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MODELLING WIND-DRIVEN OCEAN CIRCULATION

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ABSTRACT

In these notes, I describe a linear, continuously stratified ocean model that has been successfully applied to a wide variety of problems in wind-driven ocean circulation. The model is similar to the inviscid model of Lighthill (1969) in that solutions are found as expansions of vertical normal modes; however, the present formalism also allows the vertical mixing of heat and momentum in the deep ocean. These notes are intended to provide an introduction to this kind of ocean modelling. Toward this end, assumptions built into the model equations are carefully listed. Mathematical and conceptual details, not usually dealt with in the literature, are discussed. Finally, a number of solutions, encompassing many different situations of geophysical interest, are found.

Due to the presence of vertical mixing in the model, low-order and high-order modes respond quite differently to the wind. Low-order modes are not strongly affected by vertical mixing. They tend to adjust to Sverdrup balance by radiating Kelvin and Rossby waves from the region of the wind. High-order modes are strongly affected by vertical mixing. They sum to generate the Ekman flow component of the flow field.

CONTENTS

FOREWORD

Dr. Julian McCreary is one of the most notable equatorial oceanographic theoreticians in the world today. His talents are so diverse and numerous that it would be impossible to list them all here. (The editor would surely censor such a list, in any case.)

We had the privilege of a prolonged visit by Dr. McCreary during the summer of 1979. He delivered a series of lectures/seminars on equatorial dynamics. The material was so useful that we decided it should be published as a technical report. The process of producing it has been a long, hard one, and at times, I fear, frustrating for Dr. McCreary. But finally most of the mistakes have been caught, and this magnum opus is ready to go to press.

I hope the reader finds this material useful. In addition to Dr. McCreary himself, special thanks are due Karena Yee, Jan Witte, Wendy Tanaka, and Rita Pujalet for making this volume a reality.

Dennis W. Moore
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INTRODUCTION

A vast number of theoretical studies consider the response of the ocean to a driving wind stress. In many of these studies the ocean model is similar to the linear inviscid model studied by Lighthill (1969) in that solutions are found as expansions of the possible vertical normal modes of the system. This approach has had considerable success in describing transient phenomena that occur in the tropical ocean; in particular, the approach has demonstrated the importance of equatorially trapped baroclinic waves. The inviscid model, however, fails to produce realistic steady ocean circulations; all steady flows are necessarily contained in the surface mixed layer.

In this set of notes I describe an extension of the inviscid model that allows the vertical diffusion of heat and momentum into the deeper ocean. This viscid model retains all the mathematical advantages of Lighthill's approach, and also produces realistic steady solutions. My purpose is to discuss conceptual and mathematical details involved in this type of ocean model that are not usually dealt with in the literature. I hope that those readers who are not familiar with the subject will find these notes to be a useful introduction and that the readers who are familiar with the subject will find them a convenient reference source. The notes are arranged in four major sections in the following manner.

<u>Discussion of the ocean model</u>

In this section I introduce equations of motion and boundary conditions. In order to better understand the limitations of the model, the assumptions built into it are carefully listed. I choose a convenient form for eddy viscosity and diffusivity that still allows solutions to be found as sums of vertical normal modes. It is possible to introduce a "mathematical" mixed layer at the ocean surface simply by assuming that the background density field is homogeneous there. Properties of this mixed layer are discussed. I define a number of terms that are useful for describing the dynamics of the model. Finally, I point out that high-order vertical modes can affect the solutions much more than was suspected by previous workers.

<u>Simple solutions</u>

Here I find solutions to a highly simplified set of model equations. Although solutions of this set are often not a good approximation to those of the exact equations, they can be found with a minimum of mathematical effort. Therefore, they serve to illustrate mathematical techniques used in the more complicated models considered in the later sections. I show how to find the model response to a patch of zonal wind stress when the wind oscillates at frequency σ, and also when it is switched on at some initial time. Solutions are found by using Fourier and Laplace transforms, and some elementary theorems are derived. An important property is that low-order modes can adjust to Sverdrup balance when the winds are steady. Another is that high-order modes generate the Ekman drift associated with the flow field.

Equatorial solutions

The techniques useful for finding solutions in the previous section are again used here to find equatorial solutions. The difference is that now solutions must be found as expansions of Hermite functions, and algebraic nastiness results. (The algebra-to-physics ratio in this section is uncomfortably but unavoidably high!) The model response to both an oscillatory and a switched-on zonal wind field is again determined. The effect of meridional continental barriers on the equatorial flow is discussed. Solutions compare favorably with observations of the Equatorial Undercurrent (McCreary, 1980a, 1980c) and provide a possible explanation for some aspects of El Niño (McCreary, 1976, 1977, 1978). Again, for steady winds low-order modes of the model tend to be in Sverdrup balance. High-order modes tend to be in Yoshida balance, and these modes sum to generate the meridional circulation pattern characteristic of equatorial Ekman pumping.

Eastern coastal solutions

In this section I consider another simplified model that can describe the wind-driven ocean response in the presence of an eastern ocean boundary (like the coast of California). Because this model is so similar to the equatorial model, one might expect solutions to have similar features. In fact, a meridional wind field (with no curl) generates patterns of coastal upwelling and alongshore flow in good agreement with observation. In particular, the model generates a Coastal Undercurrent (McCreary, 1980b, 1980c).

DISCUSSION OF THE OCEAN MODEL

Equations and assumptions

A complete set of non-linear fluid equations is

$$u_t + \left\{ u u_x + v u_y + w u_z \right\} + \gamma w - f v + \frac{1}{\bar{\rho}} p_x = (\nu u_z)_z + E \nabla^2 u,$$

$$v_t + \left\{ u v_x + v v_y + w v_z \right\} + f u + \frac{1}{\bar{\rho}} p_y = (\nu v_z)_z + E \nabla^2 v, \tag{1a}$$

$$w_t + \left\{ u w_x + v w_y + w w_z \right\} - \gamma u + \frac{1}{\bar{\rho}} p_z = -g + (\nu w_z)_z + E \nabla^2 w,$$

$$\rho_t + \left\{ u \rho_x + v \rho_y + w \rho_z \right\} = (K \rho_z)_z + E \nabla^2 \rho.$$

A final equation is conservation of mass

$$\rho_t + \vec{\nabla} \cdot \rho \vec{V} = 0,$$

which, after subtracting out the heat conservation equation, is

$$\rho \, \vec{\nabla} \cdot \vec{V} = -(K \rho_z)_z - E \nabla^2 \rho. \tag{1b}$$

Most oceanographers agree that equations (1), or a similar set, can describe oceanic circulation. These equations, however, have some unpleasant drawbacks. First, they must be solved numerically on a computer with CPU-consuming routines. Second, solutions are often too complicated to provide much insight into the underlying dynamics. The approach that I take here is to study a simpler set of equations that allows solutions to be found analytically. This simplification requires the neglect and/or modification of many of the terms of (1). So here I give up the effort to find exact solutions for the sake of mathematical simplicity and dynamical understanding.

The linearization of equations (1) is standard. The following development is an expansion of the discussion in McCreary (1980a). Similar developments can be found in Veronis (1973), Moore and Philander (1978), and elsewhere.

ASSUMPTION #1: Drop non-linear terms from the momentum equations. This neglect is sensible because there are linear terms that play important (often dominant) roles in these equations. Moreover, only when the linear solution is known can the effects of non-linearities be appreciated. Nevertheless, there are numerous studies that do illustrate the importance of these terms. For example, two recent papers that discuss the effects of non-linearities on the Equatorial Undercurrent are Semtner and Holland (1980), and Philander and Pacanowski (1980). So I classify this assumption as QUESTIONABLE and reserve judgment as to its usefulness until the linear solutions are compared with observations.

ASSUMPTION #2: Drop horizontal Coriolis force terms, γ. There are no studies which suggest that these terms ever play an important role anywhere in the world's ocean. I classify this assumption GOOD.

ASSUMPTION #3: Neglect horizontal friction by setting $E = 0$. There is little evidence that describing horizontal mixing processes with an eddy viscosity is realistic. Models that depend critically on the presence of E are therefore suspect. I classify this assumption SENSIBLE.

ASSUMPTION #4: Keep vertical diffusion of heat and momentum. There are two good reasons for not neglecting vertical mixing as well as horizontal mixing. First, vertical mixing is necessary in order to introduce wind stress into the ocean. Second, only when heat and momentum can diffuse into the deeper ocean do realistic <u>steady</u> solutions result. I classify this assumption NECESSARY.

ASSUMPTION #5: Impose the hydrostatic relation by neglecting w_t and $(\nu w_z)_z$. The neglect of w_t affects only problems where frequencies of the order of the Väisälä frequency or above are involved, and I am not interested in these frequencies. The neglect of $(\nu w_z)_z$ filters out a very thin boundary layer near the ocean surface which is dynamically unimportant for the rest of the flow field. This assumption is GOOD.

ASSUMPTION #6: Impose incompressibility by neglecting the right-hand side of (1b). With a scale analysis, Veronis (1973) showed that for typical wind-driven ocean circulations this assumption is VERY GOOD. Essentially the neglect of these terms amounts to making the Boussinesq approximation. (Veronis also showed that the right-hand side of the heat equation cannot be neglected.)

ASSUMPTION #7: Linearize the heat equation about a steady thermohaline-driven background state. The difficulty with the heat equation is that it involves at least one non-linear term of well-known importance. The vertical advection of the density field can play a powerful role in the dynamics of equations (1) by directly affecting the pressure gradients in the deeper ocean.

It is possible to obtain an approximate linear heat conservation equation that retains this key process. Assume a steady thermohaline-driven background state $\vec{u}_B(x,y,z)$, $\rho_B(x,y,z)$, and $p_B(x,y,z)$. Then wind-driven departures from this background state are given by

$$\rho_t + \left\{u\rho_x + v\rho_y + w\rho_z\right\} + \left[u_B\rho_x + v_B\rho_y + w_B\rho_z\right] + \left(u\rho_{Bx} + v\rho_{By}\right) + w\rho_{Bz} + (\kappa\rho_z)_z . \tag{2}$$

Since the background thermohaline velocity field is not known, it is not possible to retain the terms in square brackets. I ignore the non-linear terms in curly brackets for the same reasons as in Assumption #1. Finally, ρ_B is assumed to be a function of z alone, so that the terms in parentheses are identically zero. This linearization of the heat equation is common; the scheme was first utliized by Fjeldstad (1933).

