}$
    - Small external friction $\omega \sim \tau_\Gamma^{-1} \sim \Gamma \ll \tau^{-1}$

Examples

- Navier Stokes
- Superfluid Helium (Landau ~ 1937)
- Leslie-Ericksen nematodynamics (1960-68)
- Magnetic Systems (Halperin-Hohenberg ~1969)
- Dynamic Critical Phenomena (70's)

General Approach (many contributors)

- Identify slow variables: φ
- Determine static thermodynamics: $F(\varphi)$
- Develop dynamics: Invariances or Poisson-brackets plus dissipation
- Mode Counting (Martin,Pershan, Parodi *PRA* **6**, 2401 (1972)):
  - One hydrodynamic mode for each conserved or broken-symmetry variable
  - Extra Modes for slow non-hydrodynamic
  - Friction and constraints may reduce number of hydrodynamics variables

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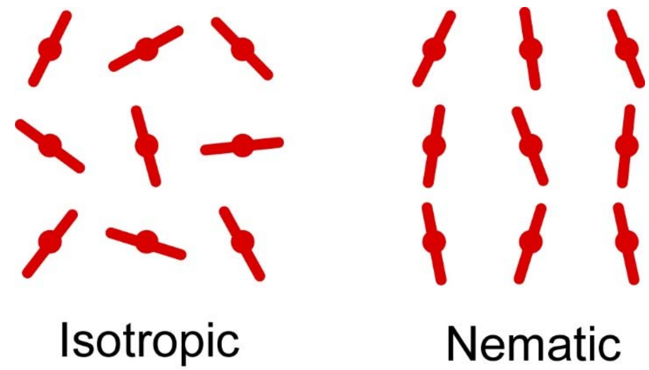
Outline

- Pure hydrodynamics
  - Rigid Rotors
  - Isotropic fluids
- Poisson bracket approach
  - Rigid Rotors again
- Crystals, Tethered Solids in isolation and in a solvent
- Nematic Liquid crystals
  - Hydrodynamics
  - Rigid-rod nematodynamics: Leslie-Ericksen with some twists
- A Chiral Granular Gas
- Force Dipoles
- Nematic Elastomers

Rigid Rotors: Disordered I



$$\nu_l = (\cos \vartheta_l, \sin\vartheta_l);$$
$$\langle Q_{ij} \rangle = \sum_l \left(\nu_i \nu_j - \tfrac{1}{3}\delta_{ij}\right)\delta\left(\mathbf{x} - \mathbf{R}_l\right)$$
$$= \left(N/V\right) S \left(n_i n_j - \tfrac{1}{3}\delta_{ij}\right)$$

$$U = -J \sum_{\langle l,l \rangle} \cos\left[2\left(\vartheta_l - \vartheta_{l'}\right)\right]$$

$$p_l = I\dot{\vartheta}_l \quad \text{angular momentum}$$
$$\ell = \tilde{I}\Omega = \quad \text{angular momentum density}$$
$$\tilde{I} = (N/V)I$$

$$s - T \text{ and } \ell - \Omega : \text{Conjugate variables}$$

Conserved Variables:

$\varepsilon =$ energy density; ℓ

$$\frac{\partial \varepsilon}{\partial t} + \vec{\nabla}\cdot\vec{j}^{\varepsilon} = 0; \qquad \frac{\partial \ell}{\partial t} + \vec{\nabla}\cdot\vec{\tau} = 0$$

$$\varepsilon = \varepsilon_0(s) + \tfrac{1}{2\tilde{I}}\ell^2$$
$$d\varepsilon = T ds + \Omega d\ell$$
$$d\varepsilon = C_\ell dT$$
$$\Omega = \left.\frac{\partial \varepsilon}{\partial \ell}\right|_s = T\left.\frac{\partial s}{\partial \ell}\right|_\varepsilon = \frac{\ell}{\tilde{I}}$$

Rigid Rotors

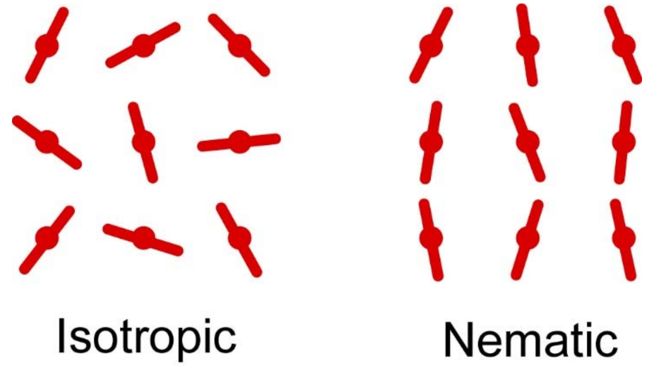
$$Tds = d\varepsilon - \Omega d\ell$$
$$T\frac{\partial s}{\partial t} = \frac{\partial \varepsilon}{\partial t} - \Omega\frac{\partial l}{\partial t} = -\vec{\nabla}\cdot\vec{j}^{\,\varepsilon} + \Omega\vec{\nabla}\cdot\vec{\tau}$$
$$= -\vec{\nabla}\cdot\left(\vec{j}^{\,\varepsilon} - \Omega\vec{\tau}\right) - \vec{\tau}\cdot\vec{\nabla}\Omega$$
$$T\left(\frac{\partial s}{\partial t} + \vec{\nabla}\cdot\left(\vec{Q} / T\right)\right) = -\vec{Q}\cdot\frac{\vec{\nabla}T}{T} - \vec{\tau}\cdot\vec{\nabla}\Omega$$

$$\vec{Q} = -\kappa\vec{\nabla}T; \quad \vec{\tau} = -\Gamma\vec{\nabla}\Omega$$
$$\frac{\partial \varepsilon}{\partial t} = -\vec{\nabla}\cdot\left(\vec{Q} + \Omega\vec{\tau}\right)$$
$$= \vec{\nabla}\left(\kappa\vec{\nabla}T + \Omega\Gamma\vec{\nabla}\Omega\right)$$
$$\approx \kappa\nabla^2 T \approx (\kappa / C_\ell)\nabla^2\varepsilon$$
$$\frac{\partial \ell}{\partial t} = \vec{\nabla}\cdot\left(\Gamma\vec{\nabla}\Omega\right) = \left(\Gamma / \tilde{I}\right)\nabla^2\ell$$

$$T\frac{\partial S}{\partial t} = T\int d^d x \frac{\partial s}{\partial t} \geq 0$$
$$= -\int d^d x\left(\vec{Q}\cdot\frac{\vec{\nabla}T}{T} + \vec{\tau}\vec{\nabla}\Omega\right)$$

