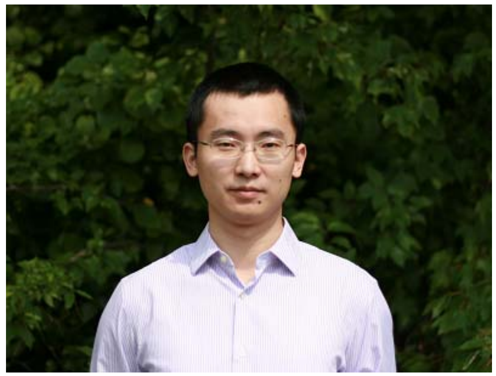
Frames near Mechanical Collapse

Xiaming Mao (Umich)

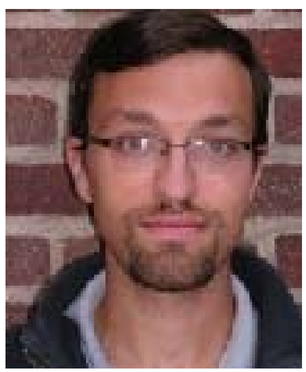


Kai Sun (Umich)

Anton Souslov (Ga Tech)

Charlie Kane (Penn)

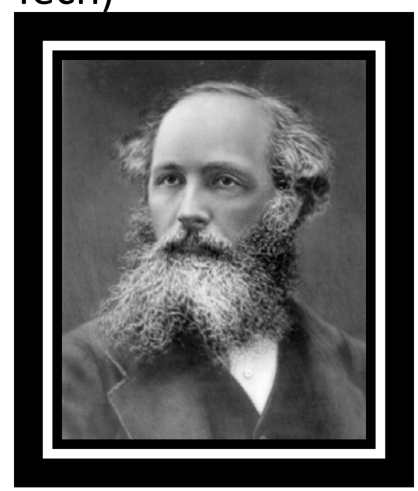
Olaf Stenull (Penn)



Zeb Rocklin (Umich)

Bryan Chen (Leiden)

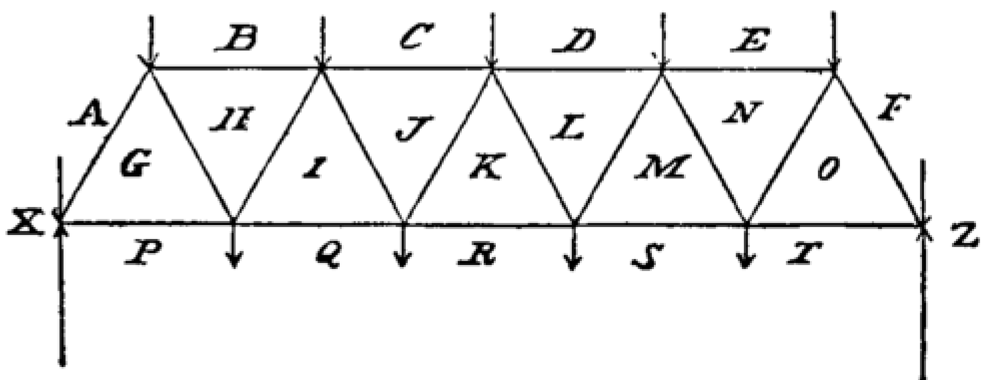
Vincenzo Vitelli (Leiden)



James Clerk Maxwell

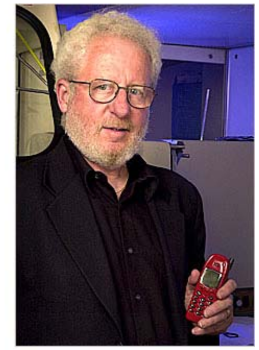
Bridges and Buildings

Warren Girder and Warren Truss bridge (1848)



Hancock Tower, Chicago

Polyacetylene and Topological Insulators

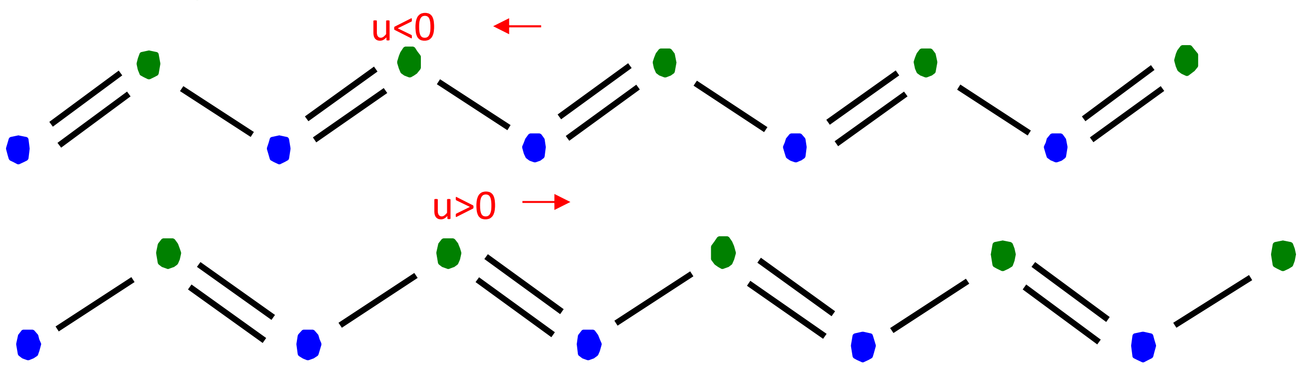


Polyacetalene: 1D chain with dimerization $u = \pm u_0$

Su, Schrieffer, Heeger ‘79

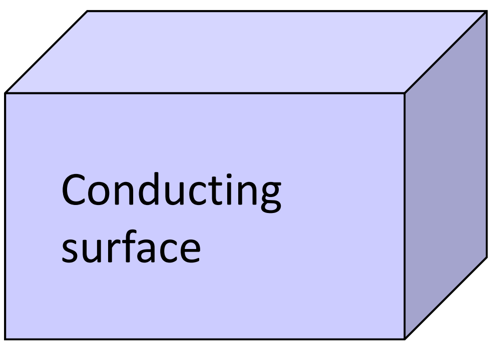
“A phase”

“B phase”



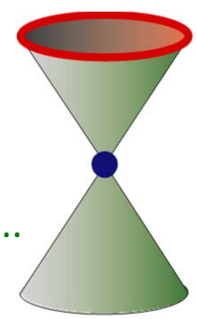
Three Dimensions: 3D TI

Fu & Kane ’06; Moore & Balents ‘06; Roy ‘06



Insulating interior

Experiment : Bi_xSb_{1-x}, Bi_2Se_3, ….
Hsieh et al. (Hasan) ‘08



Outline

- Maxwell's rules for rigidity of frames (1864)
- "Maxwell" frames and lattices
- Review of language of elasticity and dynamical modes
- Improving Maxwell: States of Self Stress
- Periodic lattices
- Soft elasticity: Guest modes
- Square and kagome lattices
- Twisted Kagome lattices and auexetic response

T. C. L., C. L. Kane, X. Mao, A. Souslov, K. Sun, R*eports Prog. Phys.* **78**, 073901 (2015);
K. Sun, A. Souslov, X. M. Mao, T. C. L., PNAS **109**, 12369-12374 (2012).

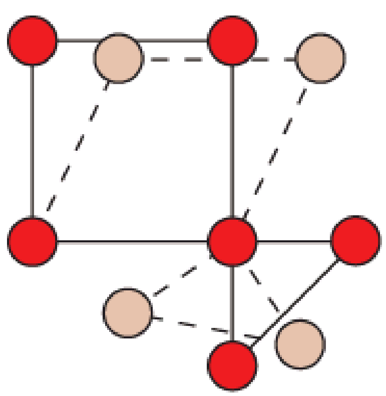
Maxwell I

- (1864) James Clerk Maxwell : mechanical stability - the number N_c of contacts between N frictionless spherical particles must exceed dN, where d is the spatial dimension. Modern era: network glass work of J.C. Phillips and Michael Thorpe [J. C. Maxwell, Phil. Mag. **27**, 598 (1864); Thorpe, *J. Non-Cryst. Solids* **57**, 355 (1983); J. C. Phillips, J. Non-Cryst. Solids 43, 37 (1981).]

- N particles have dN degrees of translational freedom and dN modes with zero energy in the absence of contacts or interparticle forces. Each contact (central force) reduces the number of zero modes by 1, leading to $N_0 = dN\text{-}N_c$ zero modes for $N_c = dN$.

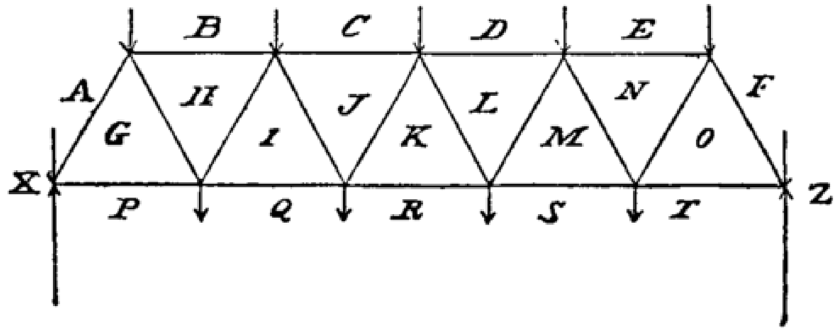
- If z is the average number of neighbors per particle, then $N_c = zN/2$ and $z_c = 2d$.

- Isostatic system: $N_0 = 0$



$$N = 6; \quad N_c = 7$$
$$N_0 = 2N - N_c = 5$$
2 translations, 1 rotation,
2 internal - FLOPPY MODES

The Warren Girder



C.R. Calladine, Int. J. Solids Structure, 14, 161 (1978)

$$\mathbf{H} \cdot \mathbf{t} = \mathbf{f}$$
$$\mathbf{t} = \mathbf{H}^{-1} \cdot \mathbf{f}$$

$N_B = 2N - 3$

$N_0 = 2N - N_B = 3$

$\mathbf{t}$ = $2N$ -dimensional vector of N_B tensions + 3 "reactions"

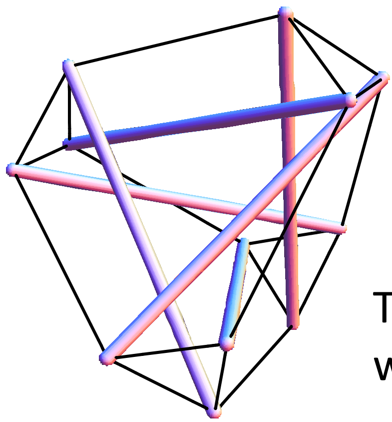
$\mathbf{f}$ = $2N$ -dimensional vector of forces at nodes

$\mathbf{H}$ = $2N \times 2N$ -dimensional non-singular "Equilibium Matrix"

Tensions are determined entirely by forces (loads) at nodes. System is "statically determinate"

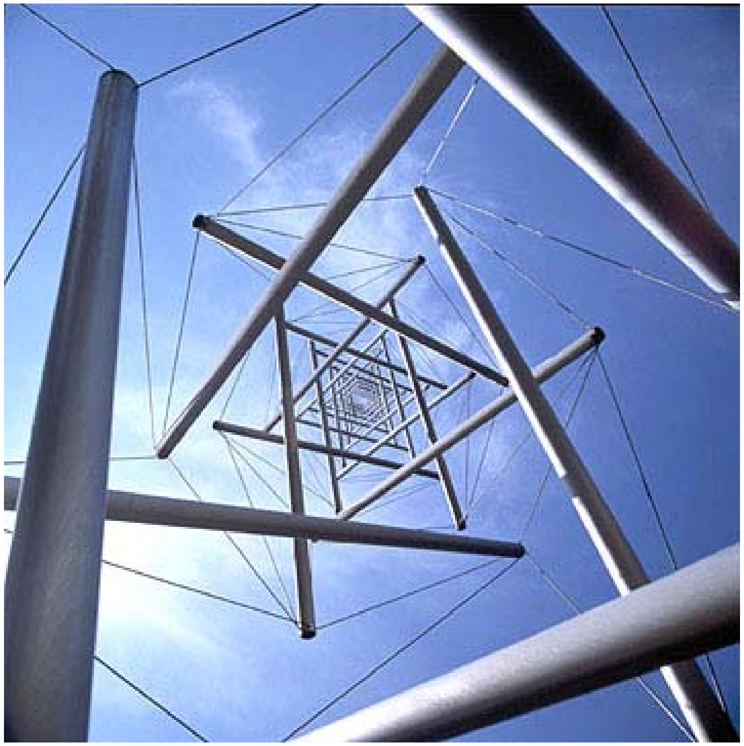
Penn

Tensegrity Structures



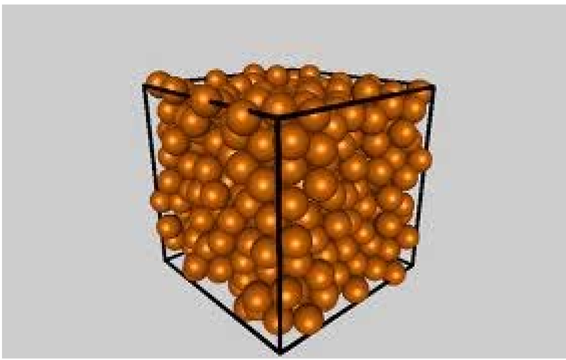
Truncated tetrahedron with cross beams

Tank Street Bridge, Brisbane, Australia (2010)