A large number of guesses are involved in this linearization scheme. Non-linear terms are ignored. The flow field of the thermohaline circulation is not known and so is neglected. Finally, a guess is made for $\rho_B(z)$. I classify the linearization QUESTIONABLE. Its worth can be judged only after a careful comparison of model results to observations.

ASSUMPTION #8: Choose a mathematically convenient form for the mixing of heat and momentum. Mixing coefficients, ν and κ, are generally taken to be constants, but in the real ocean they probably are not. Moreover, there is no reason that the mixing of heat (or of momentum, for that matter) is necessarily described by an expression like $(\kappa\rho_z)_z$. I choose depth-dependent mixing coefficients.

$$\nu = \kappa = A/N_B^2 , \tag{3a}$$

and change the form of the mixing of heat to

$$(\kappa\rho_z)_z \longrightarrow (\kappa\rho)_{zz}. \tag{3b}$$

The choices (3a,b) are crucial to the entire approach taken here. As we shall see, only if these choices are made is it possible to find solutions as expansions of vertical normal modes. Previous workers have already recognized the mathematical convenience of (3a,b); Fjeldstad (1963), and later Mork (1972), used them in their studies of internal waves. Choosing $\nu = \kappa$ amounts to setting the Prandtl number to unity. Equation (3a) states that mixing coefficients decrease wherever background stratification is large. Because stratification does inhibit mixing in the ocean (3a) is physically reasonable (Turner, 1973).

After these assumptions are imposed, equations (1) reduce to the following set:

$$\begin{aligned} u_t - fv + p_x &= \left(\frac{A}{N_B^2} u_z\right)_z, \\ v_t + fu + p_y &= \left(\frac{A}{N_B^2} v_z\right)_z, \\ p_z &= -\rho g, \\ \rho_t + \frac{1}{g} N_B^2 w &= \left(\frac{A}{N_B^2}\rho\right)_{zz}, \\ u_x + v_y + w_z &= 0, \end{aligned} \tag{4}$$

where N_B is the background Väisälä frequency defined by

$$N_B^2 = -\frac{g\rho_{Bz}}{\bar{\rho}}, \tag{5}$$

and factors of $\bar{\rho}$ (the average density of the water column) are dropped. These are the equations of motion discussed in the remainder of these notes.

Boundary conditions and more assumptions

Because there are eight z-derivatives, eight boundary conditions imposed either at the ocean surface or at the bottom are needed to solve this system. At the ocean surface I assume

$$\nu u_z = \tau^x, \quad \nu v_z = \tau^y, \quad w = 0, \quad \rho = 0, \qquad @ \; z = 0. \tag{6}$$

ASSUMPTION #9: Requiring w = 0 at the surface imposes a rigid lid. This assumption only affects the barotropic mode. The transient response of the barotropic mode is now instantaneous rather than just very rapid. The baroclinic modes are weakly affected to order $\varepsilon = \Delta\rho/\rho$. So for the low-frequency or steady problems of interest here, this requirement does not significantly alter the model response. I rate this assumption GOOD.

ASSUMPTION #10: Requiring $\rho = 0$ at the surface assumes that the atmosphere is a constant-temperature source of heat. This choice is unpleasant because it prohibits the ocean model from developing any sea surface temperature changes. A more desirable boundary condition, perhaps, would specify the heat flux at the ocean surface rather than its temperature there. Unfortunately, in order to retain the ability to expand solutions as sums of vertical normal modes the fixed temperature condition is necessary. I rate this assumption UNFORTUNATE, but NECESSARY.

At the ocean bottom I take

$$\nu u_z = 0, \quad \nu v_z = 0, \quad w = 0, \quad \rho = 0, \qquad @ \; z = -D. \tag{7}$$

ASSUMPTION #11: Requiring stress to vanish at the ocean bottom filters out a bottom Ekman layer. This assumption will only be a problem if solutions develop large bottom velocities, but in most cases they do not. For example, McPhaden (1980) has modelled the Equatorial Undercurrent using conditions of no-flow and also no-stress at the ocean bottom. The choice of bottom boundary conditions has little effect on the near-surface currents. Again this assumption is necessary in order to expand the solution into vertical modes. I rate this assumption NECESSARY, and also NOT BAD.

ASSUMPTION #12: Requiring $\rho = 0$ at the ocean bottom assumes that the bottom acts as a constant-temperature source of heat. Because the flow field is surface-trapped, the solution is not strongly affected by the nature of the bottom boundary condition. Again this choice is necessary in order to expand solutions into vertical modes. As in Assumption #11, this choice is NECESSARY, but NOT BAD.

The major point here is that the expansion into vertical normal modes requires specific boundary conditions as well as specific forms for mixing coefficients. Equations (6) and (7) are the only possible choices that allow this expansion. Most of the conditions are reasonable. The most limiting one is the requirement that $\rho = 0$ at the ocean surface, and as a result the model cannot generate sea surface temperature anomalies. An obvious topic for future research is the relaxation of this restriction.

Barotropic and baroclinic modes

It is convenient to rewrite the last three equations of (4). Then equations (4) become

$$\begin{aligned}
u_t - fv + p_x &= A\left(\partial_z \frac{1}{N_B^2}\partial_z\right)u,\\
v_t + fu + p_y &= A\left(\partial_z \frac{1}{N_B^2}\partial_z\right)v,\\
-\left(\partial_z \frac{1}{N_B^2}\partial_z\right)p_t + u_x + v_y &= -A\left(\partial_z \frac{1}{N_B^2}\partial_z\right)^2 p,\\
w = -\frac{1}{N_B^2}&\left[p_t - A\left(\partial_z \frac{1}{N_B^2}\partial_z\right)p\right]_z,\\
\rho &= -\frac{1}{g}p_z.
\end{aligned} \tag{8}$$

The first three equations form a 3 x 3 set that can be solved for u, v and p. With p known, the last two equations then give the w- and ρ-fields.

Note the similarity of the z-operators (in parentheses). This similarity was insured by the choices (3), and is necessary in order to separate out the z-dependence from the problem. The similarity suggests that we expand solutions into eigenfunctions $\psi_n(z)$ that satisfy

$$\left(\partial_z \frac{1}{N_B^2}\partial_z\right)\psi_n = \left(\frac{1}{N_B^2}\psi_{nz}\right)_z = -\frac{1}{c_n^2}\psi_n. \tag{9a}$$

Although it is not yet apparent, boundary conditions on ψ_n which are consistent with (6) and (7) are

$$\frac{1}{N_B^2}\psi_{nz} = 0 \quad @ \quad z = 0, -D. \tag{9b}$$

It is useful to write the eigenfunction problem, (9a,b), in an alternate way,

$$\begin{aligned} \psi_{nz} &= -\frac{N_B^2}{c_n^2}\int_{-D}^{z}\psi_n\,dz, \\ \frac{1}{c_n^2}\int_{-D}^{0}\psi_n\,dz &= 0. \end{aligned} \tag{10}$$

A convenient normalization of these function is

$$\psi_n(0) = 1. \tag{11}$$

Since the ψ_n's are solutions of a Sturm-Liouville equation, it follows that they are orthogonal. Moreover, the eigenfunctions can be ordered so that as n increases c_n decreases; in that case $c_n \propto 1/n$.

I now look for solutions to (8) as expansions of the form

$$\begin{aligned} &u = \sum_{n=0}^{\infty} u_n\psi_n, \qquad v = \sum_{n=0}^{\infty} v_n\psi_n, \qquad p = \sum_{n=0}^{\infty} p_n\psi_n, \\ &w_z = \sum_{n=0}^{\infty} w_n\psi_n \implies w = \sum_{n=0}^{\infty} w_n\int_{-D}^{z}\psi_n\,dz, \\ &\int_{-D}^{z}\rho\,dz = \sum_{n=0}^{\infty}\rho_n\psi_n \implies \rho = \sum_{n=0}^{\infty}\rho_n\psi_{nz}. \end{aligned} \tag{12}$$

A practical model truncates these sums at some number n = N. N must be chosen large enough so that the solutions are well converged.

To find the equations governing the expansion coefficients of (12), I use the standard technique of multiplying each equation of (8) by ψ_n and then integrating over the water column. For example, the left-hand side of the zonal momentum equation becomes

$$\int_{-D}^{0} \psi_n \left[u_t - fv + p_x \right] dz = \int_{-D}^{0} \psi_n \sum_{m=0}^{\infty} \left[u_{mt} - fv_m + p_{mx} \right] \psi_m \, dz$$

$$= \sum_{m=0}^{\infty} \left[u_{mt} - fv_m + p_{mx} \right] \int_{-D}^{0} \psi_n \psi_m \, dz$$

$$= \left[u_{nt} - fv_n + p_{nx} \right] \int_{-D}^{0} \psi_n^2 \, dz .$$

The right-hand side gives

$$\int_{-D}^{0} \psi_n A \left(\partial_z \frac{1}{N_B^2} \partial_z \right) u \, dz = A \psi_n \frac{u_z}{N_B^2} \Big|_{-D}^{0} - \int_{-D}^{0} A \frac{\psi_{nz}}{N_B^2} u_z \, dz$$

$$= A \psi_n \frac{u_z}{N_B^2} \Big|_{-D}^{0} - A u \frac{\psi_{nz}}{N_B^2} \Big|_{-D}^{0} + \int_{-D}^{0} A u \left(\partial_z \frac{1}{N_B^2} \partial_z \right) \psi_n \, dz$$

$$= \psi_n \nu u_z \Big|_{-D}^{0} - A u \frac{\psi_{nz}}{N_B^2} \Big|_{-D}^{0} - \frac{A}{c_n^2} u_n \int_{-D}^{0} \psi_n^2 \, dz$$