Two conserved variables,
Two diffusive modes:
$$\omega_\varepsilon = -i\left(\kappa / C_\ell\right)q^2$$
$$\omega_\ell = -i\left(\Gamma / \tilde{I}\right)q^2$$

Rigid Rotors – Ordered I



$$\langle Q_{ij}\rangle \neq 0; \qquad \vec{n} = \left(\cos\theta, \sin\theta\right)$$

Free energy invariant under $\theta \to \theta + \text{const.}$

Depends only on $\vec{v}_\theta = \vec{\nabla}\theta$

$$T\,ds = d\varepsilon - \Omega\, dl - \vec{h}_\theta \cdot d\vec{v}_\theta$$

$$\vec{h}_\theta = \rho_s \vec{v}_\theta$$

Steady state: $\vec{n}$ and ℓ rotate together:

$$\frac{d\theta}{dt} = \Omega - X'; \quad X' = \text{dissipative part}$$

$\vec{h}_\theta - \vec{v}_\theta$:
Conjugate variables

$$T\frac{\partial s}{\partial t} = -\vec{\nabla}\cdot\vec{j}^{\,\varepsilon} + \Omega\vec{\nabla}\cdot\vec{\tau} - \vec{h}_\theta\cdot\vec{\nabla}\left(\Omega - X'\right)$$

$$T\left(\frac{\partial s}{\partial t} + \vec{\nabla}\cdot\left(\vec{Q}/T\right)\right) = -\vec{Q}\cdot\frac{\vec{\nabla} T}{T} - \left(\vec{\tau} + \vec{h}_\theta\right)\cdot\vec{\nabla}\Omega - X'\vec{\nabla}\cdot\vec{h}_\theta$$

$$\vec{Q} = \vec{j}^{\,\varepsilon} - \Omega\vec{\tau} - X'\vec{h}_\theta$$

Rigid Rotors Ordered II

No Dissipation:

$$\vec{\tau} = -\vec{h}_\theta; \quad \vec{Q} = 0; \; X' = 0:$$

$$\frac{\partial l}{\partial t} = \vec{\nabla}\cdot\vec{h}_\theta = \rho_s \nabla^2\theta; \quad \frac{\partial \theta}{\partial t} = \Omega = \tilde{I}^{-1}\ell$$

$$\frac{\partial^2\theta}{\partial t^2} = \rho_s \tilde{I}^{-1}\nabla^2\theta \Rightarrow \omega = \pm\left(\rho_s / \tilde{I}\right)^{1/2} q$$

Note: These equations would appear to apply to polar rotators as well, but there is a complication: these systems always have a $\vec{\nabla}\cdot\vec{P}$ term that prefers a spatially inhomogeneous ground state.

Add Dissipation:

$$X' = -\gamma\vec{\nabla}\cdot\vec{h}_\theta; \quad \kappa \to \kappa(\theta); \quad \Gamma \to \Gamma(\theta)$$

$$\omega_\varepsilon = -i\kappa(\theta)q^2; \quad \omega_\theta = \pm cq - \tfrac{1}{2} iDq^2$$

$$D = D_\theta + D_\ell; \qquad D_\theta = \gamma\rho_s; \quad D_\ell = \Gamma(\theta) / \tilde{I}$$

2 conservation laws, 1 broken symmetry:

Three modes: 2 "director wave", 1 heat diffusion

Hydrodynamics: isotropic fluids I

Five convservation Laws: Mass (ρ), momentum ($\vec{g}$), energy (ε):

$$\frac{\partial \varepsilon}{\partial t} + \vec{\nabla}\cdot\vec{g}; \quad \frac{\partial \varepsilon}{\partial t} + \vec{\nabla}\cdot\vec{j}^{\varepsilon} = 0; \quad \frac{\partial g_i}{\partial t} + \nabla_j \pi_{ij} = 0$$

$\vec{v} - \vec{g}$ and $\rho - \alpha$: Conjugate variables

Thermodynamics:

$$T ds = d\varepsilon - \alpha d\rho - \vec{v}\cdot d\vec{g}; \quad \alpha = \text{chem. pot. / mass}; \ \vec{v} = \text{velocity}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{g}; \quad \frac{\partial \varepsilon}{\partial t} + \vec{\nabla}\cdot\vec{j}^{\varepsilon} = 0; \quad \frac{\partial g_i}{\partial t} + \nabla_j \pi_{ij} = 0$$

$$T\frac{\partial s}{\partial t} = -\vec{\nabla}\cdot\vec{j}^{\varepsilon} + \alpha\vec{\nabla}\cdot\vec{g} + v_j \nabla_i \pi_{ij}$$

$\vec{g}$: Momentum density
Mass current

$\vec{j}^{\epsilon}$: energy current

π_{ij} : Momentum current

$$T\vec{v}\cdot\vec{\nabla}s = \vec{v}\cdot\vec{\nabla}\varepsilon - \alpha\vec{v}\cdot\vec{\nabla}\rho - v_i v_j \nabla_i g_j$$

$$T\left[\frac{\partial s}{\partial t} + \vec{\nabla}\cdot\left(\vec{v}s + \frac{\vec{Q}}{T}\right)\right] = -\vec{Q}\cdot\frac{\vec{\nabla}T}{T} - \left(\vec{g} - \rho\vec{v}\right)\cdot\vec{\nabla}\alpha - \left(\pi_{ji} - p\delta_{ij} - v_i g_j\right)\nabla_i v_j$$

$$p = -\left(\varepsilon - \alpha\rho - Ts - \vec{v}\cdot\vec{g}\right); \quad Q_i = j_i^{\varepsilon} - \ldots$$

Hydrodynamics: isotropic fluids II

Dissipationaless: $\partial s / \partial t = 0$

$\vec{g} = \rho \vec{v}; \;\; \pi_{ij} = p\delta_{ij} + v_j g_i = -\sigma_{ij} + \rho v_i v_j$

$\vec{j}^{\,\varepsilon} = (\varepsilon + p)\vec{v}$

π_{ij} : momentum current

σ_{ij} : stress tensor

Dissipation:

$\vec{Q} = -\kappa \vec{\nabla} T; \quad \sigma'_{ij} = \eta\left(\nabla_i v_j + \nabla_j v_i - \frac{2}{3}\delta_{ij}\vec{\nabla}\cdot\vec{v}\right) + \zeta\delta_{ij}\vec{\nabla}\cdot\vec{v}$

η : transverse viscosity; $\quad \zeta$: longitudinal viscosity

Five conservation laws: five modes:

2 Long. sound : $\omega = \pm cq + iDq^2$

2 Trans. momentum diffusion : $\omega = -i(\eta / \rho)q^2$

1 Heat diffusion (mixes with Long. sound) : $\omega = -i(\kappa / C)q^2$

Poisson Bracket formalism

Harmonic Oscillator: seeds of complete formalism

Poisson bracket

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2; \qquad \{p, x\} = 1$$

$$\dot{p} = -\{p, x\}\frac{\partial H}{\partial x} - \Gamma \dot{x} = -kx - \Gamma v$$

$$\dot{x} = -\{x, p\}\frac{\partial H}{\partial p} = \frac{p}{m}$$

$$p = mv$$

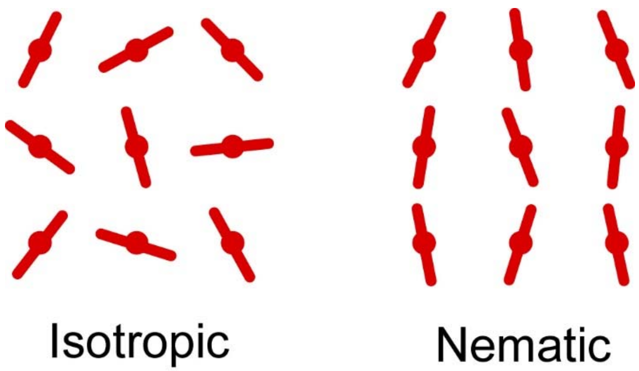
friction

Poisson brackets: mechanical coupling between variables – time-reversal invariant.

Dissipative couplings: not time-reversal invariant

Dissipative: time derivative of field (p) to its conjugate field (v)

Disordered Rotors on Rigid Lattice



Conserved densities:

Spin angular momentum: $\boldsymbol{\ell} = \boldsymbol{I\Omega}$

Energy : ε

3 Diffusive "spin" modes;

1 Diffusive temperature mode;

All modes are dissipative

Ignore energy: isothermal

$\boldsymbol{\ell}$ now a vector

Static Thermo

$$F = \int d^d x \frac{\ell^2}{2\tilde{I}}$$

Diffusive "spin" modes:

$$\omega = -i(\nu / \tilde{I})q^2$$

$$\partial_t \ell_i + \partial_j \pi_{ij} = 0$$

No Poisson bracket: purely dissipative

$$\pi_{ij} = -\nu \partial_j \Omega_i = -\nu \partial_j \frac{\delta F}{\delta \ell_i}$$

$$I \partial_t \Omega_i = \nu \nabla^2 \Omega_i = \nu \nabla^2 \frac{\delta F}{\delta \ell_i}$$

Ordered Rotors on Rigid Lattice I

Conserved densities:

Spin angular momentum: $\ell_i = I_{ij}\Omega_j$

Broken Symmetry:

Nematic director: $\mathbf{n}$

$$F = \int d^d x \left[\frac{\ell_{||}^2}{I_{||}} + \frac{\ell_{\perp}^2}{I_{\perp}} + \frac{1}{2} K (\partial_i n_j)^2 \right]$$

Static thermo

ignore ℓ-ℓ :

nonlinear corrections

$$\partial_t n_i = -\int d^d x' \{ n_i(\mathbf{x}), \ell_j(\mathbf{x}') \} \frac{\delta F}{\delta \ell_j(\mathbf{x}')} - \frac{1}{\gamma} \frac{\delta F}{\delta n_i(\mathbf{x})}$$

$$\partial_t \ell_i = -\int d^d x' \{ \ell_i(\mathbf{x}), n_j(\mathbf{x}') \} \frac{\delta F}{\delta n_j(\mathbf{x}')} + \nu_{ij} \nabla^2 \frac{\delta F}{\delta \ell_i(\mathbf{x})}$$

Ordered Rotors on Rigid Lattice II

$$\partial_t \mathbf{n} = \boldsymbol{\Omega} \times \mathbf{n} + \frac{1}{\gamma}\mathbf{h}$$

$$I_\perp \partial_t \boldsymbol{\Omega}_\perp = \mathbf{n} \times \mathbf{h} + \nu_\perp \nabla^2 \boldsymbol{\Omega}_\perp$$

$$I_{||} \partial_t \Omega_{||} = \nu_{||} \nabla^2 \Omega_{||}$$

$$h_i = -\frac{\delta F}{\delta n_i} = K \nabla^2 n_i$$

5 hydrodynamic modes: 2 pairs of propagating transverse spin modes (4) (like anti-ferromagnet) plus one longitudinal spin diffusion

$$\omega_{x,y} = \pm \sqrt{\frac{K}{I_\perp}} q - \frac{1}{2} i \left(\frac{K}{\gamma} + \frac{\nu_\perp}{I_\perp} \right) q^2$$

$$\omega_z = -i \frac{\nu_{||}}{I_{||}} q^2$$

Note linear dispersion of $\omega_{x,y}$: like anti-ferromagnet

Rigid rotors with "substrate" friction

$$\partial_t \mathbf{n} = \boldsymbol{\Omega} \times \mathbf{n} + \frac{1}{\gamma}\mathbf{h}$$

$$I_\perp \partial_t \boldsymbol{\Omega}_\perp = \mathbf{n} \times \mathbf{h} + \nu_\perp \nabla^2 \boldsymbol{\Omega}_\perp - \Gamma_\perp \boldsymbol{\Omega}_\perp$$

$$I_\parallel \partial_t \Omega_\parallel = \nu_\parallel \nabla^2 \Omega_\parallel - \Gamma_\parallel \Omega_\parallel$$

Angular momentum conservation broken but broken symmetry remains

Effective broken-symm hydrodynamic theory

$$\partial_t \mathbf{n} = \left(\frac{1}{\Gamma_\perp} + \frac{1}{\gamma}\right)\mathbf{h}$$

$$= \frac{1}{\gamma_1}\mathbf{h}$$

Parallel paths

$\text{Fast, non-hydro modes:}$

$$\omega_\perp = -i\frac{\Gamma_\perp}{I_\perp} = -i\tau_\perp^{-1} \quad (2 \text{ modes})$$

(Maybe more complex)

$$\omega_\parallel = -i\frac{\Gamma_\parallel}{I_\parallel} = -i\tau_\parallel^{-1} \quad (1 \text{ mode})$$

$\text{Slow hydro modes:} \quad \omega\tau_\perp, \omega\tau_\parallel \ll 1$

$$\omega = -i\left(\Gamma_\perp^{-1} + \gamma^{-1}\right)Kq^2 \quad (2 \text{ modes})$$