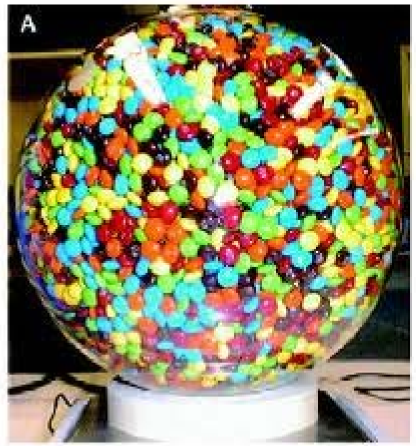


Kenneth Snelson: Needle Tower, Hirshhorn Sculpture Garden (1968)

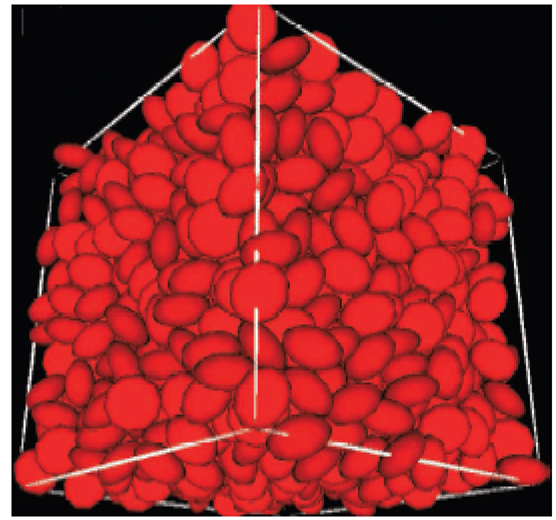
Random Close Packing



Cherrypit.princeton.edu



Chaikin

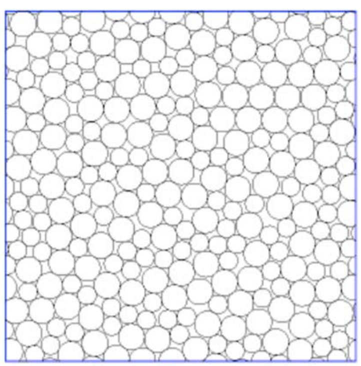


Torquato

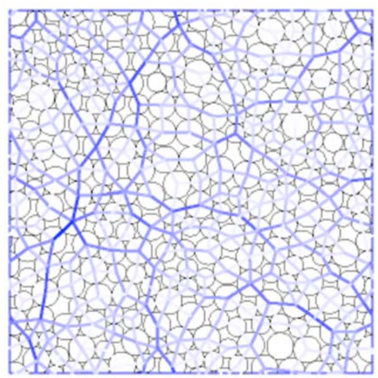
non-overlapped

V=0

p=0



T_f=0



T_f=0

overlapped

V>0

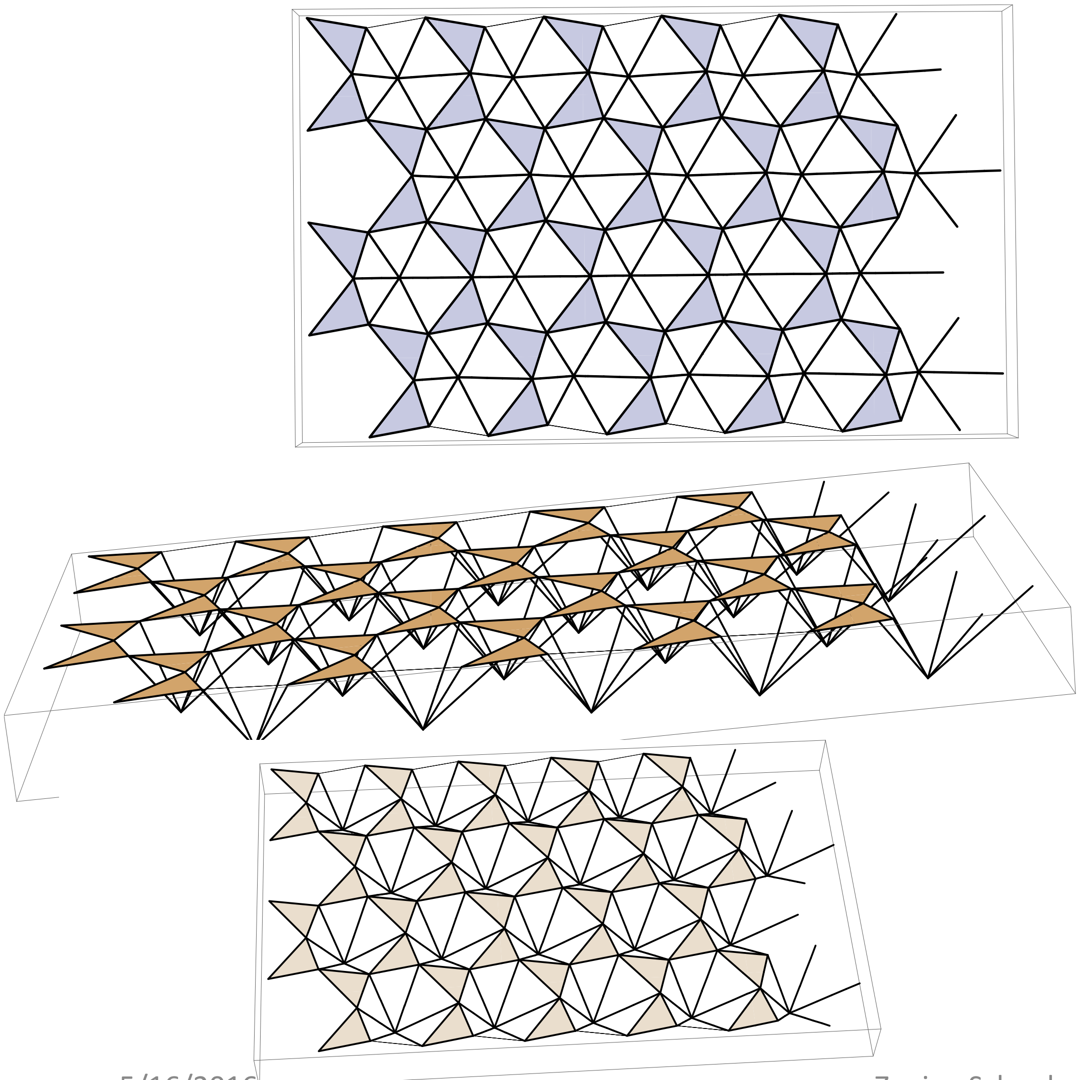
p>0

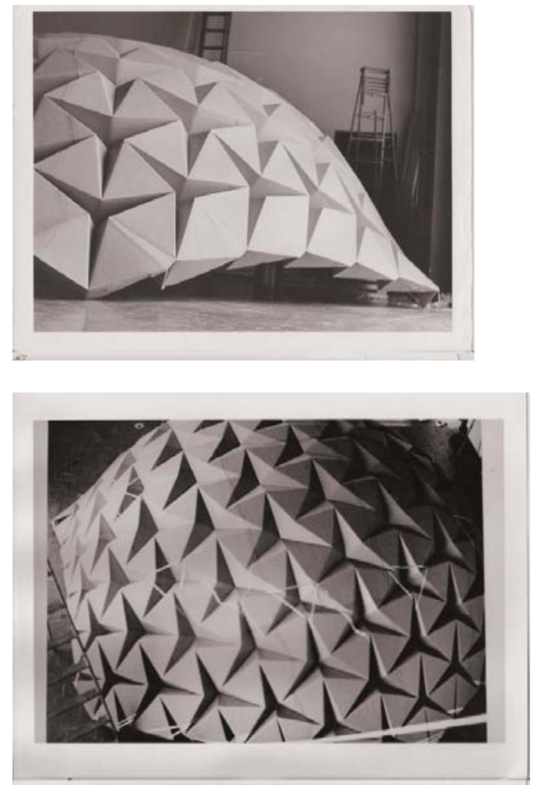
O'hern, Langer

Liu, Nagel

2d to 3D: "Origami lattices"

d=3, z=6, but a planar structure

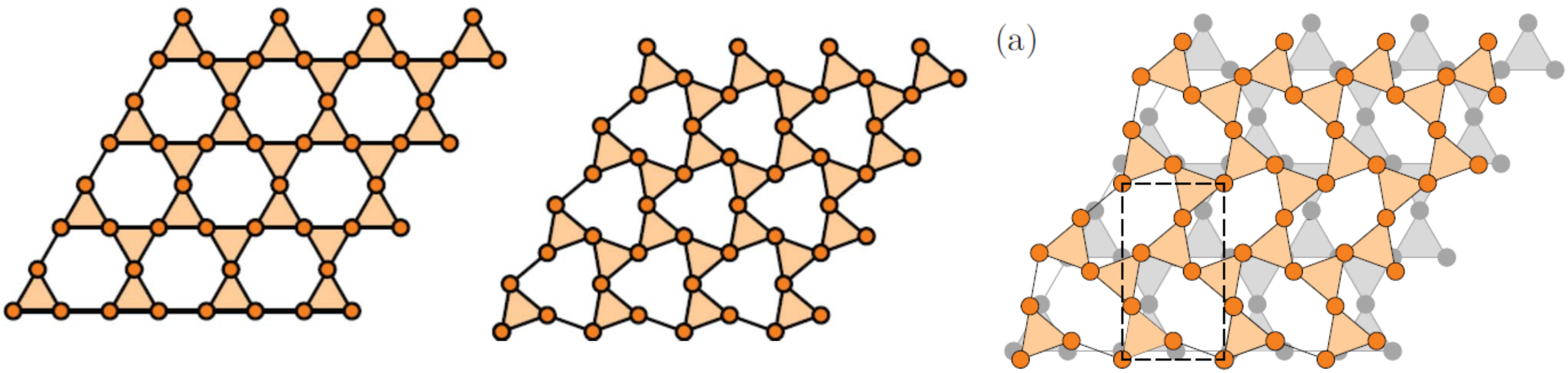




Ron Resch: Note curvature

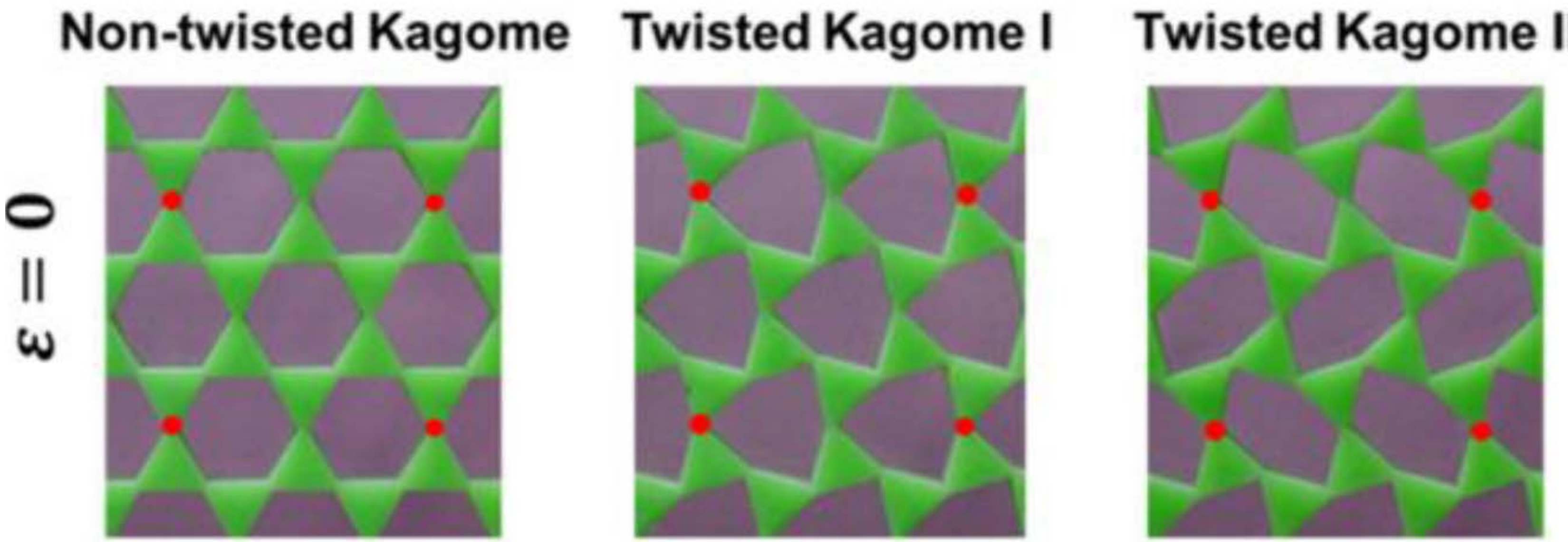
2 D Maxwell Lattices

Maxwell Lattice – one that under periodic boundary conditions have $z=2d$ exactly, i.e., $z=4$ in two-dimensions and $z=6$ in three dimension