$$= \tau^x - 0 - \frac{A}{c_n^2} u_n \int_{-D}^{0} \psi_n^2 \, dz .$$

As another example, consider the w_z-field equation

$$\int_{-D}^{0} w_z \psi_n \, dz = w_n \int_{-D}^{0} \psi_n \, dz = -\int_{-D}^{0} \psi_n \left[\left(\partial_z \frac{1}{N_B^2} \partial_z\right) p_t - A\left(\partial_z \frac{1}{N_B^2} \partial_z\right)^2 p \right] dz$$

$$= -\frac{1}{N_B^2} \left[p_t - A\left(\partial_z \frac{1}{N_B^2} \partial_z\right) p \right]_z \psi_n \Big|_{-D}^{0} + \int_{-D}^{0} \frac{1}{N_B^2} \psi_{nz} \left[p_t - A\left(\partial_z \frac{1}{N_B^2} \partial_z\right) p \right]_z dz$$

$$= \frac{1}{N_B^2} \left[p_t - A\left(\partial_z \frac{1}{N_B^2} \partial_z\right) p \right] \psi_{nz} \Big|_{-D}^{0} - \int_{-D}^{0} \left(\frac{1}{N_B^2} \psi_{nz} \right)_z \left[p_t - A\left(\partial_z \frac{1}{N_B^2} \partial_z\right) p \right] dz$$

$$= \frac{1}{c_n^2} \int_{-D}^{0} \psi_n \, p_t \, dz - \frac{1}{c_n^2} \int_{-D}^{0} \psi_n A \left[\left(\partial_z \frac{1}{N_B^2} \partial_z\right) p \right] dz$$

$$= \frac{p_{nt}}{c_n^2} \int_{-D}^{0} \psi_n^2 \, dz - \frac{g}{c_n^2} \psi_n A \frac{1}{N_B^2} \rho_z \Big|_{-D}^{0} + \frac{1}{c_n^2} \int_{-D}^{0} \frac{\psi_{nz}}{N_B^2} A \, p_z \, dz$$

$$= \frac{p_{nt}}{c_n^2} \int_{-D}^{0} \psi_n^2 \, dz + \frac{A}{c_n^2} \frac{\psi_{nz}}{N_B^2} p \Big|_{-D}^{0} - \frac{A}{c_n^2} \int_{-D}^{0} \left(\frac{\psi_{nz}}{N_B^2} \right)_z p \, dz$$

$$= \frac{p_{nt}}{c_n^2} \int_{-D}^{0} \psi_n^2 \, dz + \frac{A}{c_n^2} \frac{1}{c_n^2} \int_{-D}^{0} \psi_n \, p \, dz$$

$$= \left(\frac{p_{nt}}{c_n^2} + \frac{A}{c_n^4} p_n \right) \int_{-D}^{0} \psi_n^2 \, dz \, .$$

The equations for the coefficients are therefore

$$\left(\partial_t + \frac{A}{c_n^2}\right)u_n - f v_n + p_{nx} = F_n,$$

$$\left(\partial_t + \frac{A}{c_n^2}\right)v_n + f u_n + p_{ny} = G_n,$$

$$\left(\partial_t + \frac{A}{c_n^2}\right)\frac{p_n}{c_n^2} + u_{nx} + v_{ny} = 0, \tag{13a}$$

$$w_n = \left(\partial_t + \frac{A}{c_n^2}\right)\frac{p_n}{c_n^2}, \quad \rho_n = \frac{p_n}{g},$$

where

$$F_n = \frac{\tau^x}{\int_{-D}^{0} \psi_n^2\, dz}, \qquad G_n = \frac{\tau^y}{\int_{-D}^{0} \psi_n^2\, dz}. \tag{13b}$$

The first three equations form a 3 x 3 set that can be solved for u_n, v_n and p_n. With p_n known, the other coefficients, w_n and ρ_n, are also known. Equations (12) then specify the complete flow field.

Note that friction enters (13) in the form of simple drag, with a drag coefficient A/c_n^2. Because c_n decreases like $1/n$, even if drag is unimportant for the low-order modes, eventually it will dominate the dynamics of the high-order modes. As we shall see, the fact that drag is mode-dependent in this model is crucial for its ability to generate realistic flow fields.

PROBLEM #1: Carry out appropriate integrations of all the equations of motion and show that equations (13) result.

A possible eigenvalue of (10) is $c_o = \infty$. For this eigenvalue,

$$\psi_n(z) = 1,$$

that is, the barotropic mode is depth independent. Vertical friction has no effect on this mode since $A/c_o^2 = 0$. The other eigenvalues are finite. Equation (10) then requires that

$$\int_{-D}^{0} \psi_n(z)\, dz = 0.$$

Thus, for these modes horizontal transport vanishes. As modenumber, n, increases these modes are more and more wiggly. It is reasonable, then, that as the modenumber increases the effect of vertical friction also increases markedly.

A surface mixed layer

It is possible to introduce a surface mixed layer into the model simply by assuming that $N_B^2 = 0$ for $z > -H$. The eigenvalue problem defined in (10) is still well posed, and so the eigenfunctions as well as solutions (12) remain well behaved. [Strictly speaking, since factors of N_B^2 appear in the denominators of several quantities, one must first take $N_B(z) = \varepsilon$ for $z > -H$, find the solutions (12), and then determine the limit $\varepsilon \to 0$.]

Properties of the mixed layer. This mixed layer has been introduced entirely in a mathematical way with no appeal to physics. We should reflect a bit on its properties to see whether it is at all physically meaningful. According to (10) and (12), the mixed layer has the following properties:

1. ν and κ are infinitely large.
2. u, v, p are independent of z.
3. w varies linearly.
4. ρ is identically zero.
5. Stress, defined by $\vec{\tau} = \lim_{\varepsilon \to 0} \nu \vec{u}_z$, varies linearly from a surface value set by the wind to a value at $z = -H$ set by matching to the deeper ocean. Therefore, wind stress acts as a body force in the layer. Because the normal mode expansion for the stress is not uniformly convergent, this property is not obvious, and no proof will be attempted here.
6. Heat flux, defined by $Q = \lim_{\varepsilon \to 0} (\kappa\rho)_z$, is constant in the layer. So whatever heat is absorbed (lost) at the interface is lost (absorbed) at the surface.

All of these properties are those we might expect in a real well-mixed layer.

Relationship to the inviscid model. Now the analogy with Lighthill's inviscid model can be seen most clearly. Lighthill assumed a well-mixed surface layer just as I have done here, but took the deeper ocean to be inviscid. As a result, stress in his surface layer varies linearly from a surface value set by the wind to zero at the base of the mixed layer. It is interesting that Lighthill's model is just (13) with A (the measure of viscosity in the deeper ocean) set to zero.

Although the inviscid model has successfully described a number of time-dependent ocean circulation problems, it cannot describe steady flows at all. Reasons for this failure are evident in (13). When $\partial_t = 0$ and $A = 0$, equations (13a) imply that $w_n = 0$; thus, $w = 0$ and the inviscid model cannot describe steady upwelling. Furthermore, in these limits the first three equations of (13a) can be summed exactly. They imply that steady inviscid flow is all contained in the surface mixed layer. There is no vertical structure that in any way resembles the observed steady equatorial or coastal flow fields.

Limitations. The mixed layer introduced in this way has the unpleasant property that it can generate density inversions at the base of the mixed layer. For example, in the steady state whenever there is downwelling of fluid out of the layer a density inversion occurs. To see this, first realize that

$$\psi_{nz} > 0, \quad \int_{-D}^{z} \psi_n \, dz < 0 \qquad @ \ z = -H.$$

Then, according to (12) and (13a),

$$w_\rho > 0 \qquad @ \qquad z = -H. \tag{14}$$

Because of the linearization of the heat equation, such distortions of the density field are expected. Equation (14) occurs because the linearization keeps the depth of the mixed layer fixed during upwelling and downwelling. We can, perhaps, interpret a region of density inversion at the base of the mixed layer as a deepening of the mixed layer there.

Useful definitions

One of the benefits of theory is that it provides a language that can subsequently be used to describe the real world. For example, the terms "Ekman flow" and "Sverdrup flow" are commonly used to discuss ocean circulation. Yet neither of these terms is capable of describing the baroclinic structure of the circulation (because they emerged from the study of ocean models that ignore that structure). An adequate language does not yet exist with which baroclinic flows can be discussed. As a result it is difficult to describe the dynamics of this continuously stratified model. I have found it useful to introduce the following definitions.

SVERDRUP BALANCE: A vertical mode in which the dominant balance of terms is

$$
\begin{aligned}
-fv_n + p_{nx} &= F_n , \\
fu_n + p_{ny} &= G_n , \\
u_{nx} + v_{ny} &= 0 ,
\end{aligned}
\tag{15}
$$

is said to be in Sverdrup balance. An ocean flow in which all the modes are in Sverdrup balance is a Sverdrup flow. These definitions are appropriate because (15) is just the balance of terms of the "classic" depth-averaged Sverdrup theory, now applied to each mode.

EKMAN BALANCE: A mode in which the dominant balance of terms is

$$
\begin{aligned}
\left[\frac{A}{c_n^2} u_n\right] - fv_n &= F_n , \\
\left[\frac{A}{c_n^2} v_n\right] + fu_n &= G_n , \\
\frac{A}{c_n^4} p_n + u_{nx} + v_{ny} &= 0 ,
\end{aligned}
\tag{16}
$$

is said to be in Ekman balance. An ocean flow in which all the modes are in Ekman balance is an Ekman flow. These definitions are appropriate because the solution converges to a flow field very much like the classic Ekman spiral at mid-latitudes. Note that at the equator the response of the barotropic mode is undefined, demonstrating the singularity of mid-latitude Ekman theory as $f \to 0$.

PROBLEM #2: Assume that N_B is a constant. Then the vertical normal modes are cosines, and ν and k are constants. Assume that <u>all</u> modes of the model are in Ekman balance and find the solutions in the form (12). Finally, sum the series (a useful reference is Jolley, 1961), and show that the solutions are exactly the classic Ekman spiral.