Fluid Flow – Navier Stokes

Conserved densities:

mass: $\boldsymbol{\rho}$ Energy: ε

Momentum: $g_i = \boldsymbol{\rho} v_i$

$$F = \tfrac{1}{2}\int d^d x (g^2 / \rho) + \int d^d x f[\rho]$$

$$\partial_t \rho + \partial_i g_i = 0$$

$$\partial_t g_i = \partial_i \sigma_{ij} = -\partial_i p + \eta \nabla^2 v_i$$

$$\partial_t \varepsilon + \partial_i j_i^\varepsilon = 0$$

$$\omega = \pm cq - \frac{2\eta}{3\rho} i q^2 \qquad \text{(2 modes)}$$

$$\omega = -i \frac{\eta}{\rho} q^2 \qquad \text{(2 modes)}$$

$$\omega = -i \frac{\kappa}{C_p} q^2 \qquad \text{(1 mode)}$$

Crystalline Solid I

Mass density is periodic

$$\rho \to \rho + \sum_{\mathbf{G}} \rho_{\mathbf{G}} e^{i\mathbf{G}\cdot(\mathbf{x}-\mathbf{u})}$$

Conserved densities:

mass: ρ Energy: ε

Momentum: $g_i = \rho v_i$

Broken-symmetry field:

Phase of mass-density field: $\mathbf{u}$
describes displacement of periodic part of density

Strain $u_{ij} = (\partial_i u_j + \partial_j u_i)/2$

Free energy

$$F = \int d^d x \left[\tfrac{1}{2}(g^2/\rho) + f[\rho] - \lambda \delta\rho u_{ii} + \tfrac{1}{2} K_{ijkl} u_{ij} u_{kl} \right]$$

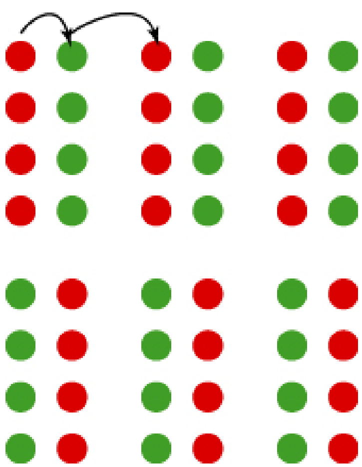
Crystalline Solid II

$$\partial_t \rho + \partial_i g_i = 0$$

$$\partial_t u_i = v_i - \frac{1}{\gamma}\frac{\delta F}{\delta u_i}$$ permeation

$$\partial_t g_i = -\partial_i p - \frac{\delta F}{\delta u_i} + \eta \nabla^2 v_i$$

Modes:
Transverse phonon: 4
Long. Phonon: 2
Permeation (vacancy diffusion): 1
Thermal Diffusion: 1



Permeation: independent motion of mass-density wave and mass: mass motion with static density wave

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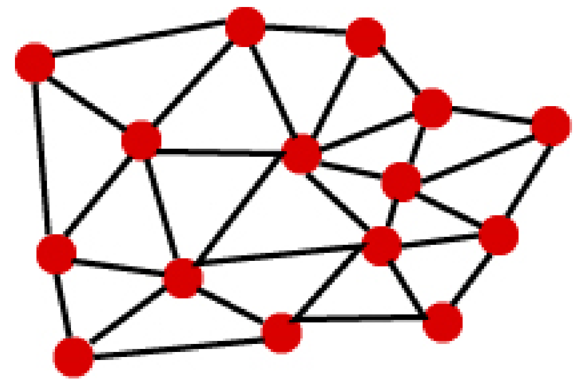
Tethered Solid

7 hydrodynamic variables: 1 density, 3 momenta, 3 displacements, 1 energy + 1 constraint = 8-1=7



Classic equations of motion for a Lagrangian solid

Isotropic elastic free energy

$$F = \frac{1}{2}\int d^d x \left(\lambda u_{ii}^2 + 2\mu u_{ij}^2 \right)$$

$$\rho \partial_t^2 u_i = -\frac{\delta F}{\delta u_i} + \eta \nabla^2 v_i$$

Note: 2nd order u_i

$$\omega_T = \pm\sqrt{\frac{\mu}{\rho}} q + i\frac{\eta}{2\rho} q^2 \qquad (4)$$

$$\omega_L = \pm\sqrt{\frac{\lambda + 2\mu}{\rho}} q + i\frac{2\eta}{3\rho} q^2 \qquad (2)$$

+ energy mode (1)

Gel: Tethered Solid in a Fluid

Tethered solid

$$\rho_s \partial_t^2 u_i = -\frac{\delta F}{\delta u_i} + \eta_s \nabla^2 \dot{u}_i - \Gamma(\dot{u}_i - v_i)$$

Friction only for relative motion- Galilean invariance

Frictional Coupling

Fluid

$$\partial_t g_i = -\partial_i p + \eta \nabla^2 v_i - \Gamma(v_i - \dot{u}_i)$$

Total momentum conserved

$$\partial_t (g_i + \rho_s \dot{u}_i) = \partial_j \sigma_{ij}^T$$

Fast non-hydro mode

$$\omega_F = -i\tau^{-1} = -i(\rho^{-1} + \rho_s^{-1})\Gamma$$

$\omega\tau \ll 1 :$ Effective Tethered Hydro.

$$(\rho + \rho_s)\ddot{u}_i = -\frac{\delta F}{\delta u_i} + (\eta + \eta_s)\nabla^2 \dot{u}_i$$

Fluid and Solid move together

Nematic Hydrodynamics: Harvard I

$$F = \tfrac{1}{2}\int d^d x (g^2 / \rho) + \int d^d x f[\mathbf{n},\rho]$$

g is the total momentum density: determines angular momentum

$$f[\mathbf{n},\rho] = \frac{1}{2}K_1(\rho)(\nabla\cdot\mathbf{n})^2 + \frac{1}{2}K_2(\rho)[\mathbf{n}\cdot(\nabla\times\mathbf{n})]^2 + \frac{1}{2}K_3(\rho)[\mathbf{n}\times(\nabla\times\mathbf{n})]^2$$

Frank free energy for a nematic

Foster,TCL,Martin, Swift, Pershan PRL, 1016 (1971)