Kagome Lattices

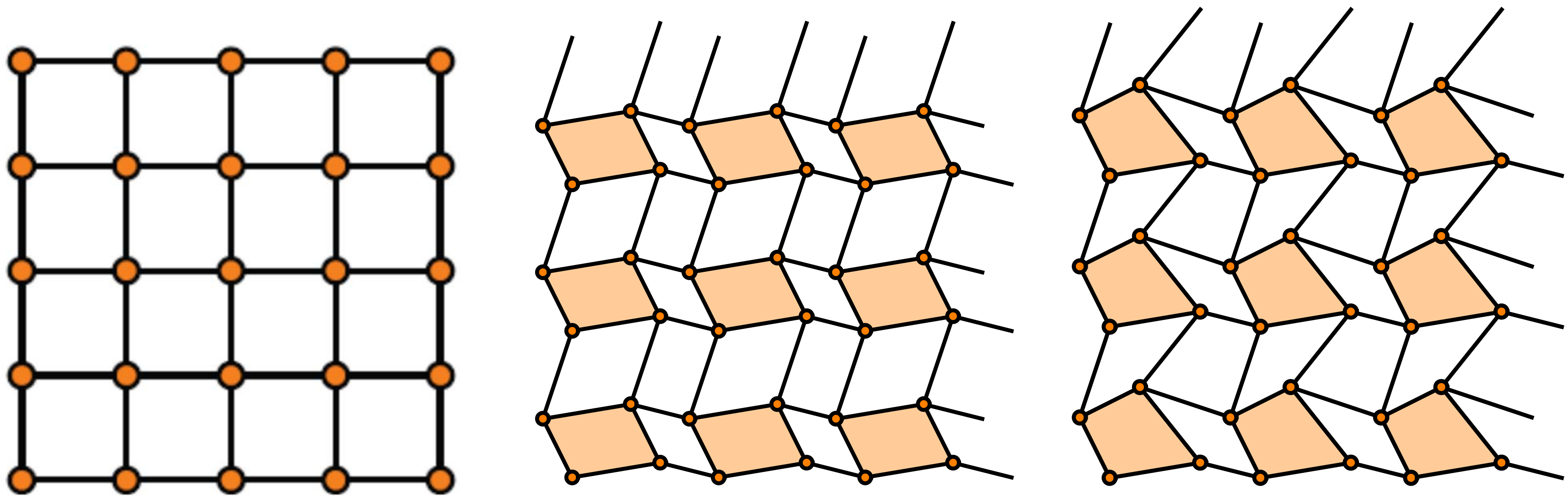
Fabricated Kagome Lattices



Silcone rubber lattices fabricated from molds fabricated with 3D printing.

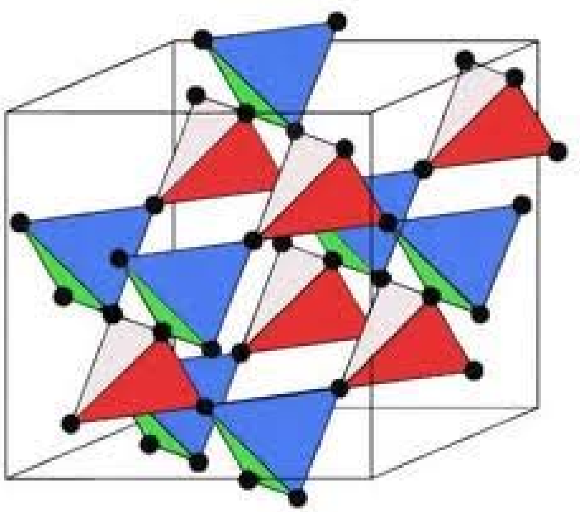
Gaoxiang Wu†, Yigil Cho†, In-Suk Choi, Dengteng Ge, Ju Li, Heung Nam Han, TCL, and Shu Yang, Advanced Materials **27,** 2747 (2015)

Square-Based 2D Maxwell Lattices

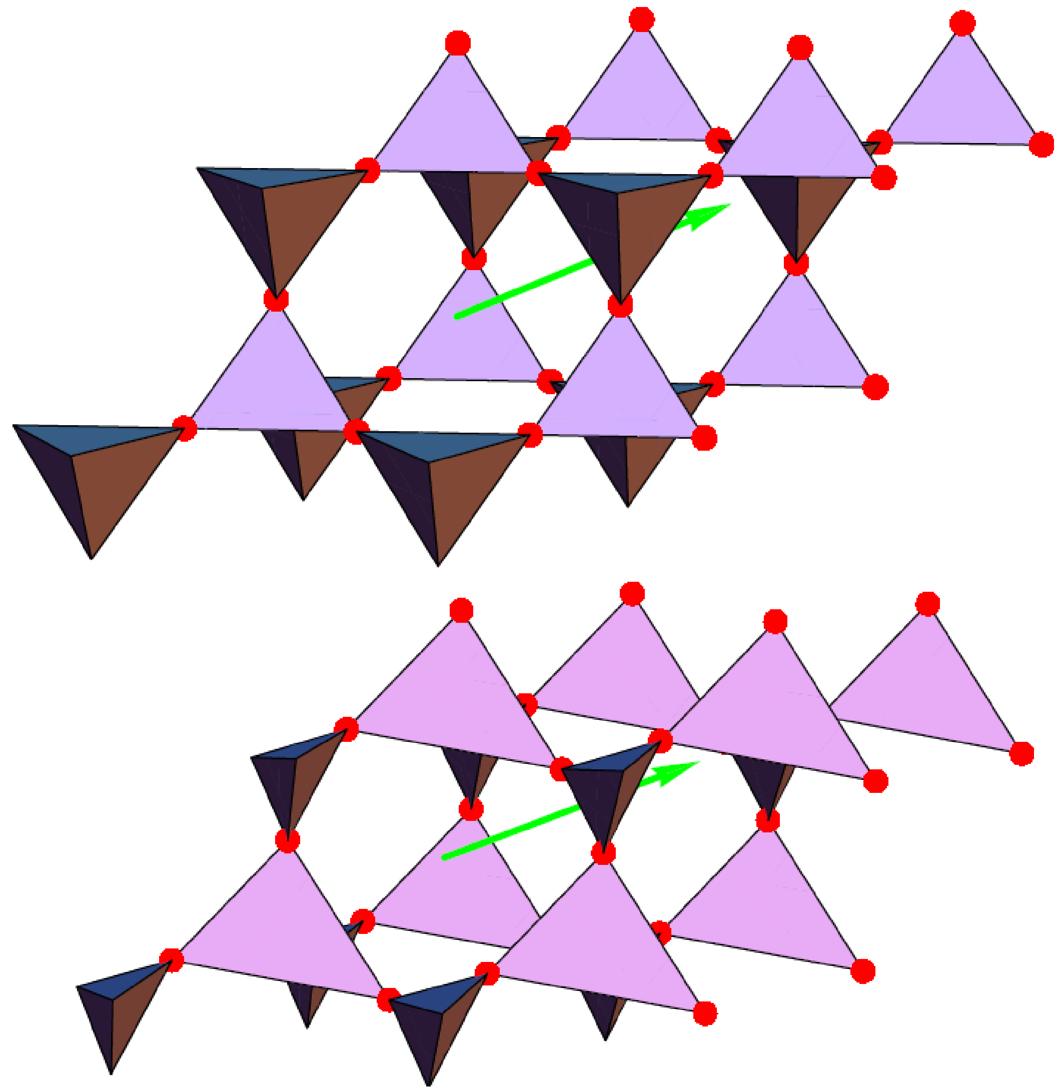


Zeb Rocklin, Bryan Chen, Martin Falk, TCL, and Vincenzo Vitelli – in preperation

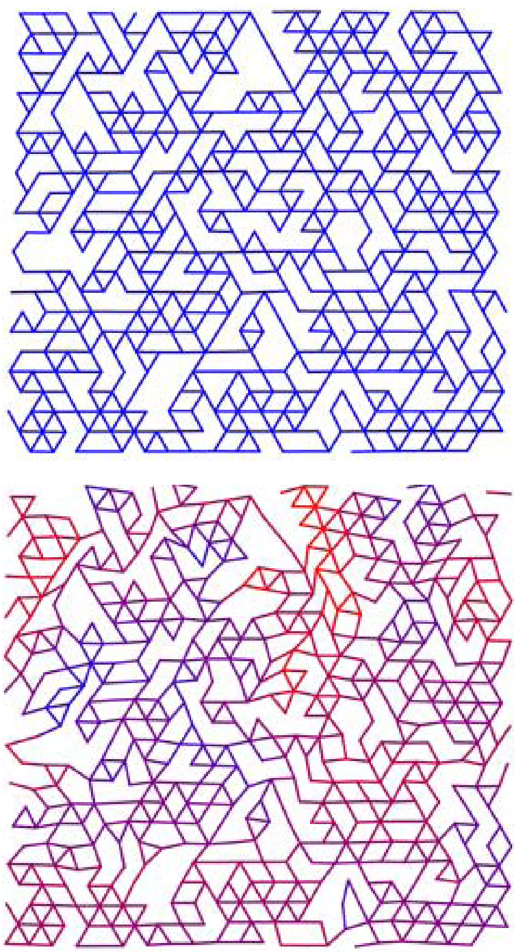
3D Maxwell Lattice: Pyrochlore and Distorted Phyrochlore

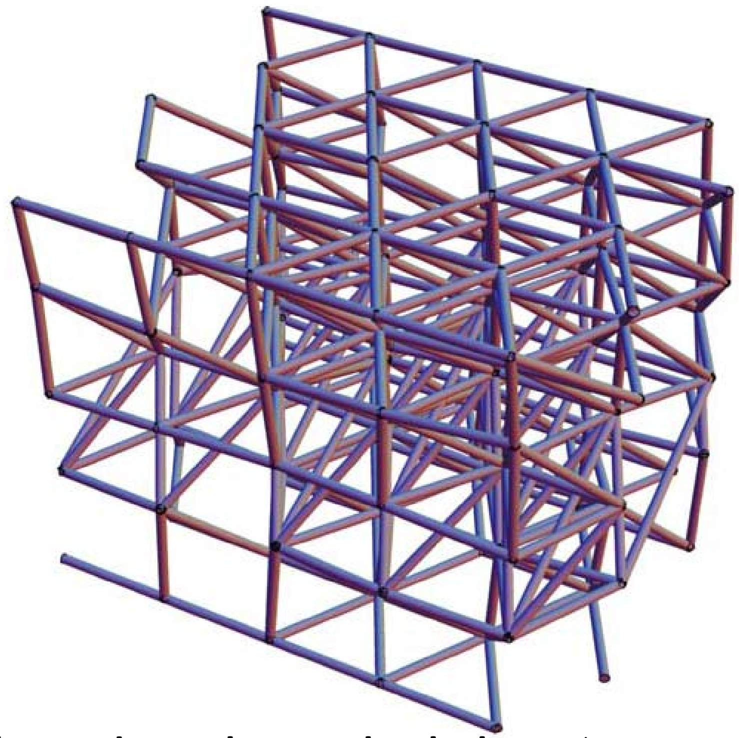


Stenull,Kane, and TCL, in preparation



Rigidity Percolation





Remove bond with probability $1-p$.