PSEUDO-EKMAN BALANCE: In this balance the terms in square brackets in (16) are absent. An ocean flow in which all the modes are in pseudo-Ekman balance is a pseudo-Ekman flow. The mid-latitute solution no longer converges to an Ekman spiral. Instead, all motion is contained in the surface mixed layer, and the layer moves as a slab at right angles to the wind. If there is no surface mixed layer, the solution does not converge. Pseudo-Ekman flow is undefined at the equator because the barotropic mode has no solution there.

YOSHIDA BALANCE, X-INDEPENDENT FLOW: An interesting balance of terms occurs when the driving wind stress is assumed to be x-independent. The flow field is therefore also x-independent and in steady state is

$$
\begin{aligned}
\frac{A}{c_n^2} u_n - f v_n &= F_n, \\
\left[\frac{A}{c_n^2} v_n\right] + f u_n + p_{ny} &= G_n, \qquad (17) \\
\frac{A}{c_n^4} p_n + v_{ny} &= 0.
\end{aligned}
$$

Away from the equator this balance is essentially the Ekman balance. Equations (17) differ fundamentally from the simpler Ekman balance in that the baroclinic pressure gradients, p_{ny}, are retained. As a consequence, strong equatorial zonal jets are in geostrophic balance, and the equatorial Rossby radius emerges as a natural width scale. I refer to equations (17) as the Yoshida balance, and an oceanic flow in which all modes are in Yoshida balance as a Yoshida flow. [Yoshida, 1959, was first to apply a system of equations like (17) in a wind-driven model of the Equatorial Undercurrent.] A Yoshida-balanced barotropic mode is undefined at the equator.

PSEUDO-YOSHIDA BALANCE: In this balance the term in square brackets in (17) is absent. An oceanic flow in which all modes are in pseudo-Yoshida balance is a pseudo-Yoshida flow. At mid-latitudes this balance is the pseudo-Ekman balance. At the equator the response of the barotropic mode is undefined.

LOCAL DYNAMICS: A balance in which the effects of radiation are not important. In such a dynamics the response of the mode looks very much like that of the driving wind. More precisely, if $\tau = \tau_0 X(x) Y(y)$, then the model response is local provided that it is proportional to

$X(x)Y(y)$. (An example is a model in which all modes are in Ekman balance.) Along ocean boundaries and the equator I relax this definition to include one or the other of $X(x)$ or $Y(y)$. In other words, along a meridional coast the dynamics is said to be local provided the model response is proportional to $Y(y)$, but not necessarily $X(x)$. At the equator the dynamics is local if the response is proportional to $X(x)$, but not necessarily to $Y(y)$.

NON-LOCAL DYNAMICS: A balance in which radiation can occur and plays a strong role. The response of a particular mode does not look like the driving wind. (For example, in the Sverdrup balance the response of u_n and p_n involves zonal integrals of X, and v_n involves Y_y.) If the forcing is confined to a wind patch of limited extent then there is a significant response outside the region of the wind.

The importance of high-order modes

Previous workers have argued that only the lowest-order vertical modes are needed to describe the response of ocean models (Lighthill, 1969). They assume that a measure of the amplitude of the response of each mode is simply F_n. Since F_n decreases rapidly with n for realistic choices of $\rho_B(z)$ (McCreary, 1977), their conclusion follows. The flaw in this argument is that the model response is not simply measured by F alone. In fact, in the ocean model discussed here, high-order modes are highly visible in the solutions.

There are two major reasons for this visibility. First, the response of the model depends on c_n as well as F_n. For example, in some of the solutions a measure of the response, q_n, is

$$q_n \propto \frac{F_n}{c_n} \quad \text{or} \quad q_n \propto \frac{F_n}{\sqrt{c_n}} .$$

Because $c_n \propto 1/n$, these measures weaken much more slowly with n than does F_n alone (Merrill and Geisler, 1980). Second, for steady winds the presence of radiation in the model allows low-order modes to come into Sverdrup balance. If the wind has no curl then there is no flow associated with these low-order modes and they never appear in the flow field.

SIMPLE SOLUTIONS

In this section I discuss a popular simplification of (13) that ignores both the drag and acceleration terms in the momentum equations of (13). The equations of motion reduce to

$$
\begin{aligned}
-fv_n + p_{nx} &= F_n, \\
fu_n + p_{ny} &= 0, \\
i\omega \frac{p_n}{c_n^2} + u_{nx} + v_{ny} &= 0,
\end{aligned}
\tag{18a}
$$

where

$$
i\omega = \partial_t + \frac{A}{c_n^2}. \tag{18b}
$$

For convenience, I ignore meridional winds throughout this section.

Many of the variables in subsequent equations should be labelled with a subscript n. For notational simplicity I ignore this subscript whenever possible. I retain it only if confusion might result from its absence.

Are equations (18) a reasonable simplification to make? To discuss this question, I compare equations involving v alone. The exact v-equation resulting from (13) is

$$
\begin{aligned}
\left[\overset{(b)}{i\omega\left(v_{xx} + v_{yy}\right)} + \overset{(a)}{\frac{i\omega^3}{c^2}v}\right] - \overset{(c)}{i\omega\frac{f^2}{c^2}v} + \beta v_x \\
= i\omega\frac{f}{c^2}F - F_{yx}.
\end{aligned}
\tag{19}
$$

The approximate v-equation resulting from (18) is

$$
-i\omega\frac{f^2}{c^2}v + \beta v_x = i\omega\frac{f}{c^2}F - F_{yx}. \tag{20}
$$

Equation (20) differs from (19) in the absence of the terms in square brackets. So, equations (18) will be a useful approximation only if the neglect of these terms does not seriously alter the nature of the solutions.

Clearly, the approximation is valid only if term (a) << term (c). This property will hold provided that

$$\partial_t = \frac{1}{T} < f \quad \text{and} \quad \frac{A}{c^2} < f.$$

The first inequality requires that the time scale T of the forcing is much longer than the local inertial period. Thus, equations (18) cannot describe the generation of inertial or gravity waves in the ocean. In fact, these waves are completely filtered out of this model. The second inequality cannot always be valid. For the low-order modes it is true that $A/c^2 < f$, but eventually for the high-order modes the sense of this inequality is reversed. This error is simply an indication that the high-order modes of (18) are in pseudo-Ekman balance, whereas those of (13) are in Ekman balance. Therefore, the distortions caused by the absence of term (a) are well understood; its neglect is not serious.

It is also necessary that term (b) << term (c), which requires that

$$L_x > \alpha^{-1} \quad \text{and} \quad L_y > \alpha^{-1},$$

where $\alpha^{-1} = c/f$. In other words, the horizontal length scales of solutions to (18) must always be greater than the local Rossby radius, α^{-1}. Solutions almost always fail to satisfy one or the other of these requirements. For example, near an eastern ocean boundary, solutions develop narrow coastal jets for which $L_x \lesssim \alpha^{-1}$. Even in the ocean interior, well away from ocean boundaries, these inequalities are not always satisfied [see discussion following (25)]. Therefore, we can expect that distortions caused by the absence of term (b) will be quite severe, and the use of (18) to model the real ocean is suspect.

<u>Oscillating winds</u>

I show here how to solve (18) when the wind field has the form

$$\tau^x = \tau_0 X(x) Y(y) e^{i\sigma t}$$
$$\Rightarrow \quad F_n = \frac{\tau_0 X(x) Y(y)}{\int_{-D}^{0} \psi_n^2 \, dz} e^{i\sigma t} \equiv \tau_{on} X(x) Y(y) e^{i\sigma t}, \tag{21}$$

where τ_{on} is obviously defined. The functions X(x) and Y(y) are assumed to be zero for both x and y sufficiently large, so this wind field describes a wind patch of limited extent. Note that with this choice of wind field $i\omega$ is no longer an operator; rather it is just the complex number

$$i\omega = i\sigma + \frac{A}{c_n^2}. \tag{22}$$

A straightforward method of solution is to solve (18) by taking Fourier transforms in x. Denote any transformed variable by a tilde, that is, the Fourier transform of q(x) is $\tilde{q}(k)$. Then according to (20) $\tilde{v}$ is given by

$$-i\omega \frac{f^2}{c^2}\tilde{v} + \beta ik\tilde{v} = i\omega\frac{f}{c^2}\tilde{F} - \tilde{F}_{yx}$$

$$\Longrightarrow \quad \tilde{v} = \frac{\frac{\omega f}{c^2}\tilde{F} + i\tilde{F}_{yx}}{\beta\left(k - \frac{\omega}{\beta}\frac{f^2}{c^2}\right)} . \tag{23}$$

To invert this Fourier transform I use the easily proved theorem

$$\frac{\tilde{g}}{k-a} \leftrightarrow i\, e^{iax}\int_L^x e^{-iax} g\, dx . \tag{24}$$

The choice of lower limit, L, is not specified in (24). We must appeal to the physics of the problem to set this lower limit [see discussion of (28)]. With the aid of (24) it follows that

$$v = i\frac{\omega f}{\beta c^2}\, e^{i\frac{\omega}{\beta}\alpha^2 x}\int_L^x e^{-i\frac{\omega}{\beta}\alpha^2 x} F\, dx$$

$$- \frac{1}{\beta}\, e^{i\frac{\omega}{\beta}\alpha^2 x}\int_L^x e^{-i\frac{\omega}{\beta}\alpha^2 x} F_{yx}\, dx . \tag{25}$$

It is easy to see that (25) is not accurate for sufficiently large values of x or y. For example, assume that the ocean is inviscid and that $f = \beta y$. I rewrite the exponential in the alternate ways

$$e^{i\frac{\sigma}{\beta}\alpha^2 x} = e^{i\left(\frac{\sigma y}{c}\alpha\right)x} = e^{i\left(\frac{\sigma x}{c}\alpha\right)y} .$$

Thus, (25) oscillates with wavelengths $\lambda_x = 2\pi/(\frac{\sigma y}{c}\alpha)$ and $\lambda_y = 2\pi/(\frac{\sigma x}{c}\alpha)$ in the zonal and meridional directions, respectively. If $x > c/\sigma$, then $L_y < \alpha^{-1}$; if $y > c/\sigma$, then $L_x < \alpha^{-1}$. The distance c/σ need not be too large. A typical value of c for the n = 2 vertical mode is 100 cm/sec. At the annual frequency, then, c/σ is just 5000 km. Schopf et al. (1980) discuss this breakdown of (25) in greater detail.