Nematic Hydrodynamics: Harvard II

$$\partial_t n_i = \lambda_{ijk}\partial_k v_j - \frac{1}{\gamma}\frac{\delta F}{\delta n_i}$$ permeation

$$\partial_t g_i = -\partial_i p + \partial_j\left(\lambda_{kij}\frac{\delta F}{\delta n_k}\right) + \partial_j \sigma'_{ij}$$

$$A_{ij} = \tfrac{1}{2}\left(\partial_i v_j + \partial_j v_i\right)$$

$$\omega_i = \tfrac{1}{2}\varepsilon_{ijk}\partial_j v_k$$

ω – fluid vorticity not spin frequency of rods

$$\sigma'_{ij} = \eta_{ijkl}A_{kl};$$

$$\lambda_{ijk} = \tfrac{1}{2}\left(\delta^T_{ij}n_k - \delta^T_{ik}n_j\right) + \tfrac{1}{2}\lambda\left(\delta^T_{ij}n_k + \delta^T_{ik}n_j\right)$$

Modes: 2 long sound, 2 "slow" director diffusion. 2 "fast" velocity diff.

$$\partial_t \mathbf{n} = \boldsymbol{\omega}\times\mathbf{n} + \lambda\mathbf{n}:\mathbf{A} + \frac{1}{\gamma}\mathbf{h}$$

Anti-sym strain rate rotates $\mathbf{n}$

Stress tensor can be made symmetric

Rigid-rod Nematodynamics I

$g_i = \rho v_i =$ C.M. Momentum Density: conserved
$\ell_i = I\Omega_i =$ "spin" angular momemtum density: not conserved

$$\mathbf{L}_T = \int d^d x \left(\mathbf{x} \times \mathbf{g} + \boldsymbol{\ell} \right) = \text{Total ang. mom.: conserved}$$

$$F = \frac{1}{2} \int d^d x \left(\frac{g^2}{\rho} + \frac{l^2}{I} \right) + \int d^d x f[\mathbf{n}, \rho]$$

H. Stark, TCL, Phys. Rev. E 72, 051714 (2005).

Rigid-rod Nematodynamics II

$$h_i = -\frac{\delta F}{\delta n_i}$$

$$\partial_t \mathbf{n} = \boldsymbol{\Omega} \times \mathbf{n} + \frac{1}{\gamma}\mathbf{h}$$

$$\partial_t \ell_i = (\mathbf{n}\times\mathbf{h})_i + \nu_\perp \nabla^2 \Omega_i$$

$$- \Gamma_{ij}(\Omega_j - \omega_j) - \tfrac{1}{2}\Gamma^A(\varepsilon_{ijl} n_l n_k + \varepsilon_{ikl} n_l n_j) A_{jk}$$

$$\partial_t g_i = -\partial_i p + \partial_j(\sigma'^A_{ij} + \sigma'^S_{ij})$$

Like friction in tethered solid in a fluid

$$\sigma'^A_{ij} = \tfrac{1}{2}\varepsilon_{ijk}\Gamma_{kl}(\Omega_l - \omega_l)$$

$$+ \tfrac{1}{2}\Gamma^A(n_j A_{ip} n_p - n_i A_{jp} n_p)$$

$$\sigma'^S_{ij} = \eta_{ijkl} A_{kl}$$

Anti-symmetric part of stress tensor guarantees conservation of total angular momentum

Rigid-rod Nematodynamics III

$$\partial_t \mathbf{n} = \boldsymbol{\Omega} \times \mathbf{n} + \frac{1}{\gamma}\mathbf{h}$$

Part perpendicular to $\mathbf{n}$;
Compact notation

$$I\partial_t \boldsymbol{\Omega}_\perp = \mathbf{n} \times \mathbf{h} + \nu_\perp \nabla^2 \Omega_i - \Gamma_\perp(\boldsymbol{\Omega}_\perp - \boldsymbol{\omega}_\perp) - \Gamma^A (\mathbf{A} : \mathbf{n}) \times \mathbf{n}$$

$$I\mathbf{n} \times \ddot{\mathbf{n}} = \mathbf{n} \times (\mathbf{h} - \gamma_2 \mathbf{A} : \mathbf{n} - \gamma_1 \mathbf{N})$$

$$\mathbf{N} = \dot{\mathbf{n}} - \boldsymbol{\omega} \times \mathbf{n}; \quad \gamma_1 = \Gamma_\perp; \quad \gamma_2 = -\Gamma^A$$

$$\boldsymbol{\Omega}_\perp = \mathbf{n} \times (\boldsymbol{\Omega} \times \mathbf{n})$$

$$\boldsymbol{\omega}_\perp = \mathbf{n} \times (\boldsymbol{\omega} \times \mathbf{n})$$

$$F_{\text{kin}} = \frac{1}{2I}\int d^3x (\mathbf{n} \times \dot{\mathbf{n}})^2$$

$$\omega_f = -i\frac{\Gamma_\perp}{I}$$

Non-hydro mode

Hydrodynamics limit: ignore inertial term

$$\mathbf{h} = \gamma_1 \mathbf{A} : \mathbf{n} + \gamma_2 \mathbf{N}$$

Rigid-rod Nemotodynamics IV

Regain LE form: solve $\mathbf{n}$ equation for $\mathbf{\Omega}$ in terms of $\mathbf{n}$ and use in angular momentum Eq.

$$I\left(\mathbf{n}\times\ddot{\mathbf{n}}-\frac{1}{\gamma}\dot{\mathbf{h}}\right)=\alpha\mathbf{n}\times\left(\mathbf{h}+\gamma_1\mathbf{A}:\mathbf{n}+\gamma_2\mathbf{N}\right)$$

γ_1 : contributions from friction and permeation

$$\alpha=\left(1+\frac{\Gamma_\perp}{\gamma}\right);\qquad \frac{1}{\gamma_1}=\frac{1}{\gamma}+\frac{1}{\Gamma_\perp};\qquad \gamma_2=-\frac{\Gamma^A}{\Gamma_\perp}\gamma_1\equiv-\lambda\gamma_1$$

Hydrodynamic limit: ignore inertial term. Same as hydro limit of LE, but with different values of coefficients

$$\mathbf{h}=\gamma_1\mathbf{A}:\mathbf{n}+\gamma_2\mathbf{N}$$

Ericksen, Arch. Ratl. Mech. Anal. 4, 231 (1960); Trans. Soc. Rheo. 4, 29 (1960); Leslie, Q. J. of Mech. and App. Math. 19, 357 (1966); Arch. Rat. Mech. Anal. 28, 265 (1968)

Rigid-rod Nematodynamics V

Regain Harvard: Solve $\mathbf{\Omega}$ equation for $\mathbf{\Omega}\times\mathbf{n}$

$$I\partial_t\mathbf{\Omega}_\perp = \mathbf{n}\times[\mathbf{h}-\Gamma_\perp(\mathbf{\Omega}\times\mathbf{n}-\boldsymbol{\omega}\times\mathbf{n})-\Gamma^A(\mathbf{A}:\mathbf{n})]$$