Probility of being in rigid cluster: $(p-p_c)^{\beta}$

Shear modulus: $G \sim (p-p_c)^{t}$

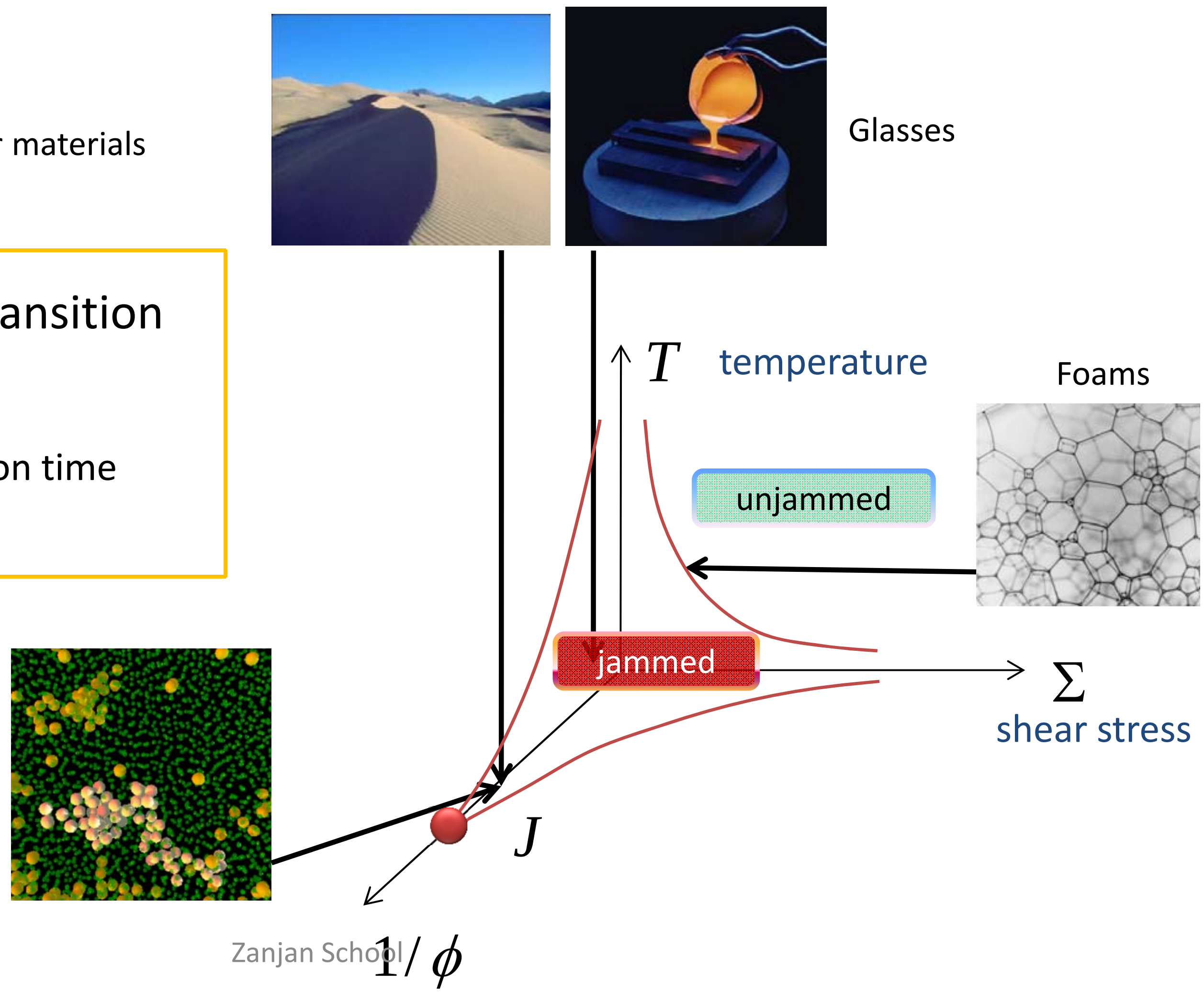
M. Thorpe, P. Sen, S. Feng, L. Schwartz, Halperin, and others (1980's)

The Jamming Phase Diagram

- Model for dynamic slowing down and emergence of rigidity in glasses, colloidal glasses, granular materials, foams, etc.

Granular materials

Glasses



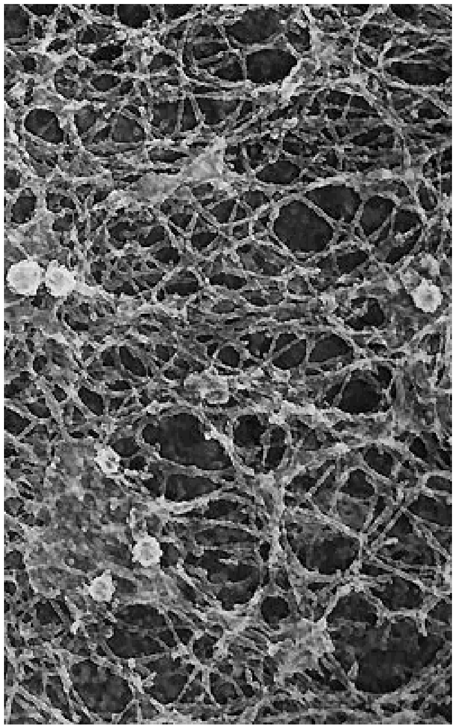
Colloidal suspensions

Common features at the transition

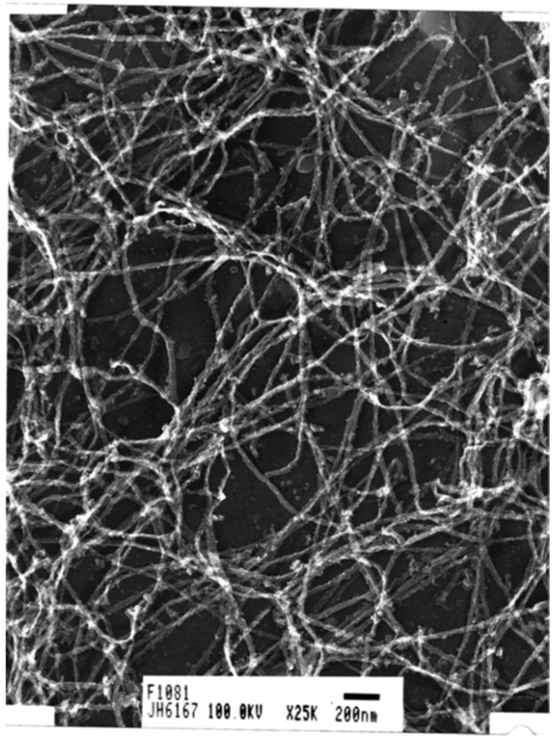
- No obvious structural change
- Dramatic increase of relaxation time
- Kinetic heterogeneities

- A. J. Liu and S. R. Nagel, Nature **396** N6706, 21 (1998).
- C. S. O'Hern, et al., Phys. Rev. E **68**, 011306 (2003).

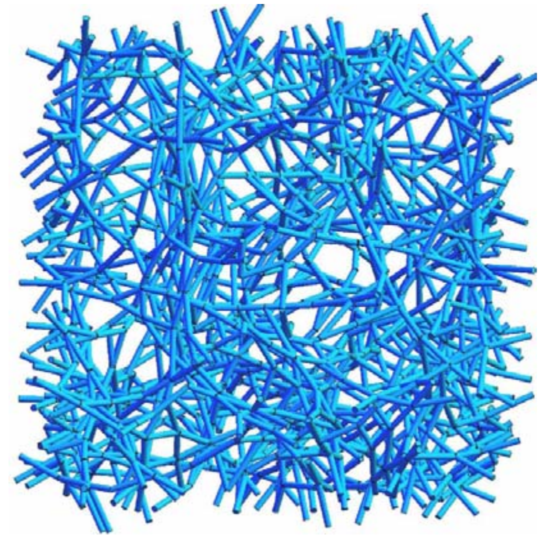
Networks of Semi-Flexible Polymers



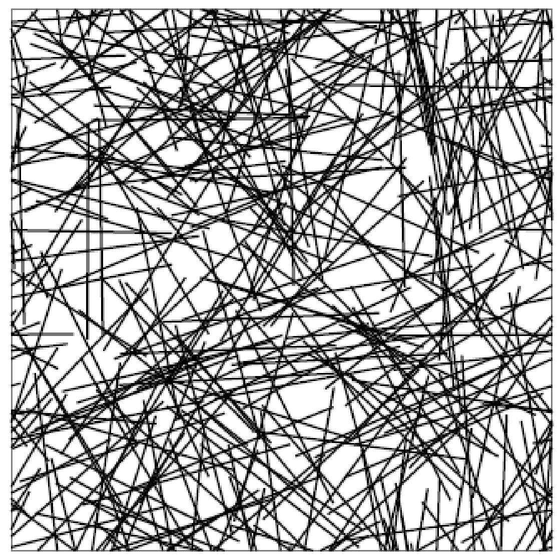
cortical
actin gel



neurofilament
network

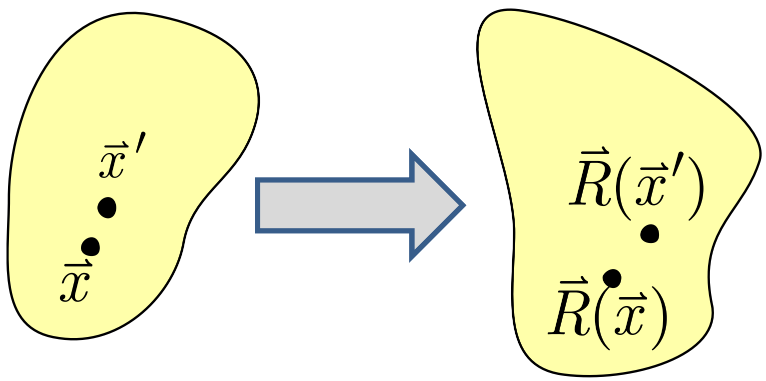


Mikado
lattice



Jamney, MacKintosh, Head, Frey, Wilhelm
Levine, Huessinger, Huisman

Strain and stress



$$\Lambda_{ij} = \frac{\partial R_i}{\partial x_j} = \delta_{ij} + \eta_{ij}$$

Cauchy deformation tensor

$$\vec{R}(\vec{x}) = \vec{x} + \vec{u}(\vec{x})$$

$$dR^2 - dx^2 = 2u_{ij}dx_i dx_j$$

$$\underset{\sim}{u} = \tfrac{1}{2}(\underset{\sim}{\Lambda}^T \underset{\sim}{\Lambda} - \underset{\sim}{\delta}) \approx \tfrac{1}{2}(\underset{\sim}{\eta} + \underset{\sim}{\eta}^T)$$

Linear strain tensor

$$u_{ij} = \tfrac{1}{2}\left(\partial_i u_j + \partial_j u_i + \partial_i u_k \partial_j u_k\right)$$