Evidence of radiation. Each term of the solution (25) has the form

$$\left\{ (\text{const}) \int_L^x e^{-ikx} X \, dx \right\} e^{ikx + i\sigma t} , \tag{26}$$

where

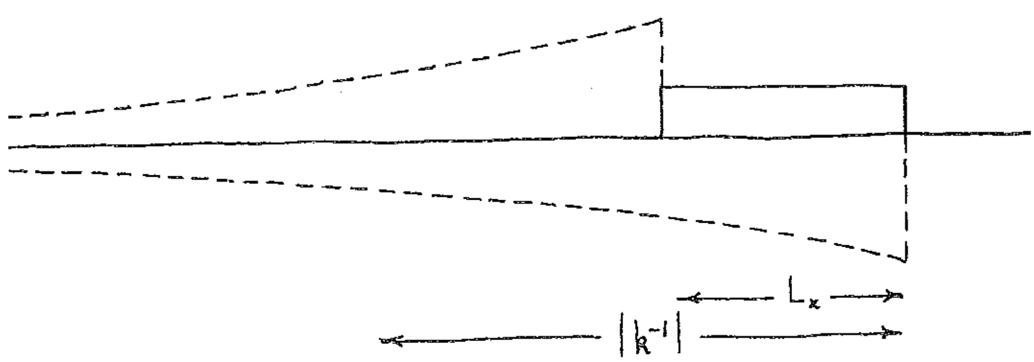
$$k = \frac{\omega}{\beta}\alpha^2 = \frac{\sigma}{\beta}\alpha^2 - i\frac{A\alpha^2}{\beta c^2} . \tag{27}$$

Equation (26) describes a wave with amplitude given in curly brackets. The wave number of the radiation, $(\sigma/\beta)\alpha^2$, is just that of a nondispersive Rossby wave. In contrast to the case of inviscid models, the radiation damps as it propagates with an e-folding scale, $(\beta c^2)/(A\alpha^2)$. Since the integral in the amplitude need not be zero outside the region of the wind patch, (26) describes the excitation and radiation of a damped Rossby wave from the patch. Note that (26) still exists when $\sigma \to 0$. In this steady limit radiation does not propagate away from the wind patch, but simply decays away from it. It is useful to continue to refer to such steady solutions as Rossby waves.

The choice of lower limit in (25) can now be made. The group velocity as well as decay direction of the Rossby waves is westward. To insure that no radiation shows up east of the patch, L must be chosen to be any position entirely to the east of the wind patch. In this unbounded ocean it is sufficient to take

$$L = +\infty . \tag{28}$$

Dynamics. Consider the steady weak-drag $(A/c^2 \to 0)$ limit of (25).



In this limit the e-folding decay scale of the Rossby waves, $|k^{-1}|$, is much longer than the width scale of the wind stress, L_x; that is,

$$|k| L_x \ll 1.$$

This situation is shown schematically in the diagram for a "top hat" wind stress distribution. The solid curve shows the locally driven response, and the dashed curves show the form of the Rossby waves excited at the edges of the wind patch. In this case, the integrals of (25) can be approximated by

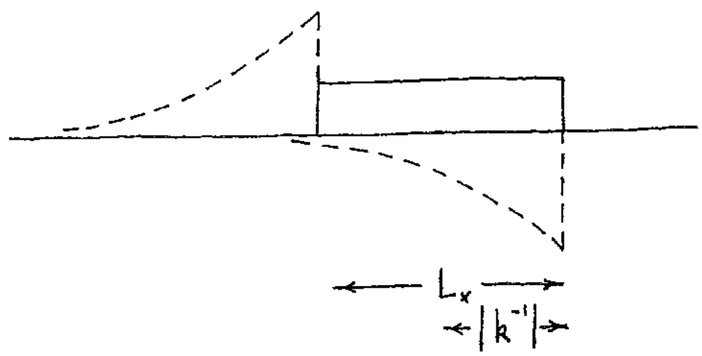
$$e^{ikx} \int_{\infty}^{x} e^{-ikx} X \, dx \longrightarrow \int_{\infty}^{x} X \, dx + O(|k| L_x). \tag{29}$$

Ignoring the small correction term, (25) reduces to

$$v = -\frac{1}{\beta} \int_{\infty}^{x} F_{yx} \, dx = \frac{F_y}{\beta}, \tag{29'}$$

which is just the Sverdrup balance.

Consider the steady strong-drag ($A/c^2 \to \infty$) limit of (25). As is shown in the figure,



the Rossby waves now decay before they can cross the region of the wind. As a result,

$$|k| L_x \gg 1,$$

and the integrals of (25) reduce to

$$\int_{+\infty}^{x} e^{ikx} F \, dx = \frac{F}{ik} + O\left(\frac{1}{|k| L_x}\right). \tag{30}$$

Again ignoring the small correction term, (25) becomes

$$v = -\frac{F}{f} - i \frac{F_{yx}}{\omega \alpha^2} \longrightarrow -\frac{F}{f}, \tag{31}$$

which is just pseudo-Ekman balance.

Recall that the size of A/c^2 increases markedly with modenumber, n. Therefore, in steady state the high-order modes are always in pseudo-Ekman balance, and only the low-order modes have the possibility of being in Sverdrup balance. It is possible to choose a value for A so large that none of the low-order modes are ever in Sverdrup balance; however, for realistic choices of eddy viscosity (where $1 \lesssim \nu \lesssim 100$) a few of the low-order modes ($n \lesssim 5$-10) always tend to be in Sverdrup balance.

PROBLEM #3: I have shown here how to solve (18) to obtain the v_n-field. Find the u_n- and p_n-fields using similar techniques. Show that these fields are also in Sverdrup balance when drag is weak and in pseudo-Ekman balance when drag is strong.

Transient forcing

I show here how to solve (18) when the forcing is a patch of wind turned on at time $t = 0$. That is, the wind has the form

$$\tau^x = \tau_0 X(x) Y(y) H(t),$$
$$F_n = \frac{\tau_0 X(x) Y(y) H(t)}{\int_{-D}^{0} \psi_n^2 \, dz}, \tag{32}$$

where

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t \le 0 \end{cases}.$$

A straightforward way of solving (18) in this case is to take Fourier transforms in space and Laplace transforms in time. Denote any Laplace-transformed variable by a carat, that is, the Laplace transform of q(t) is $\hat{q}(s)$. Then according to (20), $\hat{\tilde{v}}$ is given by

$$\hat{\tilde{v}} = -\tau_0 \frac{\left(s+\frac{A}{c^2}\right)\frac{Y}{f} - ik\frac{Y_y}{\alpha^2}}{s\left(s+\frac{A}{c^2} - ik\frac{\beta}{\alpha^2}\right)}\tilde{X}.$$

It is useful to rewrite this expression as

$$\hat{\tilde{v}} = -\tau_0 \frac{Y_y}{\beta}\frac{\tilde{X}}{s} - \tau_0\left(\frac{Y}{f} - \frac{Y_y}{\beta}\right)\frac{s+\frac{A}{c^2}}{s\left(s+\frac{A}{c^2} - ik\frac{\beta}{\alpha^2}\right)}\tilde{X}. \tag{33}$$

The Laplace inversion of (33) is easy with the aid of the transform pair,

$$\frac{s+b}{s(s+a)} \longleftrightarrow \frac{b}{a} - \frac{b-a}{a}e^{-at}.$$

It follows that

$$\tilde{v} = -\tau_0\frac{Y_y}{\beta}\tilde{X}H(t) - \tau_0\left(\frac{Y}{f} - \frac{Y_y}{\beta}\right)\frac{\frac{A}{c^2} - ik\frac{\beta}{\alpha^2}e^{ik\frac{\beta}{\alpha^2}t - \frac{A}{c^2}t}}{\frac{A}{c^2} - ik\frac{\beta}{\alpha^2}}\tilde{X}H(t),$$

or, after rewriting,

$$\begin{aligned}\tilde{v} = &-\tau_0\frac{Y_y}{\beta}\tilde{X}H(t) - \tau_0\left(\frac{Y}{f} - \frac{Y_y}{\beta}\right)\tilde{X}\, e^{ik\frac{\beta}{\alpha^2}t - \frac{A}{c^2}t} H(t) \\ &-\tau_0\left(\frac{Y}{f} - \frac{Y_y}{\beta}\right)\tilde{X}\frac{A}{c^2}\,\frac{1 - e^{ik\frac{\beta}{\alpha^2}t - \frac{A}{c^2}t}}{A/c^2 - ik\beta/\alpha^2}H(t).\end{aligned} \tag{34}$$

To complete the solution I now find the inverse Fourier transform of (34). Appropriate transform pairs can be found by using the convolution theorem of Fourier transforms

$$\tilde{F}\tilde{G} \longleftrightarrow F * G = \int_{-\infty}^{\infty} F(x-x')\,G(x')\,dx'. \tag{35}$$

Consider the following examples.

1. Suppose that

$$\tilde{F} = e^{ika} \longleftrightarrow F = \delta(x+at),$$

and $G = X$. Then (35) gives

$$e^{ikb}\,\tilde{X} \longleftrightarrow X(x+bt). \tag{36a}$$

2. Suppose that

$$\tilde{F} = \frac{1}{k-ia},\ a>0 \longleftrightarrow -i\,e^{ax}\,H(-x),$$

and $G = X$. Then (35) gives

$$\begin{aligned} \frac{\tilde{X}}{k+ia} &\longleftrightarrow \int_{-\infty}^{\infty} dx' \left[-i\,e^{a(x-x')} H(x-x')\right] X(x') \\ &= -i\,e^{ax}\int_{x}^{\infty} e^{ax'} X(x')\,dx'. \end{aligned} \tag{36b}$$

Equation (24) is a slightly different version of this transform pair.