Hydrodynamics limit: ignore inertial term

$$\mathbf{\Omega}\times\mathbf{n} = \frac{1}{\Gamma_\perp}(\mathbf{h}+\Gamma_\perp\boldsymbol{\omega}\times\mathbf{n}+\Gamma^A\mathbf{A}:\mathbf{n})$$

$$\partial_t\mathbf{n} = \mathbf{\Omega}\times\mathbf{n}+\frac{1}{\gamma}\mathbf{h}$$
$$=\boldsymbol{\omega}\times\mathbf{n}+\frac{1}{\gamma_1}\mathbf{h}+\lambda\mathbf{A}:\mathbf{n}$$

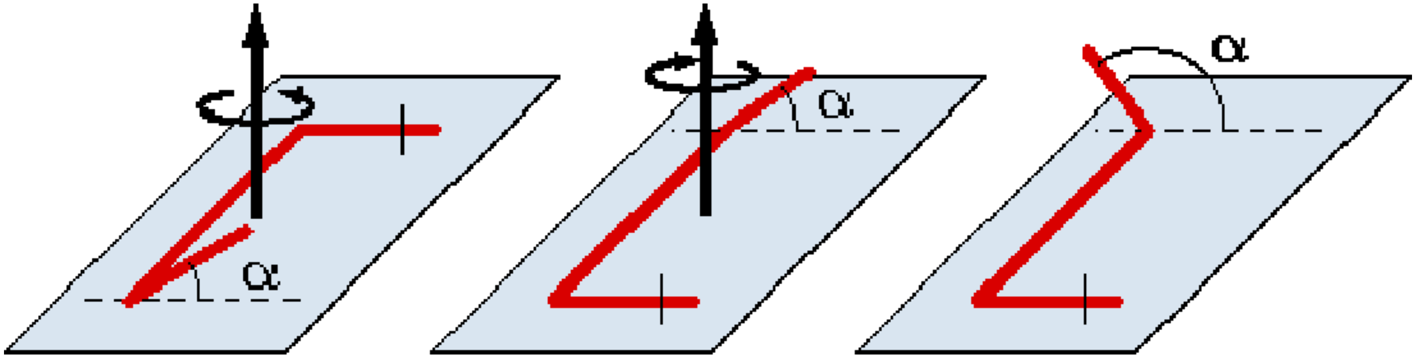
$$\frac{1}{\gamma_1} = \frac{1}{\gamma}+\frac{1}{\Gamma_\perp}$$
$$\lambda = \frac{\Gamma^A}{\Gamma_\perp} = -\frac{\gamma_2}{\gamma_1}$$

Rattleback gas

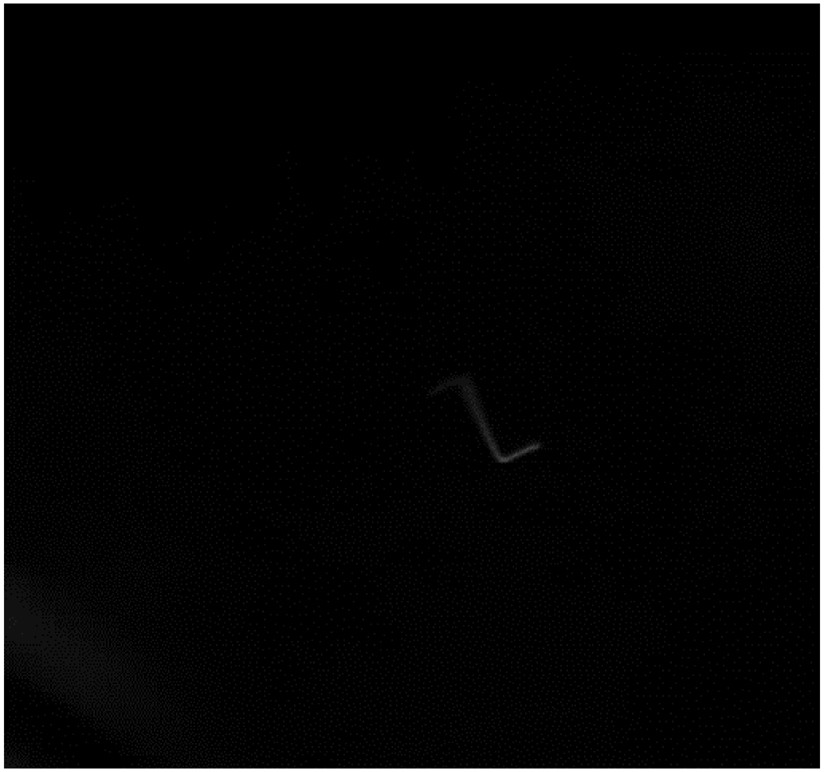
Tsai, Ye, Jimenez, Gollub, Lubensky, PRL 214301 (2005)

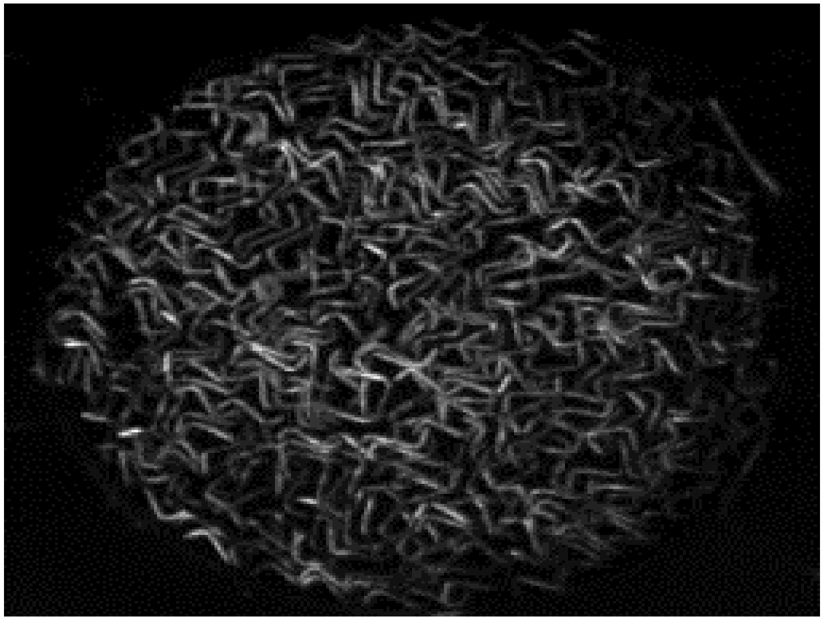
Chiral Rattlebacks spin in a preferred direction; Achiral ones do not.