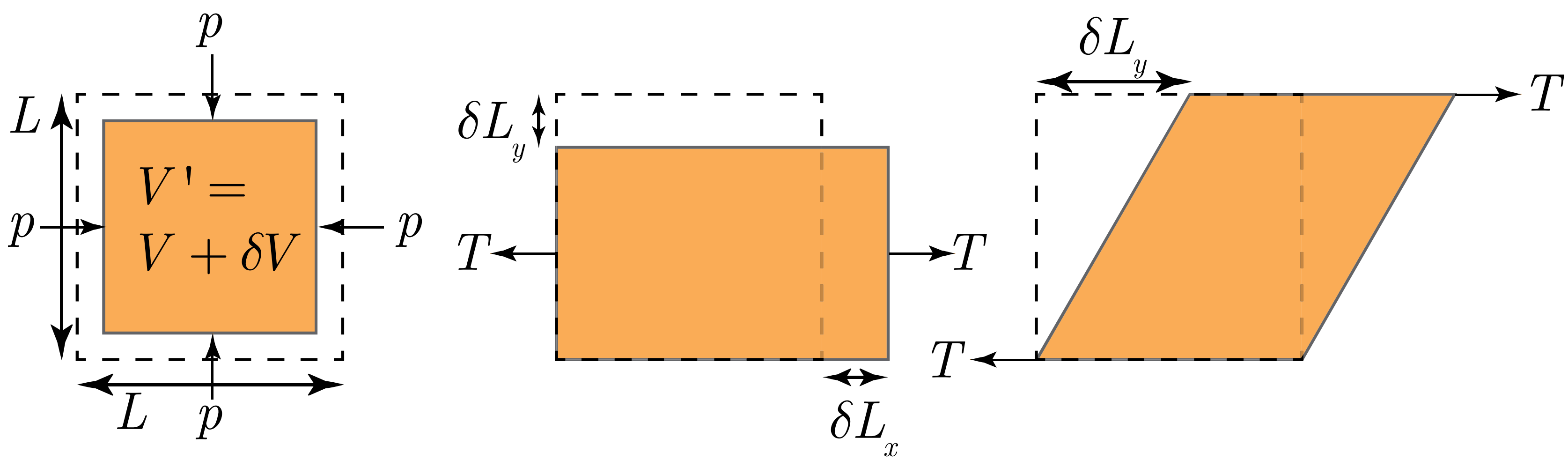
Non-linear strain tensor

Stress tensor →

$$\sigma_{ij} = C_{ijkl} u_{kl}$$

↑ Elastic tensor

Elasticity



Isostatic pressure

$$p = B\frac{\delta V}{V} = Bu_{ii}$$

$B =$ Bulk modulus

Uniaxial tension

Positive Poisson ratio: $\delta L_y \,/\, \delta L_x < 0$

Pure shear

$$T = G_1\frac{\delta L_x}{L} = G_1 u_{xx}$$

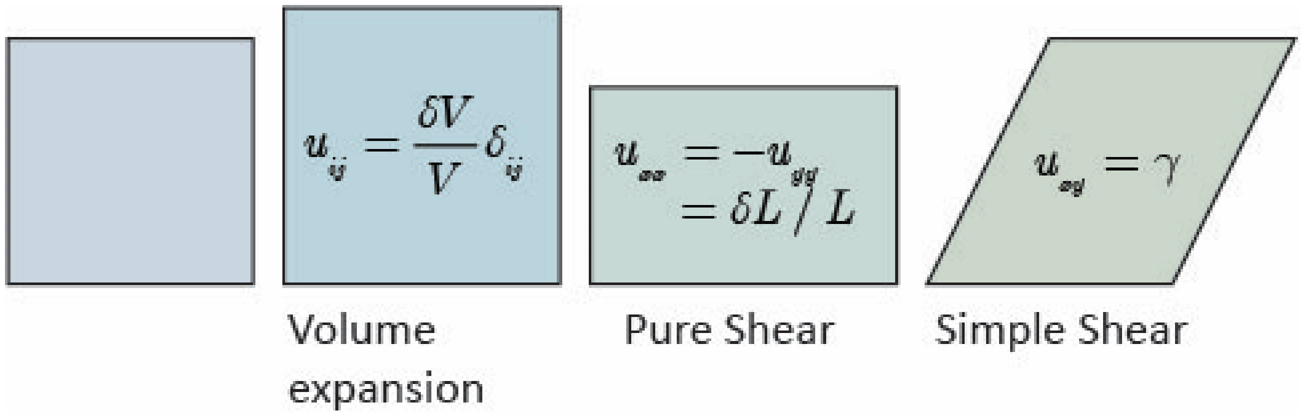
$G_1 =$ Shear modulus

Simple Shear

$$T = G_2\frac{\delta L_y}{L} = G_2 u_{xy}$$

$G_2 =$ Shear Modulus

Elastic Energies and Stresses



u_{ij}= strain tensor

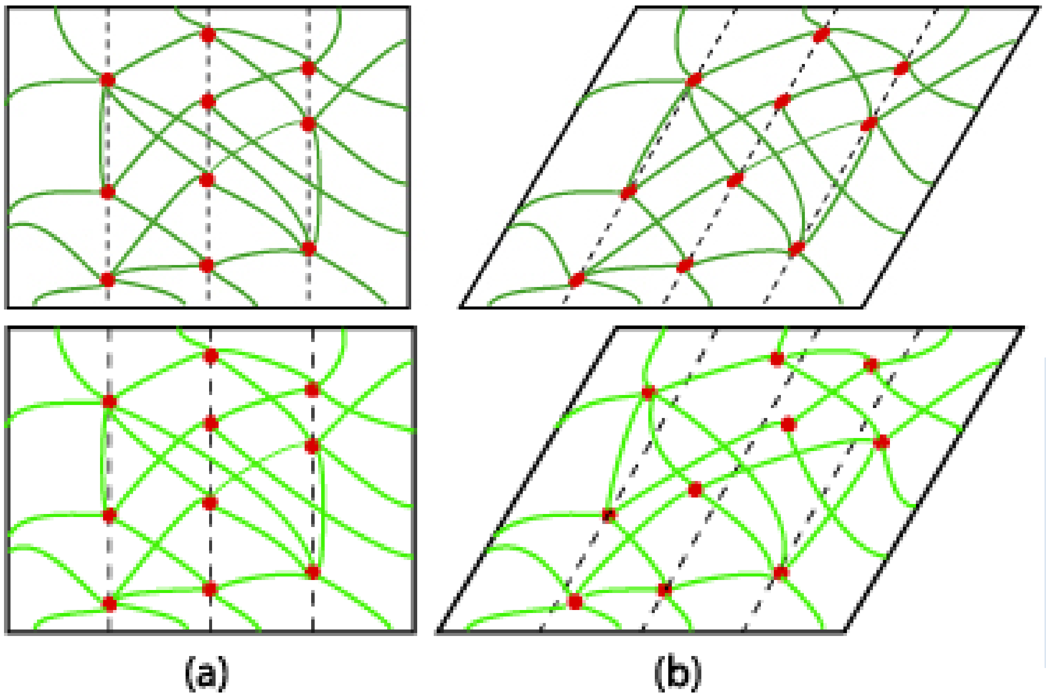
$$F = \frac{1}{2}\int d^d x\, C_{ijkl} u_{ij} u_{kl}$$

$$= \frac{1}{2}\int d^d x [B u_{ii}^2 + 2\mu (u_{ij} - \tfrac{1}{d}\delta_{ij} u_{kk})^2];$$ isotropic

$$= \frac{1}{2}\int d^d x [C_{11}(u_{xx}^2 + u_{yy}^2) + 2C_{12} u_{xx} u_{yy} + 4C_{44} u_{xy}^2];$$ Square

$$\sigma_{ij} = C_{ijkl} u_{kl}; \quad B = \lambda + \frac{2\mu}{d}$$

Nonaffine Response



Affine response: microscopic strain is the same as macroscopic strain. Response to uniform stress in Bravais lattices and homogeneous solids.

Non affine response: local and macroscopic strains differ. Response in and multi-atom periodic unit cells and in random systems

$$R_{i,\text{affine}} = \Lambda_{ij} x_j = x_i + u_{i,\text{affine}}$$

$$\delta u_i(\mathbf{x}) = u_i(\mathbf{x}) - u_{i,\text{affine}}$$

$$\Gamma = \langle (u_i(\mathbf{x}) - u_{i,\text{affine}})^2 \rangle$$

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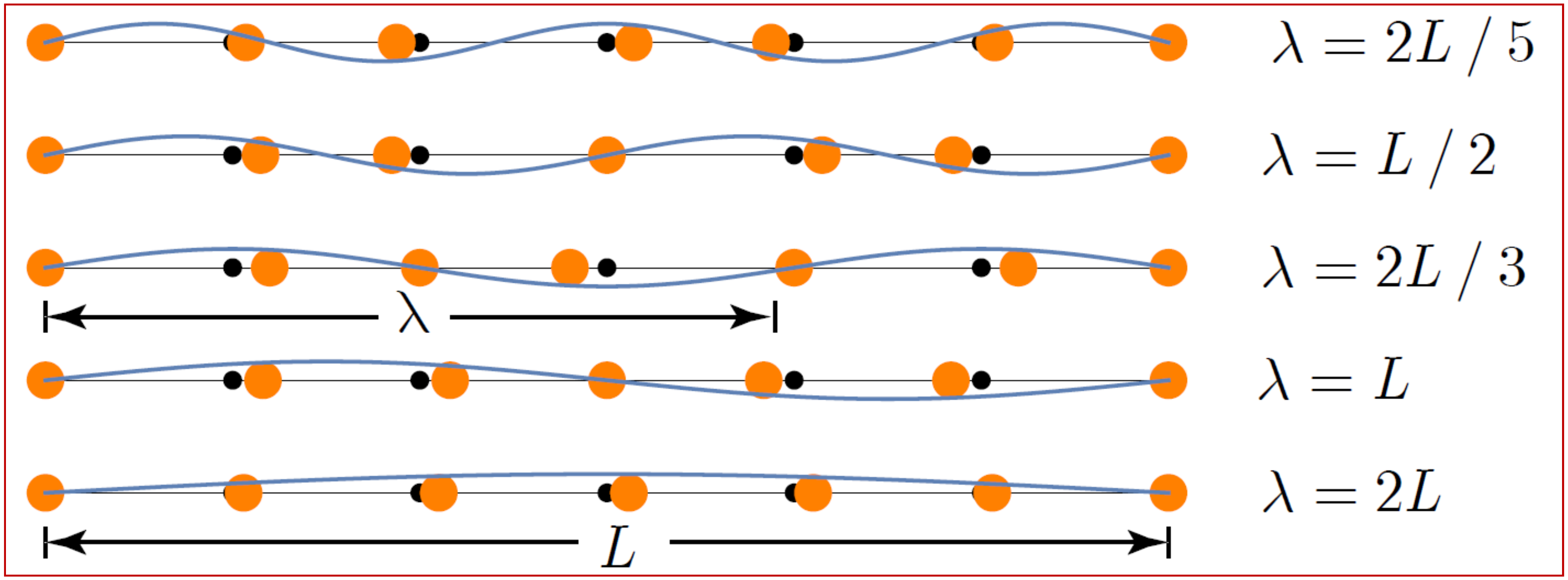
Normal Modes

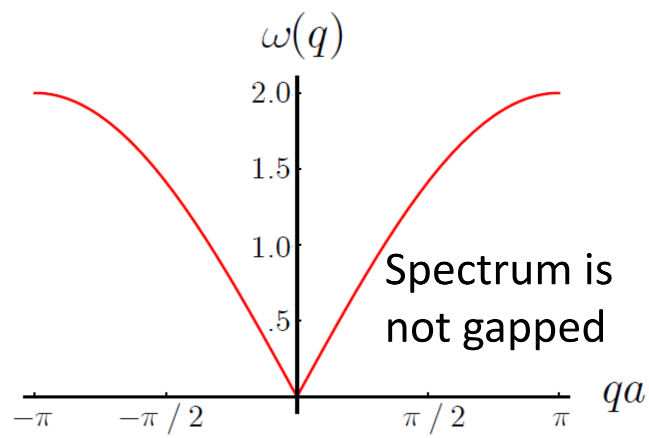


ω : angular frequency $= 2\pi$ / Period

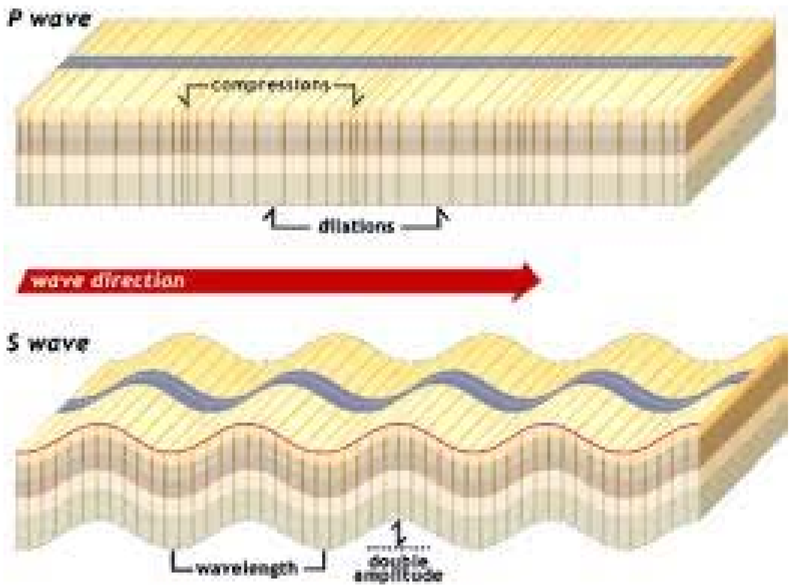
q : wavenumber $= 2\pi$ / wavelength

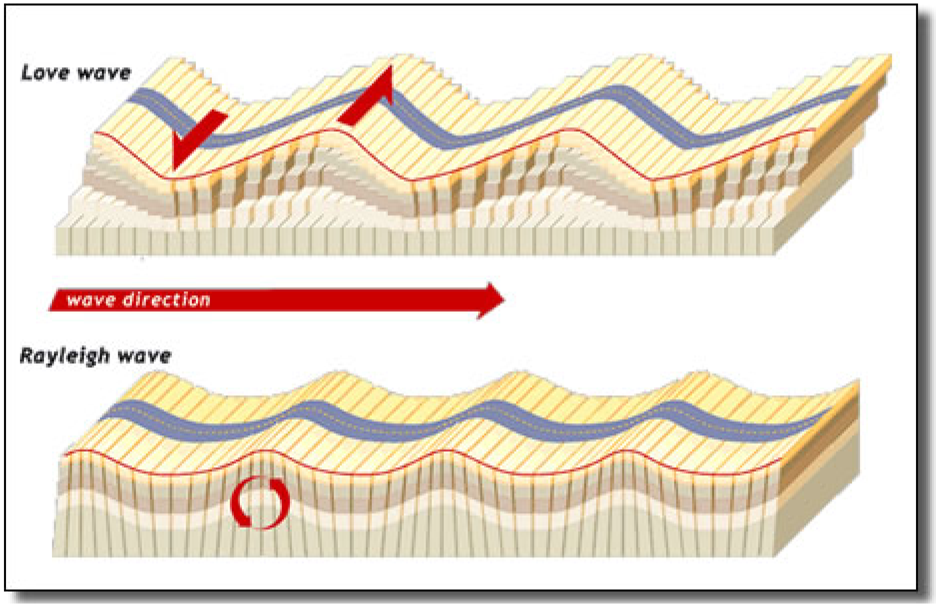
λ : wavelength

$$\omega \to \omega(q) = \begin{cases} cq & \text{sound wave} \\ 2\,\omega_0 \,|\sin(qa/2)| & \text{Lattice wave} \end{cases}$$



Long-wavelength Elastic Waves





Bulk Modes

P Wave - transverse motion:

$\omega_T(q) = c_T q$

S Wave - longitudinal motion:

$\omega_L = c_L q$

Surface Modes

Love - transverse motion:

$\omega_T^L(q) = c_T^L q$

Rayleigh - longitudinal motion:

$\omega_L^R = c_L^R q$

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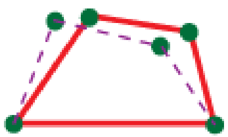
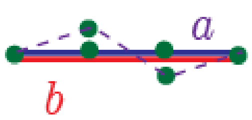
Zero Modes and Self-Stress

(a) 

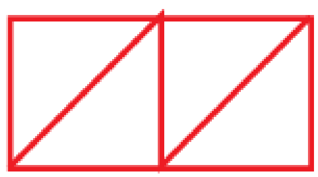
$$N = 4; \quad N_B = 4;$$

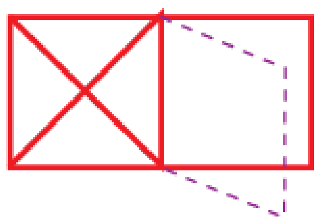
$$N_0 = 2N - N_B = 2 \times 4 - 4 = 4 = 3 + 1$$

There is one finite“floppy” modes in agreement with Maxwell

(b) 

$b{=}3a$: There are now 2 “infinitesimal floppy modes.”
$b{<}3a$: Red is under compression, purple under tension.
There is a state of Self-stress and no floppy modes at all.