3. Suppose that

$$\tilde{F} = e^{ikb}\,\tilde{X}, \qquad \tilde{G} = \frac{1}{k+ia}.$$

Then according to (35),

$$\frac{e^{ikb}\,\tilde{X}}{k+ia} \longleftrightarrow -i\,e^{ax}\int_{x}^{\infty} e^{-ax'} X(x'+bt)\,dx'. \tag{36c}$$

With the aid of (36a,b,c) the v-field is easily shown to be

$$\begin{aligned} v = &-\tau_0 \frac{Y_4}{\beta} X(x)\,H(t) - \tau_0\left(\frac{Y}{f}-\frac{Y_4}{\beta}\right) X\!\left(x+\frac{\beta}{\alpha^2}t\right) e^{-\frac{A}{c^2}t} H(t) \\ &+ \frac{A}{c^2}\frac{\alpha^2}{\beta}\tau_0\left(\frac{Y}{f}-\frac{Y_4}{\beta}\right) e^{\frac{A}{c^2}\frac{\alpha^2}{\beta}x}\int_{\infty}^{x} e^{-\frac{A}{c^2}\frac{\alpha^2}{\beta}x'} X(x')\,dx'\ H(t) \\ &- \frac{A}{c^2}\frac{\alpha^2}{\beta}\tau_0\left(\frac{Y}{f}-\frac{Y_4}{\beta}\right) e^{\frac{A}{c^2}\frac{\alpha^2}{\beta}x}\int_{\infty}^{x} e^{-\frac{A}{c^2}\frac{\alpha^2}{\beta}x'} X\!\left(x'+\frac{\beta}{\alpha^2}t\right) dx'\ e^{-\frac{A}{c^2}t} H(t), \end{aligned} \tag{37}$$

Lighthill (1969) obtained this solution for the case A = 0.

Dynamics. Does (37) reduce to pseudo-Ekman balance when drag is strong? In this case $A/c^2 \to \infty$ and I make the approximation

$$e^{\frac{A}{c^2}\frac{\alpha^2}{\beta}x}\int_{\infty}^{x} e^{-\frac{A}{c^2}\frac{\alpha^2}{\beta}x'}\,X(x')\,dx' \quad \longleftrightarrow \quad \frac{-X(x)}{\frac{A}{c^2}\frac{\alpha^2}{\beta}}\,.$$

Equation (37) becomes

$$\lim_{A/c^2\to\infty} v = -\tau_0\frac{Y_y}{\beta}X(x)H(t) - \tau_0\left(\frac{Y}{f}-\frac{Y_y}{\beta}\right)X\!\left(x+\frac{\beta}{\alpha^2}t\right)e^{-\frac{A}{c^2}t}H(t)$$

$$+\frac{A}{c^2}\frac{\alpha^2}{\beta}\tau_0\left(\frac{Y}{f}-\frac{Y_y}{\beta}\right)\left[\frac{-X(x)}{\frac{A}{c^2}\frac{\alpha^2}{\beta}} + \frac{X\!\left(x+\frac{\beta}{\alpha^2}t\right)}{\frac{A}{c^2}\frac{\alpha^2}{\beta}}\,e^{-\frac{A}{c^2}t}\right]H(t)$$

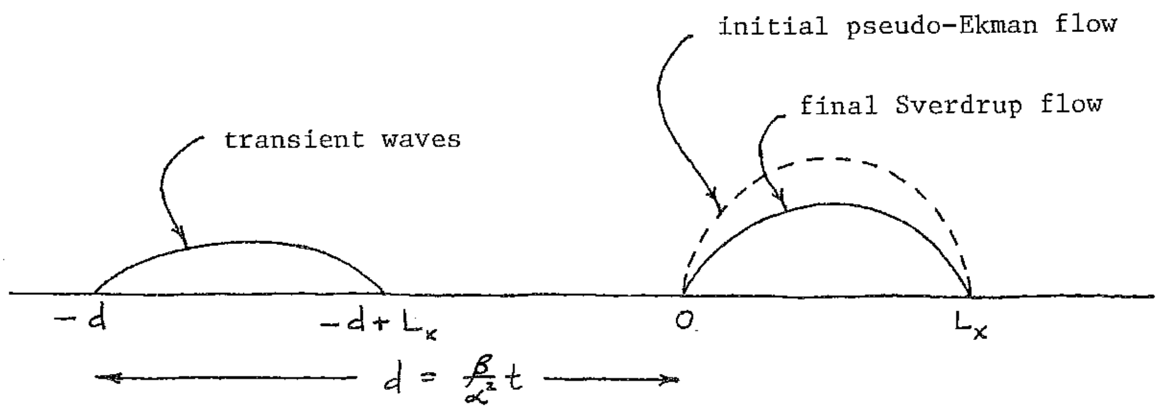
$$= -\tau_0\frac{Y}{f}X(x)\,H(t). \tag{38}$$

So, the high-order modes come instantly into pseudo-Ekman balance after the winds turn on, and thereafter remain in that balance. The set-up of pseudo-Ekman flow occurs instantly because inertial waves have been filtered out of this model.

Does (37) reduce to the Sverdrup balance for low-order modes where drag is weak? In the limit $A/c^2 \to 0$ (37) reduces to

$$\lim_{A/c^2\to 0} v = -\tau_0\frac{Y_y}{\beta}X(x)\,H(t) - \tau_0\left(\frac{Y}{f}-\frac{Y_y}{\beta}\right)X\!\left(x+\frac{\beta}{\alpha^2}t\right)H(t). \tag{39}$$

Just after the winds turn on (that is, when $t = 0^+$), (39) is just the pseudo-Ekman balance. At later times, however, the radiation of Rossby waves becomes important. The last term of (39) propagates westward as a packet of non-dispersive Rossby waves. The second term is just the expected Sverdrup balance. So (39) describes the set-up of Sverdrup balance via the radiation of Rossby waves from the wind patch. The mode comes into Sverdrup balance only <u>after</u> these transient waves have passed out of the system. The following diagram illustrates schematically this response for a wind patch confined to the region $0 \le x \le L_x$. The response is shown initially (dashed curve) and also at a later time t (solid curves).



PROBLEM #4: Find the u_n- and p_n-fields generated by a switched-on wind. Again show that the solution consists of a local piece and a propagating piece; however, show that the propagating piece advances as a front (rather than as a blob separate from the locally forced region). Show that the weak- and strong-drag limits are again the Sverdrup and pseudo-Ekman balances, respectively. Draw a schematic diagram of the solution.

EQUATORIAL SOLUTIONS

In this section I find solutions to the full set of mode equations

$$\begin{aligned} i\omega u_n - f v_n + p_{nx} &= F_n, \\ i\omega v_n + f u_n + p_{ny} &= 0, \\ i\omega \frac{p_n}{c_n^2} + u_{nx} + v_{ny} &= 0, \end{aligned} \tag{40a}$$

and also another simpler "filtered" set

$$\begin{aligned} i\omega u_n - f v_n + p_{nx} &= F_n, \\ f u_n + p_{ny} &= 0, \\ i\omega \frac{p_n}{c_n^2} + u_{nx} + v_{ny} &= 0, \end{aligned} \tag{40b}$$

where $i\omega$ is the operator given by (18b). Both sets differ fundamentally from (18a) in that they retain the term $i\omega u_n$. Keeping this term avoids singularities at the equator. In the remainder of this section I show how to find solutions to these sets of equations on the equatorial β-plane, that is, when

$$f = \beta y.$$

In this case, the solutions can be written in a mathematically convenient form and in the tropical ocean they are insignificantly altered. Again, for convenience I ignore meridional winds. I retain the subscript, n, only when confusion might result from its absence.

To see the conditions under which (40b) is a sensible description of the ocean, I again compare v-equations. The exact v-equation is

$$\Big[\overset{(b)}{i\omega v_{xx}} + \overset{(a)}{i\frac{\omega^3}{c^2} v}\Big] + i\omega \Big\{\overset{(c)}{\partial_{yy} - \frac{f^2}{c^2}}\Big\} v + \beta v_x = i\omega \frac{f}{c^2} F - F_{yx}, \tag{41a}$$

whereas (40b) gives

$$i\omega \left\{ \partial_{yy} - \frac{f^2}{c^2} \right\} v + \beta v_x = i\omega \frac{f}{c^2} F + F_{yx}. \qquad (41b)$$

Equation (41b) differs from (41a) in that the terms in square brackets are absent, and will be a useful approximation only if this absence does not seriously alter the character of the solutions.

Just as before, I compare term ⓐ and ⓑ to term ⓒ. In order to carry out this comparison I need a measure of the size of the operator in curly brackets. That measure is β/c [see discussion following (47)]. So, the approximation is valid provided that

$$\frac{1}{T} < \omega_o, \quad \frac{A}{c^2} < \omega_o, \quad L_x > \alpha_o^{-1},$$

where $\omega_o = \sqrt{\beta c}$ is the equatorial inertial frequency, and $\alpha_o = \sqrt{\beta/c}$ is the reciprocal of the equatorial Rossby radius. The first and third inequalities require that the wind forcing is low-frequency and large-scale. For sufficiently large values of n, the sense of the first two inequalities can reverse. These errors are minor at mid-latitudes and are associated with the replacement of an Ekman flow by a pseudo-Ekman flow. An analogous distortion of the solution occurs at the equator in that the Yoshida balance is modified to a pseudo-Yoshida balance. Near an eastern ocean boundary, solutions can develop coastal jets for which $L_x \lesssim \alpha_o^{-1}$; a consequence of this error is discussed further in the last part of this section. Note that there is no longer any limit on the meridional length scale, L_y, and so (40b) can describe the typically narrow equatorial jets.

Much of the material in this section summarizes mathematical techniques introduced in greater detail in other papers. The response of the equatorial ocean to oscillating and steady winds is considered by McCreary (1980a). The response to switched-on winds is discussed by Lighthill (1969), Moore (1974), Cane (1974), Cane and Sarachik (1976, 1977) and Moore and Philander (1978). A number of examples of oceanic response to various transient wind fields are found in McCreary (1976, 1977, 1978). All of these studies also consider the effects of meridional boundaries on the flow field. Studies which explicitly consider such effects are Moore (1969) and Anderson and Rowlands (1976).