Chiral wires spin in a preferred direction on a vibrating substrate

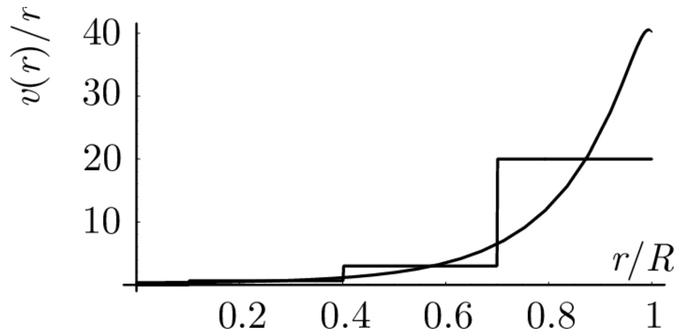


Rattleback gas II





Rattleback gas III



$l = I\Omega =$ Spin angular momentum

$g_i = \rho v_i =$ Center-of-mass mometum

$\Omega =$ Spin angular frequency

$\omega = (\nabla \times \mathbf{v})_z / 2 =$ CM angular frequency

$$\partial_t l = -\partial_j (l v_j) - \Gamma^\Omega \Omega - \Gamma(\Omega - \omega) + D_\Omega \nabla^2 \Omega + \tau$$

Substrate friction | Spin-vorticity coupling | Vibrational torque

$$\partial_t g_i = -\partial_j (g_i v_j) - \partial_i p + \eta \nabla^2 v_i - \Gamma^v v_i + \tfrac{1}{2} \varepsilon_{ij} \partial_j \Gamma(\Omega - \omega)$$

Active nematic in isotropic phase

Active particles with mass density ρ_A and solvent particles with mass density ρ_S

$$\partial_t \rho_A = -\vec{\nabla}\cdot\vec{g}_A;\ \partial_t \rho_S = -\vec{\nabla}\cdot\vec{g}_S$$

$$\rho = \rho_A + \rho_S$$

$$\partial_t \rho = -\vec{\nabla}\cdot(\vec{g}_A + \vec{g}_S) = \vec{\nabla}\cdot\vec{g}$$

$$\partial_t g_i = \nabla_j \left[\frac{\partial f}{\partial \nabla_j Q_{kl}} \nabla_i Q_{kl} + \lambda_{klij} \frac{\delta \mathfrak{F}}{\delta Q_{kl}} + \sigma_{ij} \right]$$

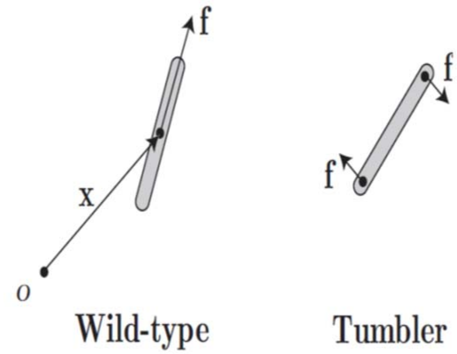
$$\partial_t Q_{ij} = -\vec{v}\cdot\vec{\nabla} Q_{ij} + \lambda_{ijkl} \partial_k v_k - L_{ijkl} \frac{\delta \mathfrak{F}}{\delta Q_{kl}}$$

Force dipoles

$$f_i^\alpha = \int dS^\alpha \tilde{f}_i^\alpha; \quad \tau_i^\alpha = \epsilon_{ijk} \int dS^\alpha r_j^\alpha \tilde{f}_k^\alpha$$

$\tilde{f}_i^\alpha$: Surface force density on part. α

f_i^α : Force; τ_i^α : torque on part. α



$$g_{Ai} = \sum_\alpha p_i^\alpha \delta\left(\vec{x} - \vec{x}^\alpha(t)\right) + \tfrac{1}{2}\epsilon_{ijk}\nabla_j \sum_\alpha \ell_i^\alpha \delta\left(\vec{x} - \vec{x}^\alpha(t)\right)$$

$$\partial_t g_{Ai} = \sum_\alpha \left(f_i^\alpha + \tfrac{1}{2}\epsilon_{ijk}\nabla_j \tau_k^\alpha\right)\delta\left(\vec{x} - \vec{x}^\alpha(t)\right)$$

Force on S=minus force on A

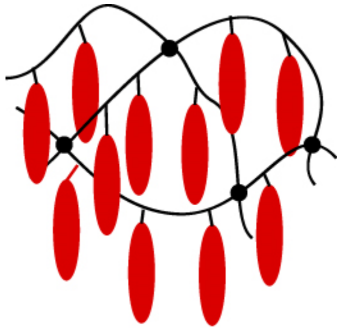
$$\partial_t g_{Si} = -\sum_\alpha \int dS^\alpha \tilde{f}_i^\alpha \delta\left(\vec{x} - \vec{x}^\alpha(t) - \vec{r}^\alpha(t)\right)$$

$$\partial_t g_i = \tfrac{1}{2}\nabla_j \int dS^\alpha \left(\tilde{f}_i^\alpha r_j^\alpha + \tilde{f}_j^\alpha r_i^\alpha\right)\delta\left(\vec{x} - \vec{x}^\alpha(t)\right)$$

Force dipole contribution: no counterpart in *Q* equation.

A. W. C. Lau, T. C. Lubensky, *PRE* **80**, (2009); D. T. N. Chen *et al.*, *PRL* **99**, (2007).

Nematic Elastomers



- Homogeneous Elastic media with broken rotational symmetry (uniaxial, biaxial)

- Crosslinked nematic polymers

- Most interesting - systems with broken symmetry that develops spontaneously from a homogeneous, isotropic elastic state

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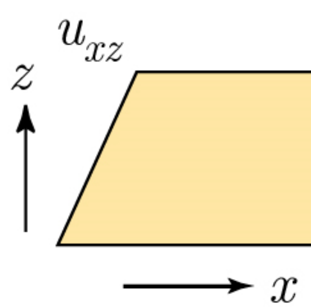
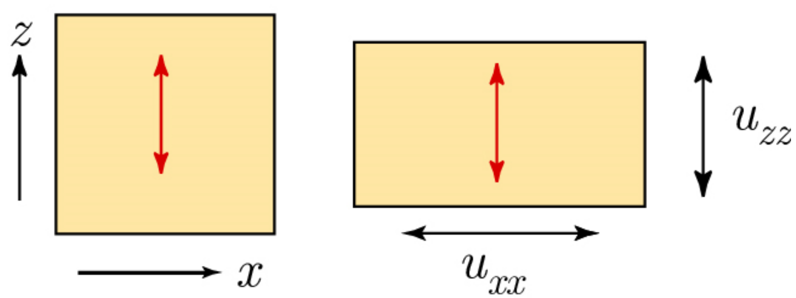
Nematic Elastomers II