(c) 

(d) 

$$N = 6; \quad N_B = 9; \quad dN - N_B = 2 \times 6 - 9 = 3$$

The system is isostatic and there should be no floppy modes. This is the case for (c), but (d) clearly has a floppy mode; it also has a state of self-stress in the crossed square.

New Rule: $S =$ number of states of self-stress
(=dimension of the Null Space of $\mathbf{H}$ in $\mathbf{H}.\mathbf{t} = \mathbf{f}$)

$$N_0 = dN - N_B + S$$

C.R. Calladine, Int. J. of Solid Structures **14**, 161 (1978)

Calladine-Maxwell Theorem

$\mathbf{t} \;=\; N_B$ tensions

$\mathbf{f} \;=\; dN$ -dimensional vector of forces at nodes

$\mathbf{H} \;=\; dN \times N_B$ -dimensional "Equilibium Matrix"

$\mathbf{H} \cdot \mathbf{t} = \mathbf{f}$

$\mathbf{u} \;=\; dN$-dimensional vector of displacements

$\mathbf{e} \;=\; N_B$-dimensional vector of bond stretches

$\mathbf{C} = N_B \times dN$-dimensional

$\mathbf{C} \cdot \mathbf{u} = \mathbf{e}$

$\mathbf{C} = \mathbf{H}^T$

$\mathbf{t} = k\mathbf{e}$

$\mathbf{H} \cdot \mathbf{t} = k\mathbf{H} \cdot e = k\mathbf{H} \cdot \mathbf{C} \cdot \mathbf{u} = \mathbf{f}$

$\mathbf{D} \cdot \mathbf{u} = \mathbf{f}$

$\mathbf{D} = k\mathbf{H} \cdot \mathbf{H}^T$

$= dN \times dN$-dimensional dynamical matrix

$r = Rank(\mathbf{H}) = Rank(\mathbf{H}^T)$

$S = Nullity(\mathbf{H});\ N_0 = Nullity(\mathbf{H}^T)$

$r + S = N_B; \quad r + N_0 = dN$

$N_0 = dN - N_B + S$

Periodic lattices

Reduced RN-theorem for every wavevector

$$Q \to Q_{\alpha\beta}(\ell,\ell') = \frac{1}{N_c}\sum_{\ell} Q_{\alpha\beta}(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{R}_\ell}; \quad Q_{\alpha\beta}(\mathbf{q}): \ dn \times n_b \ \text{ matrix}$$

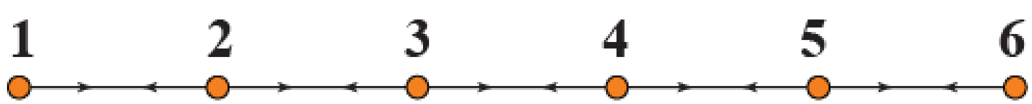
$n =$ No. of sites per cell; $\quad n_b =$ No. of bonds per cell

$N_c =$ No. of cells; $\quad N = N_c n$

$$n_0(\mathbf{q}) - s(\mathbf{q}) = dn - n_b$$

Maxwell Lattice $d\,n = n_b$

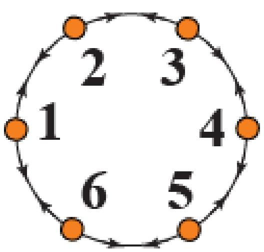
Linear Chain:



$N_b = N - 1; \quad S = 0$

$N_0 = N - (N-1) = 1$

Periodic chain:



$N_b = N; \quad S = 1$

$N_0 = N - N + 1 = 1$

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Voigt Elastic Energy

Strain ε_a has $d(d+1)/2$ independent components

$$f_{el} = \tfrac{1}{2}\varepsilon_a K_{ab} \varepsilon_b = \tfrac{1}{2} K_\lambda \varepsilon_\lambda^2 \quad \text{(Voigt notation)}$$

$$\varepsilon_a = (u_{xx}, u_{yy}, u_{xy}) \quad (2d)$$

$$K_{ab} = \begin{pmatrix} K_{xxxx} & K_{xxyy} & K_{xxxy} \\ K_{xxyy} & K_{yyyy} & K_{yyxy} \\ K_{xxxy} & K_{yyxy} & K_{xyxy} \end{pmatrix}$$

$$K_\lambda = \text{ Eigenvalue of } K; \quad \lambda = 1, \ldots, d(d+1)/2$$

Elastic Energy and States of SS

System requires states of self stress under periodic boundary conditions to support stresses

$$F_{\mathrm{el}} = \frac{1}{2} k \sum_p (\mathbf{e}^{\mathrm{aff}} \cdot \hat{\mathbf{t}}_p)^2;$$

$\mathbf{e}^{\mathrm{aff}}$: Vector of affine bond stretches

$\hat{\mathbf{t}}_p$: Orthnormal basis for $\mathbf{q} = 0$ states of self stress.

Periodic "isostatic" system has at least d $\mathbf{q}=0$ zero modes and as many states of self stress. No extra modes: d vectors in nullspace and only d nonzero eignevalues of K: $d(d+1)/2\text{-}d = d(d\text{-}1)/2$ elastic distrotions of zero energy

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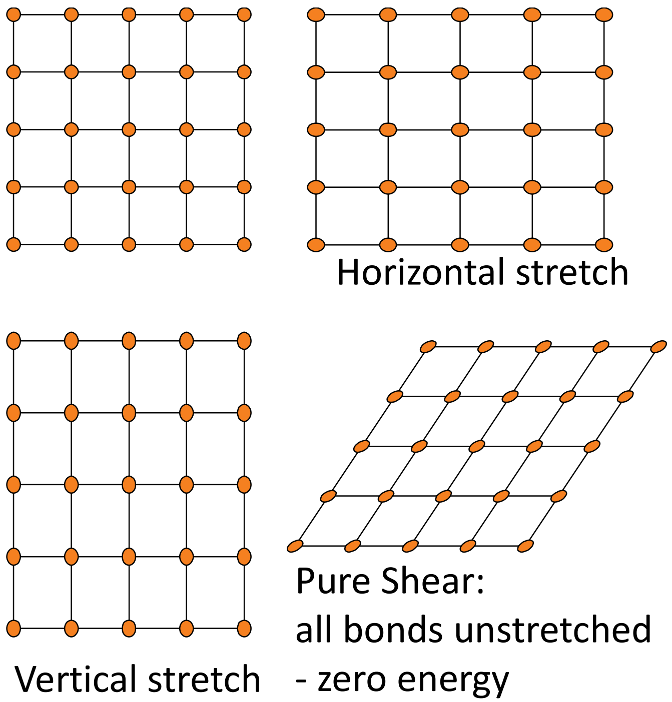
Elasticity of Maxwell Lattices

States of self stress (SSS) at q=0 stabilize the system against elastic distortions. Three independent strains in 2d ($d(d$-1)/2 in general d): Need at least three SSSs. If SSS=2 (or 1), there is 1 (or 2) zero-energy elastic distortions.



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Periodic Maxwell Lattices

Maxwell Lattices : $n_0(\mathbf{q}) = s(\mathbf{q})$

$\mathbf{q} = 0:$ 2 Zero translation modes

$\Rightarrow n_0(0) = s(0) \geq 2$ for $2d$ Maxwell Lattices

Fully Gapped Spectrum $\Rightarrow n_0(0) = s(0)$

and $n_0(\mathbf{q}) = 0$ for all $\mathbf{q} \neq 0$.

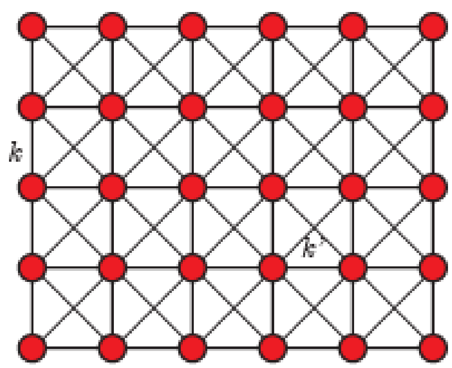
When $s(0) = 2$, There are

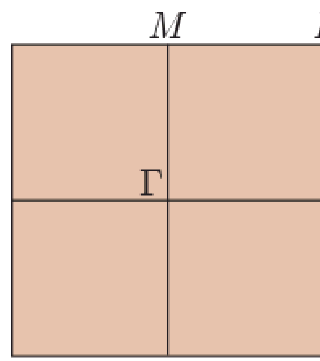
3-2=1 soft elastic distortion: Guest modes.

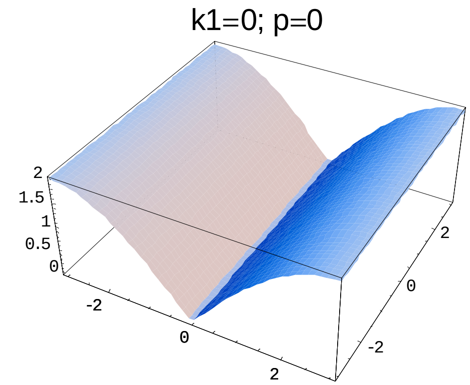
In d dimensions:

$d(d+1)/2 - d = d(d-1)/2$ Guest Modes.

Square Lattice II







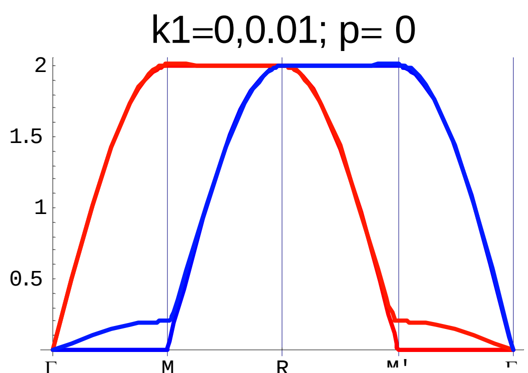
$$C_{11} = (k + k');$$

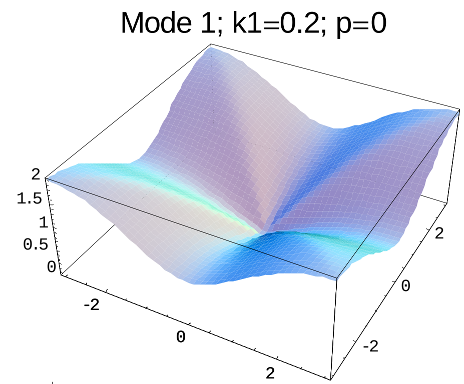
$$C_{44} = k'; \quad C_{12} = k'$$

$$\omega_M^2\left(\frac{G_0}{2}, q_x\right) = 4k' + kq_x^2a^2$$

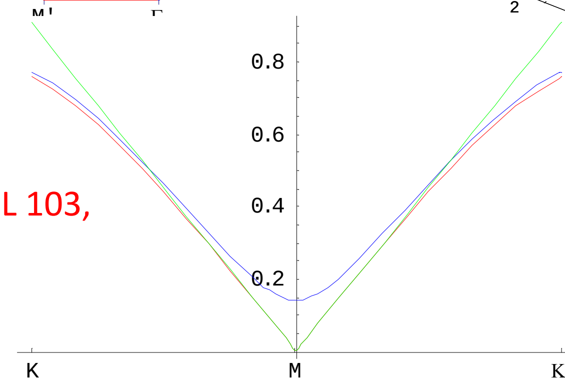
$$\xi = \ell^* \sim \sqrt{\frac{k}{k'}}$$

$$\omega^* = 2\sqrt{k'} = \sqrt{k}(a / \xi)$$



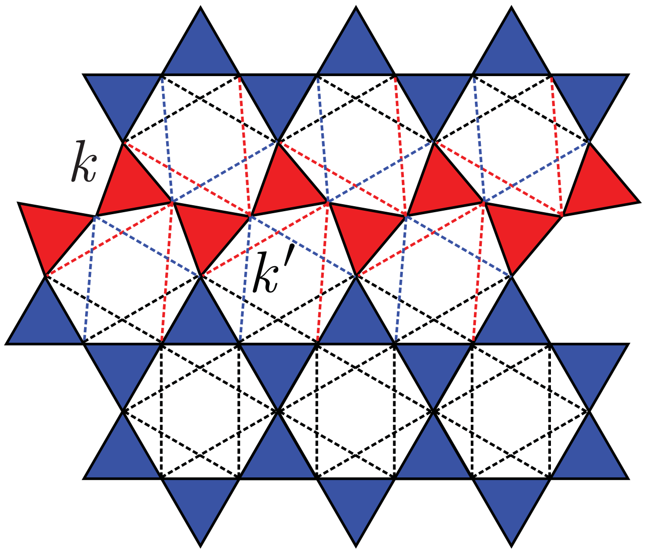


Souslov, Liu, TCL, PRL 103, 205503 (2009)