The Hermite functions

Solving (41a) or (41b) is now more difficult because the equations involve y-derivatives; the operator in curly brackets, however, does not involve x or t. This property suggests that it is possible to separate out the y-dependence of the problem just as the z-dependence was separated out. I look for solutions as expansions in the eigenfunctions of the operator in curly brackets. These eigenfunctions satisfy

$$\phi_{myy} - \frac{f^2}{c^2}\phi_m = -\alpha_m^2\,\phi_m\,, \tag{42a}$$

subject to the boundary conditions

$$\phi_m(y) = 0 \quad \text{as} \quad y \to \pm\infty. \tag{42b}$$

One representation of the solutions to (42) is

$$\phi_m(\eta) = \frac{(-1)^m}{\left[2^m m/\sqrt{\pi}\right]^{1/2}}\, e^{\eta^2/2}\,\frac{d^m}{d\eta^m}\, e^{-\eta^2}, \tag{43}$$

where

$$\alpha_m^2 = \alpha_0^2(2m+1), \quad m = 0, 1, 2, \cdots,$$

$$\alpha_0^2 = \beta/c, \quad \eta = \alpha_0 y.$$

The functions are normalized so that

$$\int_{-\infty}^{\infty} \phi_m^2\, d\eta = 1. \tag{44}$$

These eigenfunctions are convenient to work with because they have simple recursion relations. With the aid of (43) it follows that

$$\begin{aligned}\eta\phi_m &= \sqrt{\frac{m+1}{2}}\,\phi_{m+1} + \sqrt{\frac{m}{2}}\,\phi_{m-1}\,,\\ \phi_{m\eta} &= -\sqrt{\frac{m+1}{2}}\,\phi_{m+1} + \sqrt{\frac{m}{2}}\,\phi_{m-1}\,.\end{aligned} \tag{45}$$

Let q_m, $[\eta q]_m$, and $[q_\eta]_m$ be the Hermite expansion coefficients of the functions $q(\eta)$, ηq, q_η, respectively. That is,

$$q(\eta) = \sum_{m=0}^{\infty} q_m \phi_m(\eta), \qquad q_\eta = \sum_{m=0}^{\infty} [q_\eta]_m \phi_m(\eta), \qquad \eta q = \sum_{m=0}^{\infty} [\eta q]_m \phi_m(\eta).$$

Then the coefficients are related by

$$[\eta q]_m = \sqrt{\frac{m+1}{2}}\, q_{m+1} + \sqrt{\frac{m}{2}}\, q_{m-1},$$

$$[q_\eta]_m = \sqrt{\frac{m+1}{2}}\, q_{m+1} - \sqrt{\frac{m}{2}}\, q_{m-1}. \tag{46}$$

PROBLEM #5: Show that properties (44), (45), and (46) are valid. They all follow directly from (43).

Equation (43) shows that a characteristic width scale for these functions is α_o^{-1}. Because c varies roughly inversely proportional to n, α_o^{-1} decreases with increasing vertical modenumber roughly like $n^{-\frac{1}{2}}$. Typically for n = 1, $\alpha_o^{-1} \approx 350$ km; for n = 8, $\alpha_o^{-1} \approx 350 / \sqrt{8} \approx 125$ km. We can expect that solutions found as series expansions of these functions will have features with these narrow meridional scales.

Oscillating winds

I choose the wind stress as in (21), so that $i\omega$ is now just a complex number given by (22), and look for a solution to the exact v-equation (41a) as an expansion of the form

$$V = \sum_{m=0}^{\infty} V_m(x)\, \phi_m(\eta). \tag{47}$$

A practical model must truncate this series at some number m = M. M must be chosen large enough so that the solutions are well converged.

At this point it is easy to see why β/c is an appropriate measure of the operator in curly brackets in (41a) or (41b). According to (42) and (43) it follows that

$$\left|\left\{\partial_{yy} - \frac{\beta^2}{c^2} y^2\right\} V\right| = \left|\sum_{m=0}^{\infty} V_m \left\{\partial_{yy} - \frac{\beta^2}{c^2} y^2\right\} \phi_m\right|$$

$$= \left|\sum_{m=0}^{\infty} \alpha_m^2 V_m \phi_m\right| \geq \alpha_0^2 \left|\sum_{m=0}^{\infty} V_m \phi_m\right| = \frac{\beta}{c}|V|.$$

I find the equation governing the expansion coefficients, $v_m(x)$, by multiplying (41a) by $\phi_m(\eta)$ and integrating from $-\infty$ to $+\infty$

$$\int_{-\infty}^{\infty} \phi_m \left[v_{xx} + \frac{\omega^2}{c^2} v + \left\{ v_{yy} - \frac{f^2}{c^2} \right\} v + \frac{\beta}{i\omega} v_x = \frac{f}{c^2} F - \frac{1}{i\omega} F_{yx} \right] d\eta. \tag{48}$$

It is necessary to integrate the term, v_{yy}, by parts twice. These integrations are simple because of the boundary conditions (42b). Equation (48) becomes

$$v_{mxx} + \frac{\beta}{i\omega} v_{mx} - \left(\alpha_m^2 - \frac{\omega^2}{c^2}\right) v_m = \tau_{on} \alpha_o \left(\frac{1}{c} [\eta Y]_m X - \frac{1}{i\omega} [Y_\eta]_m X_x \right). \tag{49}$$

To solve (49) I Fourier transform the equation and solve algebraically for $\tilde{v}_m$ to get

$$\left[-k^2 + \frac{k\beta}{\omega} - \left(\alpha_m^2 - \frac{\omega^2}{c^2}\right) \right] \tilde{v}_m = \tau_{on} \alpha_o \left(\frac{1}{c} [\eta Y]_m - \frac{k}{\omega} [Y_\eta]_m \right) \tilde{X},$$

$$\Longrightarrow \quad \tilde{v}_m = -\tau_{on} \alpha_o \frac{\frac{1}{c} [\eta Y]_m - \frac{k}{\omega} [Y_\eta]_m}{(k - k_1^m)(k - k_2^m)} \tilde{X}, \tag{50}$$

where

$$k^m_{\binom{1}{2}} = \frac{\beta}{2\omega} \left[1 \mp \sqrt{1 - 4 \frac{\omega^2}{\beta^2} \left(\alpha_m^2 - \frac{\omega^2}{c^2}\right)} \right]. \tag{51}$$

I now expand the denominator of (50) into partial fractions. A useful theorem is

$$\frac{\sum_{j=0}^{n} a_j (k)^j}{\prod_{i=1}^{N} (k - b_i)} = \sum_{i=1}^{N} \frac{\sum_{j=0}^{n} a_j (b_\ell)^j}{(k - b_\ell) \prod_{\substack{i=1 \\ (\ell \neq i)}}^{N-1} (b_\ell - b_i)}, \quad n < N. \tag{52}$$

Equation (52) states mathematically the following simple rule. To find the coefficient of the term $1/(k - b_\ell)$ in the expansion, replace k with b_ℓ everywhere else that it appears. The particular case of (52) appropriate here has N = 2, n = 1, so that

$$\frac{a_0 + a_1 k}{(k-b_1)(k-b_2)} = \frac{a_0 + a_1 b_1}{(k-b_1)(b_1-b_2)} + \frac{a_0 + a_1 b_2}{(k-b_2)(b_2-b_1)}$$

$$= \sum_{j=1}^{2} \frac{a_0 + a_1 b_j}{(k-b_j)(b_j-b_{j'})}\,, \quad j' \neq j. \tag{53}$$

With the aid of (53), (50) becomes

$$\tilde{v}_m = - \sum_{j=1}^{2} P_j^m \frac{\tilde{X}}{k - k_j^m}\,, \tag{54a}$$

where

$$P_j^m = \tau_{on} \alpha_0 \frac{\frac{1}{c}[\eta Y]_m - \frac{k_j^m}{\omega}[Y_\eta]_m}{k_j^m - k_{j'}^m}\,, \quad j' \neq j. \tag{54b}$$

The inverse Fourier transform of (54a) is now easily found with the aid of the transform pair (24). The result is

$$v_m(x) = -i \sum_{j=1}^{2} P_j^m \int_{L_j}^{x} e^{-ik_j^m x} X \, dx \; e^{ik_j^m x + i\sigma t}\,, \tag{55}$$

where

$$L_{\binom{1}{2}} = \pm \infty\,, \tag{56}$$

and L_j is chosen to satisfy the appropriate radiation conditions for the problem (see below).

PROBLEM #6: Find the u_n- and p_n-fields as expansions in the Hermite functions by using similar techniques. Show that

$$u_m(x) = \sqrt{\tfrac{m+1}{2}}\,R + \sqrt{\tfrac{m}{2}}\,S + \delta_{m0}\,T$$

$$= \alpha_0\sqrt{\tfrac{m+1}{2}}\sum_{j=1}^{2}\frac{P_j^{m+1}}{k_j^{m+1}-\frac{\omega}{c}}\int_{L_j}^{x} e^{-ik_j^{m+1}x}\,X\,dx\; e^{ik_j^{m+1}x+i\sigma t}$$

$$-\alpha_0\sqrt{\tfrac{m}{2}}\sum_{j=1}^{2}\frac{P_j^{m-1}}{k_j^{m-1}+\frac{\omega}{c}}\int_{L_j}^{x} e^{-ik_j^{m-1}x}\,X\,dx\; e^{ik_j^{m-1}x+i\sigma t}$$

$$+\delta_{m0}\,\frac{\tau_{0n}}{2c}\int_{L_2}^{\infty} e^{i\frac{\omega}{c}x}\,X\,dx\; e^{-i\frac{\omega}{c}x+i\sigma t}, \qquad (57a)$$

where R, S and T are obviously defined, and

$$u_n = \sum_{m=0}^{\infty} u_m\,\phi_m(\eta).$$

Also show that

$$p_m(x) = c\left(-\sqrt{\tfrac{m+1}{2}}\,R + \sqrt{\tfrac{m}{2}}\,S + \delta_{m0}\,T\right), \qquad (57b)$$

where

$$p_n = \sum_{m=0}^{\infty} p_m\,\phi_m(\eta).$$

<u>HINT</u>: For convenience, drop factors of τ_0, α_0, c, β. Then solve (40a) for a single equation in u_n (use the first and third). Show that

$$\tilde{u}_m = i\sqrt{\tfrac{m+1}{2}}\,\frac{\tilde{v}_{m+1}}{k-\omega} - i\sqrt{\tfrac{m}{2}}\,\frac{\tilde{v}_{m-1}}{k+\omega} + i\omega\,\frac{Y_m}{(k-\omega)(k+\omega)}\,\tilde{X}.$$