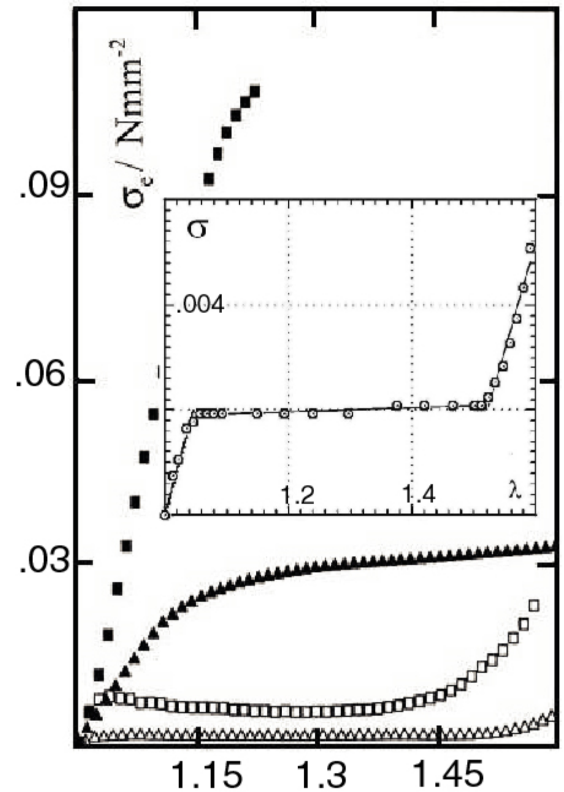
- **Soft or "Semi-soft" elasticity**



Vanishing *xz* shear modulus



Soft stress-strain for stress perpendicular to order



Warner
Finkelmann

Nematic Elastomers: Director-strain Energy

$$F = F_u + F_n + F_{u-n}$$

$$F_u = \int d^3x[\tfrac{1}{2}C_1 u_{zz}^2 + C_2 u_{zz} u_{aa} + \tfrac{1}{2}C_3 u_{ii}^2 + C_4 u_{ab}^2 + C_5 u_{az}^2]$$

$$F_n = \int d^3x[\tfrac{1}{2}K_1(\partial_a n_a)^2 + \tfrac{1}{2}K_2(\varepsilon_{ab}\partial_a n_b)^2 + \tfrac{1}{2}K_3(\partial_z n_b)^2]$$

$$F_{u-n} = \int d^3x[\tfrac{1}{2}D_1 Q_a^2 + D_2 Q_a u_{az}]$$

$$Q_a = n_a - \tfrac{1}{2}(\partial_z u_a - \partial_a u_z)$$

NE: Relaxed elastic energy

Hydrodynamic modes from effective free energy in terms of strain only

$$F_u^{\text{eff}} = \int d^3x[\tfrac{1}{2}C_1 u_{zz}^2 + C_2 u_{zz}u_{aa} + \tfrac{1}{2}C_3 u_{ii}^2$$

$$+ C_4 u_{ab}^2 + C_5^R u_{az}^2 + \tfrac{1}{2}K_1^R(\partial_a^2 u_z)^2 + \tfrac{1}{2}K_3^R(\partial_z^2 u_a)^2]$$

$$C_5^R = C_5 - \frac{D_2^2}{2D_1};$$

Soft : $C_5^R = 0;$ Semi - Soft : $C_5^R \neq 0$

$$K_1^R = \frac{1}{4}\left(1 + \frac{D_2}{D_1}\right)^2 K_1; \quad K_3^R = \frac{1}{4}\left(1 - \frac{D_2}{D_1}\right)^2 K_3$$

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NE: Director-displacement dynamics

Tethered anisotropic solid plus nematic

Stenull-Lubensky PRE (2004)

$$\partial_t n_i = \lambda_{ijk} \partial_k v_j - \frac{1}{\gamma} \frac{\delta F}{\delta n_i}$$

$$\dot{u}_i = \frac{\delta F}{\delta g_i} = \frac{1}{\rho} g_i$$

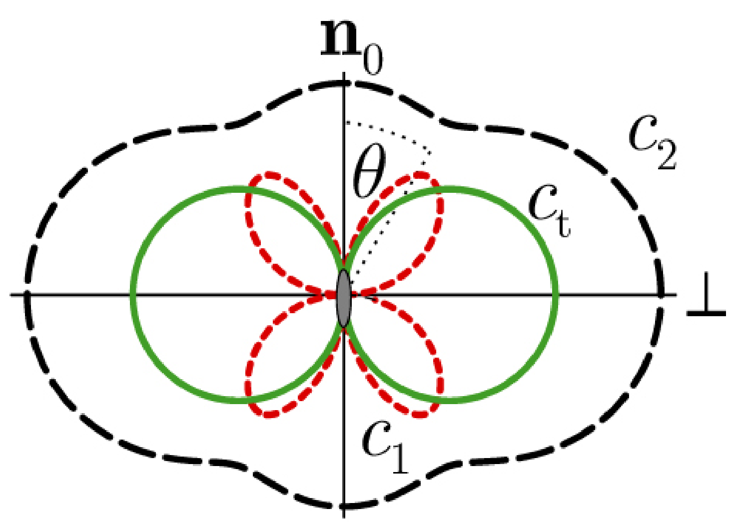
$$\partial_t g_i = \partial_j \left(\lambda_{kij} \frac{\delta F}{\delta n_k} \right) - \frac{\delta F}{\delta u_i} + \partial_j \sigma'_{ij}$$

Director relaxes in a microscopic time to the local shear – nonhydrodynamic mode

$$\omega_f = -i \frac{D_1}{\gamma} = -\frac{i}{\tau_1}$$

Soft Elastomer Hydrodynamics

$$\rho \ddot{u}_i = -\frac{\delta F_u^{\text{eff}}}{\delta u_i} + \eta_{ijkl} \partial_j \partial_l v_k$$



Same mode structure as a discotic liquid crystal: 2 "longitudinal" sound, 2 columnar modes with zero velocity along $\mathbf{n}$, 2 smectic modes with zero velocity along both symmetry directions

Slow and fast diffusive modes along symmetry directions

$$\omega_s = -i \frac{2K}{\eta_5} q^2$$

$$\omega_f = -i \frac{2\eta_5}{\rho} q^2$$

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Summary

- Phenomenogical Dynamics – our best tool for describing slow collective dynamics
- Well tested rules to determined apply to all equilibrium systems
- Provide a language for non-equilibrium systems, but must be used with care