Kagome Lattice I

z=4 with periodic boundary conditions



NN: spring constant k
NNN: spring constant k'

$k'=0$: $z=2d = 4$: isostatic
No. of zero modes ~ perimeter:
As in square lattice, expect 1D modes

Uniform Elasticity: kagome supports both compression and shear, even though there are $N^{1/2}$ zero modes! And deformations are affine!

$$N_c = N_x N_y; N_s = 3N_c$$
$$N_B = 6(N_x - 1)(N_y - 1) + 4(N_x - 1) + 4(N_y - 1) + 3$$
$$N_0 = 2N_s - N_b = 2(N_x + N_y) - 1$$

$$f_{\text{hex}} = \tfrac{1}{2}\lambda u_{ii}^2 + \mu u_{ij}^2$$
$$\lambda = \mu = \frac{3}{16}k$$
$$B = \lambda + \mu; \quad G = \mu$$

Souslov, Liu, TCL, PRL 103, 205503 (2009)

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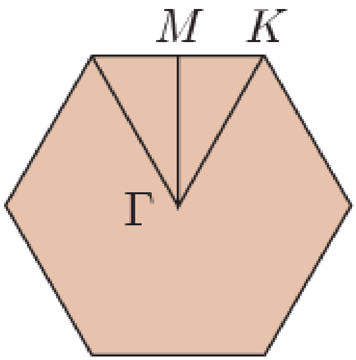
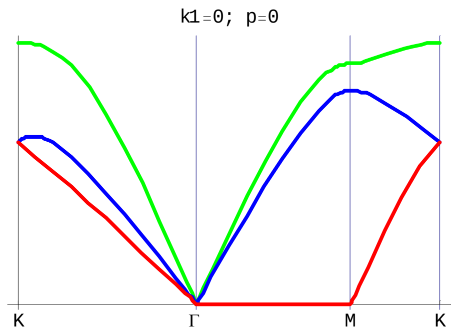
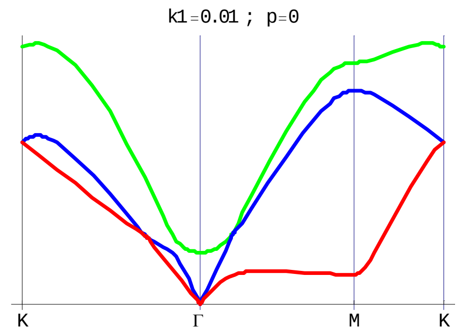
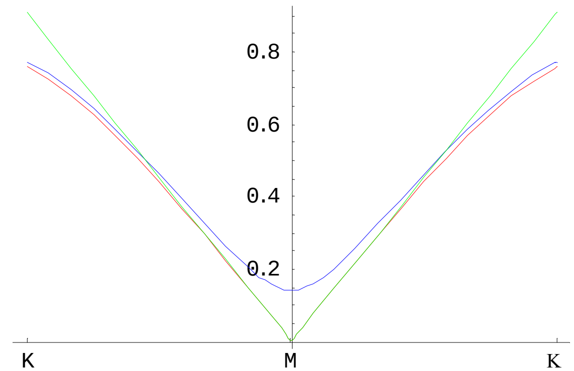
Kagome Lattice II

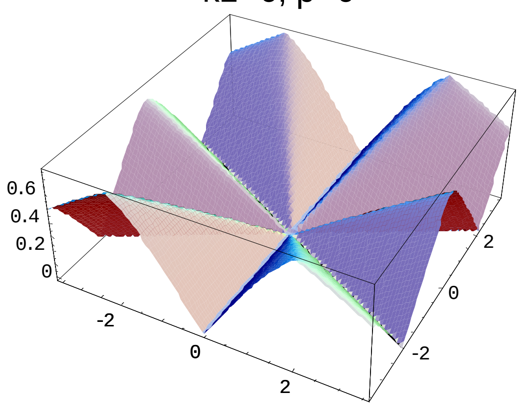




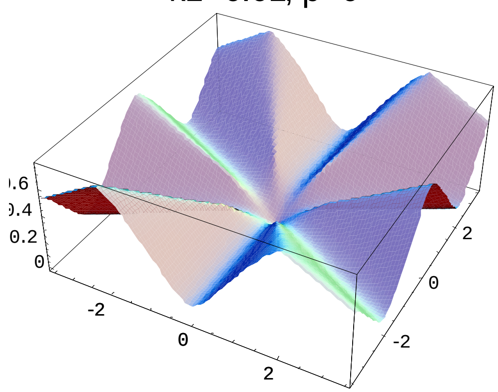




k1=0; p=0



k1=0.01; p=0



$$\omega_M^2\left(\frac{G_0}{2}, q_x\right) = 4k' + \frac{3}{16} k q_x^2$$

$$\omega_\Gamma^2(q) = 6k' + \frac{1}{16} k q^2$$

$$\xi = \ell^* \sim \sqrt{\frac{k}{k'}}; \quad \omega^* = 2\sqrt{k'}$$

Souslov, Liu, TCL, PRL **103**, 205503 (2009)

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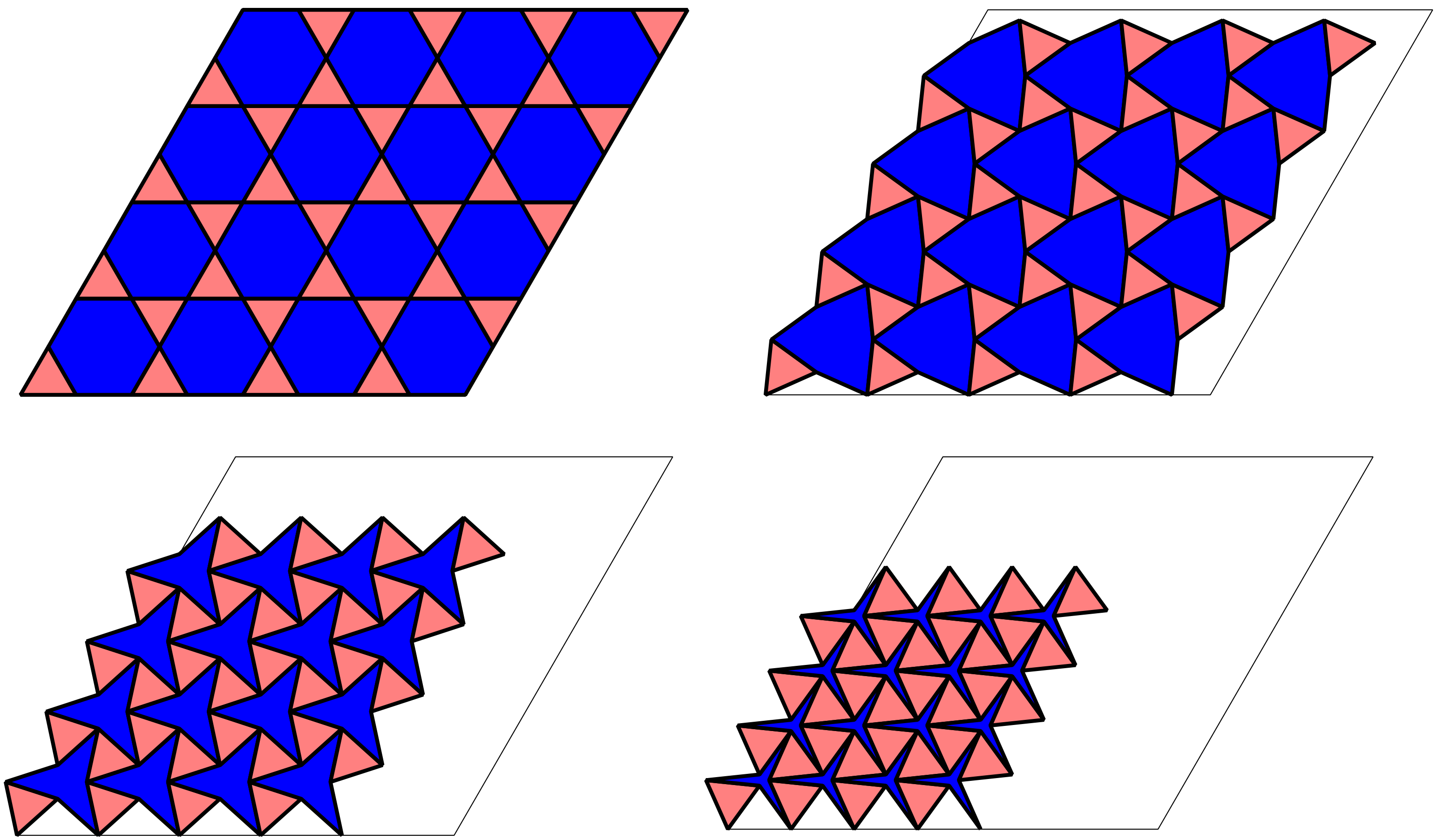
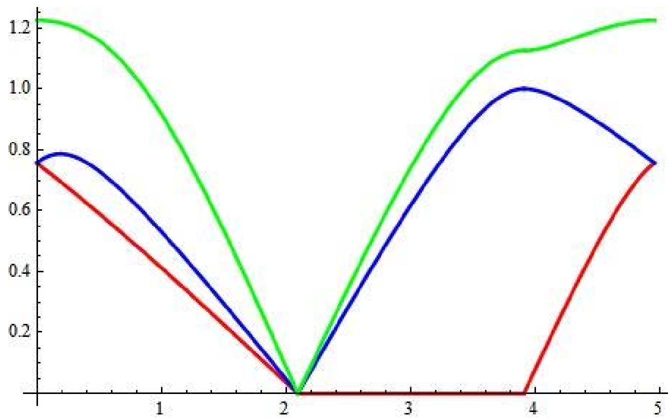
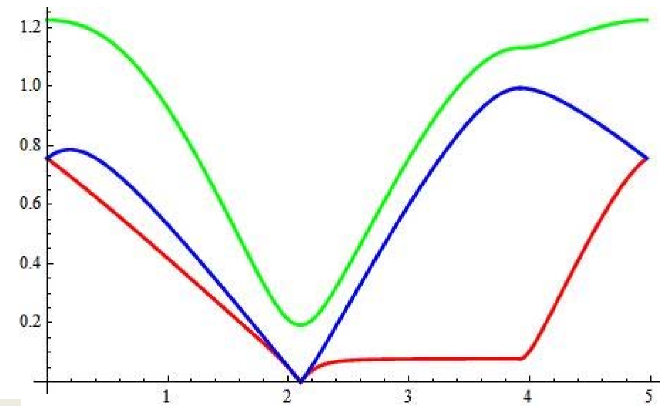
Twisted Kagome I

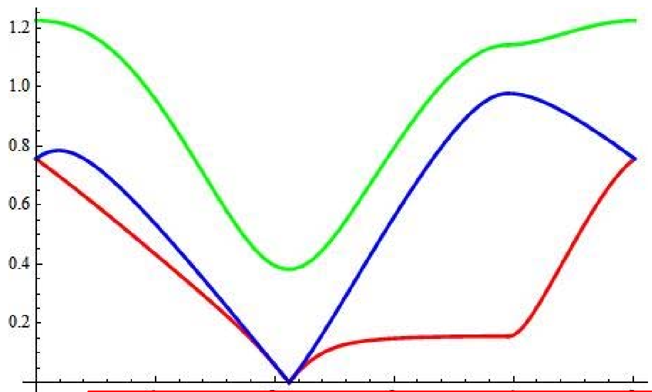


Grima, Alderson, Evans phys. stat. sol. **242**, 561–575 (2005)

Twisted Kagome Phonon Spectrum

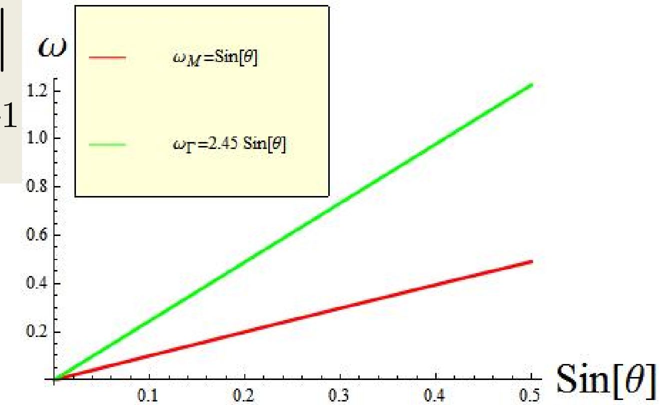






$$\omega_\theta \sim |\sin\theta|$$

$$l_\theta \sim |\sin\theta|^{-1}$$



There are no zero modes in the periodic spectrum even though Maxwell rule for free BC says there should be $\sim N^{1/2}$.

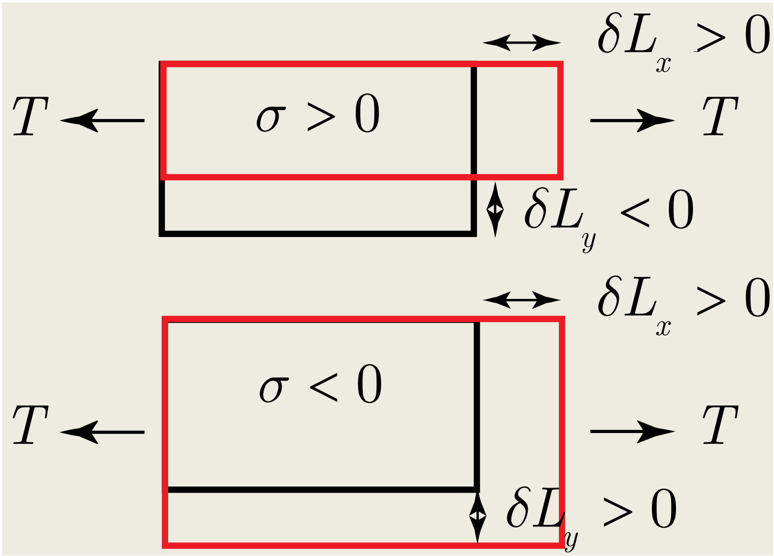
$\sin^2[\theta]$ acts like k' of NNN spring!

Poisson Ratio-General

$$u_{zz} = \frac{T}{Y}; \quad Y = \frac{2d^2 \mu B}{2\mu + d(d-1)B}$$

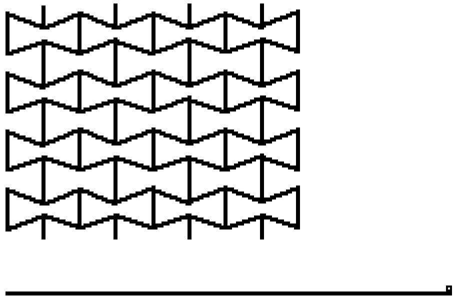
$$u_{xx} = -\sigma u_{zz}; \sigma = \frac{dB - \mu}{2\mu + d(d-1)B}$$

$$\sigma_{2d} = \frac{B-\mu}{\mu + B};$$ negative if $B<\mu$



Negative-Poisson-ratio or Auxetic materials expand perpendicular to stretch

Auxetic Materials

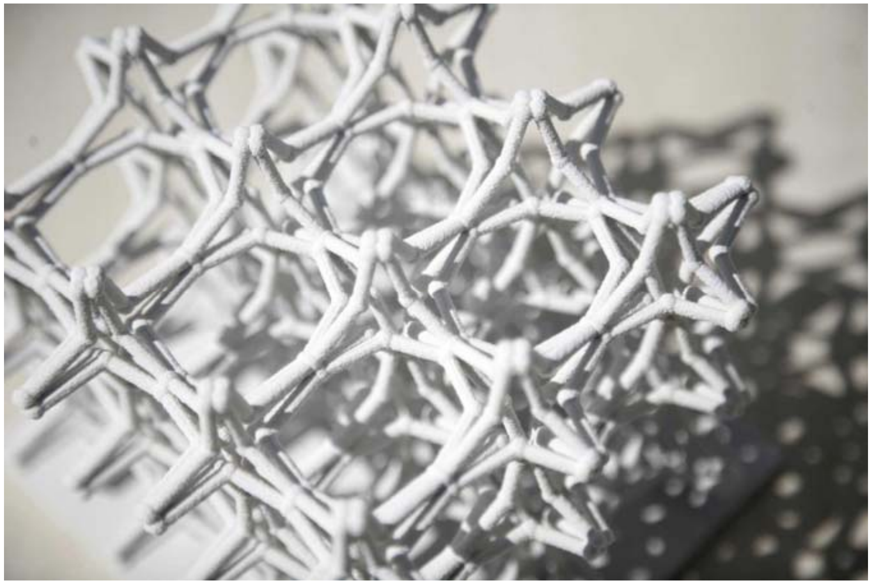


"Classic" inverted honeycomb auxetic lattice

Wine bottle corks have $\sigma \sim 0$; they do not expand when compressed

Rod Lakes, Wisconsin:
Auxetic foam: Nature **235**, 1038 (1987)

purplephoton.com

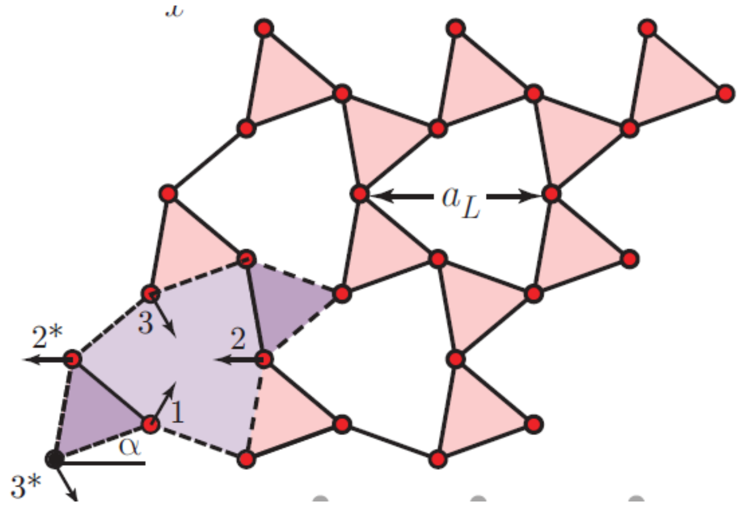


Grasshopped3d.com

Auxetic Response

Gaoxiang Wu†, Yigil Cho†, In-Suk Choi, Dengteng Ge, Ju Li, Heung Nam Han, Tom Lubensky, and Shu Yang, Advanced Materials **27,** 2747 (2015)

Effective Long-wavelength theory



$$f = \tfrac{1}{2} B u_{ii}^2 + \mu (u_{ij} - \tfrac{1}{2}\delta_{ij} u_{kk})^2 + \tfrac{1}{2} A(\alpha - r u_{ii}/2)^2$$
$$\tfrac{1}{2} K (\vec{\nabla}\alpha)^2 + W \Gamma^{ijk} \partial_i u_j \partial_k \alpha$$

When $\alpha > 0$ and $k' = 0$; $B = 0$, $\quad A \sim \sin^2 \alpha$, $\quad r \sim 1/\sin\alpha$

Length: $l_\alpha^2 = K/A \sim |\sin\alpha|^{-2}$;

α relaxes to $r u_{ii}/2$ and only shear energy survives

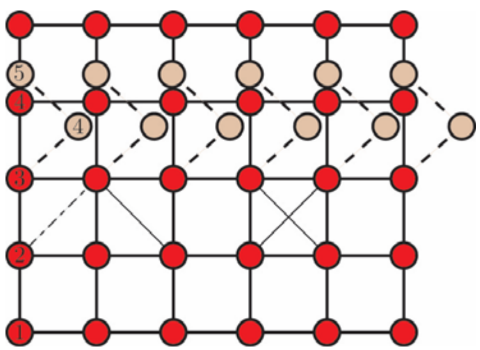
Periodic BC: Missing zero modes

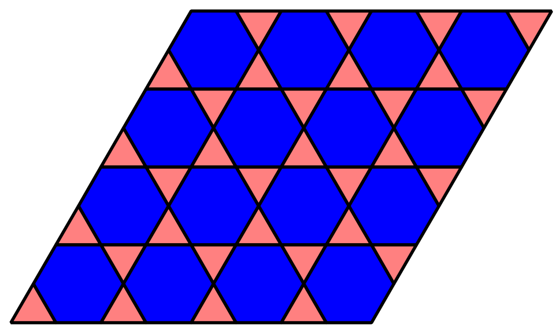
•The square and kagome lattices with periodic boundary conditions: exactly $2d=4$ neighbors per site.

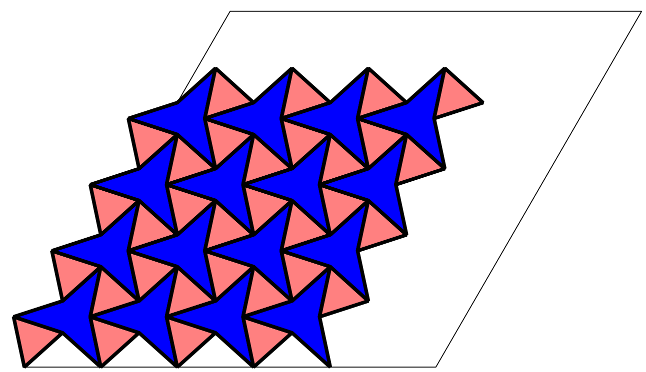
• Maxwell Rule give $N_0=0$ zero modes.

• Square and untwisted kagome lattices: states of self-stress to account for all of the zero modes seen. All zero modes reflect states of self-stress

• Twisted kagome supports only two states of self stress. Stress just bends bonds. There are thus only two zero modes for periodic BC's as observed.







But where are the zero modes for free boundary conditions?
ANSWER – THEY RESIDE IN SURFACE MODES.

Conformal Invariance

$B = 0;$ Long wavelength limit

$$f = \mu(u_{ij} - \tfrac{1}{2}\delta_{ij}u_{kk})^2$$

$u_{ij} = (g_{ij} - \delta_{ij}) / 2$ $\quad g_{ij} =$ metric tensor

$$u_{ij} - \tfrac{1}{2}\delta_{ij}u_{kk} = (g_{ij} - \tfrac{1}{2}\delta_{ij}g_{kk}) / 2$$

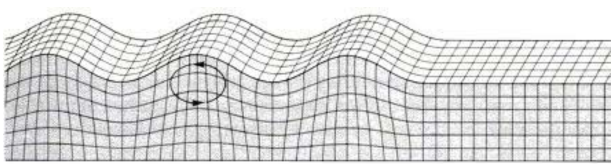
Invaraint under conformal transformation

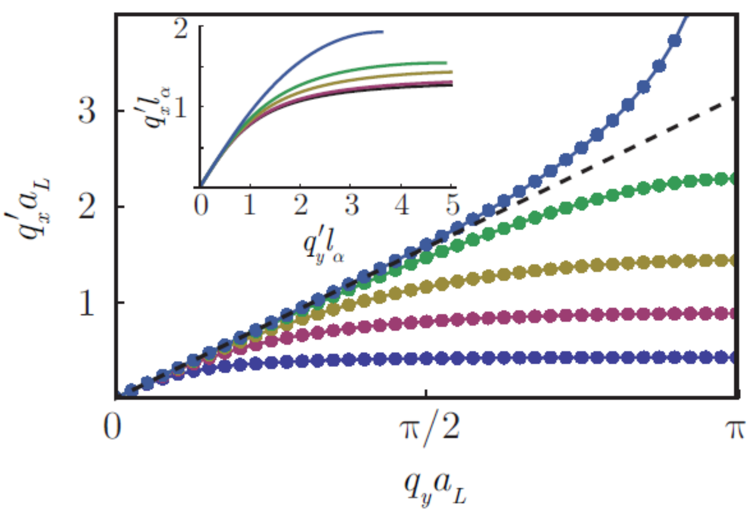
from flat state

$$g_{ij} = h(z)\delta_{ij}$$

where $h(z)$ is any holomorphic function

Surface Modes for Twisted kagome





Zero-frequency surface waves when $B = 0$ with

$$u \sim e^{-q_x y} \cos(q_y x); \quad q_x = g(q_y l_\alpha) / l_\alpha;$$

$$k_y \to \begin{cases} k_x, & s \to 0 \\ 1 / l_\alpha & s \to \infty \end{cases}$$

for $\alpha \to 0$. $\alpha > 0 : q_x$ is complex and depends on both $q_y l_\alpha$ and $q_y a$ unless $l_\alpha >> a$.

Rayleigh Wave at long wavelength:
Number proportional to perimeter.

Kai Sun, Anton Souslov, Xiaoming Mao, TCL, PNAS **108**, 11804 (2011)

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Other Maxwell Lattices