Plugging (50) into this expression gives

$$\tilde{u}_m = -i\sqrt{\frac{m+1}{2}}\,\frac{[\eta Y]_{m+1} - \frac{k}{\omega}[Y_\eta]_{m+1}}{(k-\omega)(k-k_1^{m+1})(k-k_2^{m+1})}\tilde{X} + i\epsilon_{mo}\sqrt{\frac{m}{2}}\,\frac{[\eta Y]_{m-1} - \frac{k}{\omega}[Y_\eta]_{m-1}}{(k+\omega)(k-k_1^{m-1})(k-k_2^{m-1})}\tilde{X}$$
$$+ i\omega\,\frac{Y_m}{(k-\omega)(k+\omega)}\tilde{X}$$

$$= \left\{ -i\sqrt{\frac{m+1}{2}}\sum_{j=1}^{2}\frac{P_j^{m+1}}{k_j^{m+1}-\omega}\,\frac{\tilde{X}}{k-k_j^{m+1}} + i\epsilon_{mo}\sqrt{\frac{m}{2}}\sum_{j=1}^{2}\frac{P_j^{m-1}}{k_j^{m-1}-\omega}\,\frac{\tilde{X}}{k-k_j^{m-1}} \right\}$$

$$-i\sqrt{\frac{m+1}{2}}\,\frac{[\eta Y]_{m+1} - [Y_\eta]_{m+1}}{(\omega-k_1^{m+1})(\omega-k_2^{m+1})}\,\frac{\tilde{X}}{k-\omega} + i\epsilon_{mo}\sqrt{\frac{m}{2}}\,\frac{[\eta Y]_{m-1} + [Y_\eta]_{m-1}}{(\omega+k_1^{m-1})(\omega+k_2^{m-1})}\,\frac{\tilde{X}}{k+\omega}$$

$$+\frac{i}{2}\frac{Y_m}{k-\omega}\tilde{X} - \frac{i}{2}\frac{Y_m}{k+\omega}\tilde{X},$$

where

$$\epsilon_{mo} = \begin{cases} 0, & m = 0 \\ 1, & m \neq 0 \end{cases}.$$

ϵ_{mo} is related to the familiar Kronecker δ-function by $\epsilon_{mo} = 1 - \delta_{mo}$. I have also used the relations

$$(\omega \pm k_1^m)(\omega \pm k_2^m) = \omega^2 \pm \omega(k_1^m + k_2^m) + k_1^m k_2^m$$
$$= \omega^2 \pm 1 + (2m+1-\omega^2) = \begin{cases} 2(m+1) \\ 2m \end{cases}.$$

With the aid of these identities $\tilde{u}_m$ becomes

$$\tilde{u}_m = \{\quad\} - i\sqrt{\frac{m+1}{2}}\,\frac{2\sqrt{\frac{m+1}{2}}\,Y_m}{2(m+1)}\,\frac{\tilde{X}}{k-\omega} + i\epsilon_{mo}\sqrt{\frac{m}{2}}\,\frac{2\sqrt{\frac{m}{2}}\,Y_m}{2m}\,\frac{\tilde{X}}{k+\omega}$$
$$+\frac{i}{2}\frac{Y_m}{k-\omega}\tilde{X} - \frac{i}{2}\frac{Y_m}{k+\omega}\tilde{X}$$

$$\overset{a}{=} \{\quad\} + \frac{i}{2}\epsilon_{mo}Y_m\frac{\tilde{X}}{k+\omega} - \frac{i}{2}Y_m\frac{\tilde{X}}{k+\omega}$$

$$= \{\quad\} - \frac{i}{2}\delta_{mo}\frac{Y_m}{k+\omega}\tilde{X}.$$

The reason for introducing ε_{mo} is evident in term (a). All factors of m cancel out, and without ε_{mo} we could easily forget that term (a) should be zero when m = 0. The expression for the p_n-field is found similarly.

Evidence of radiation. Each term of the solution, (55) and (57), has the form

$$\left\{ (\text{const}) \int_{\pm\infty}^{x} e^{-ikx} X\, dx \right\} e^{ikx + i\sigma t} ,$$

and so describes a wave with amplitude given in curly brackets and wavenumber, k. The wave numbers are either $k = -(\omega/c)$ or set by (51). In the inviscid model, then, the waves are the familiar undamped, equatorially trapped Kelvin and Rossby waves. In the present viscid model the waves also damp as they propagate out of the wind patch. Again solutions still exist when $\sigma \to 0$. In this steady limit the solutions do not propagate but simply decay away from the wind patch. They have the same meridional structure as the inviscid waves. I will continue to refer to these steady solutions as equatorially trapped Kelvin and Rossby waves.

The choices of lower limits of integration in (56) govern whether radiation appears to the east, west, or on both sides of the wind patch. Waves with eastward (westward) group velocity can only appear east (west) of the patch. Equivalently, in the steady state, waves which decay eastward (westward) can appear only east (west) of the patch. The choices (56) satisfy these requirements.

Dynamics. Consider the steady strong-drag ($A/c^2 \to \infty$) limit of (55). In this limit the roots of (51) are

$$k^m_{\binom{1}{2}} \longrightarrow \mp i\sqrt{\alpha_m^2 - \frac{\omega^2}{c^2}} ,$$

and it follows that

$$|k_1^m| L_x \gg 1 , \quad |k_2^m| L_x \gg 1 ,$$

for geophysically appropriate choices of parameters. After I approximate integrals as in (30), equation (55) reduces to

$$V_m(x) = X \sum_{j=1}^{2} \frac{P_j^m}{k_j^m} = \frac{\tau_{on}\alpha_o X}{k_1^m - k_2^m} \left\{ \frac{\frac{i}{c}[\eta Y]_m}{k_1^m} - \frac{\frac{i}{c}[\eta Y]_m}{k_2^m} - \frac{[Y_\eta]_m}{\omega} + \frac{[Y_\eta]_m}{\omega} \right\}$$

$$= \frac{\tau_{on}\alpha_o X}{k_1^m - k_2^m} \left\{ \frac{k_1^m - k_2^m}{k_1^m k_2^m} \frac{i}{c}[\eta Y]_m \right\} = -\tau_{on}\frac{\alpha_o}{c} X \frac{[\eta Y]_m}{\alpha_m^2 - \frac{\omega^2}{c^2}}$$

$$\Longrightarrow V_n = -\tau_{on}\frac{\alpha_o}{c} X \sum_{m=0}^{\infty} \frac{[\eta Y]_m}{\alpha_m^2 + A^2/c^6} \phi_m(\eta) .$$

This v_n-field results from the solution of (17). So, the high-order modes of the model are in Yoshida balance.

PROBLEM #7: Find the limits of u_n and p_n as $A/c^2 \to \infty$. Show that

$$u_n = \frac{c^2}{A}(\alpha_0 \eta v_n + F_n),$$

$$p_n = -\frac{c^4}{A} v_{ny},$$

which are the u_n- and p_n-fields that result from the Yoshida balance.

Consider the steady weak-drag ($A/c^2 \to 0$) limit of (55). In this limit the roots of (51) are

$$k_1^m \longrightarrow \frac{\omega}{\beta}\alpha_m^2 \longrightarrow 0, \qquad k_2^m \longrightarrow \frac{\beta}{\omega} \longrightarrow \infty.$$

So, eventually for sufficiently small values of the drag

$$|k_1^m|L_x \ll 1, \quad |k_2^m|L_x \gg 1.$$

After I approximate integrals with the aid of both (29) and (30), equation (55) reduces to

$$v_m(x) = -iP_1^m \int_{+\infty}^{x} X\,dx + \frac{1}{k_2^m} P_2^m X$$

$$= -i\tau_{0n}\alpha_0 \frac{\frac{1}{c}[\eta Y]_m - \frac{\alpha_m^2}{\beta}[Y_\eta]_m}{\frac{\omega}{\beta}\alpha_m^2 - \frac{\beta}{\omega}} \int_{+\infty}^{x} X\,dx + \frac{\omega}{\beta}\tau_{0n}\alpha_0 \frac{\frac{1}{c}[\eta Y]_m - \frac{\beta}{\omega^2}[Y_\eta]_m}{\frac{\beta}{\omega} - \frac{\omega}{\beta}\alpha_m^2} X$$

$$= \quad O(\omega) \quad - \quad O(\omega) \quad + O(\omega^2) \quad - \quad O(1).$$

Only the O(1) term remains, and so

$$v_m(x) = -\tau_{0n}\alpha_0 \frac{1}{\beta}[Y_\eta]_m X,$$

$$\Longrightarrow \quad v_n = -\tau_{on}\alpha_o \frac{1}{\beta} X \sum_{m=0}^{\infty} [Y_\eta]_m \phi_m = -\tau_{on}\alpha_o \frac{1}{\beta} X Y_\eta$$

$$= -\tau_{on}\frac{1}{\beta} X Y_y = -\frac{1}{\beta} F_y .$$

Just as for the solutions to (18), the low-order modes can be in Sverdrup balance. For realistic choices of eddy viscosity a few of them ($n \lesssim 5$-10) always tend to be in Sverdrup balance.

PROBLEM #8: Find the limits of u_n and p_n (from PROBLEM #6) as $A/c^2 \rightarrow 0$. Show that

$$u_n = \frac{1}{\beta}\int_{+\infty}^{x} F_y \, dx + \left\{ \frac{\phi_o}{2c} \tau_{on} Y_o \int_{-\infty}^{\infty} X \, dx \right\},$$

$$p_n = \int_{+\infty}^{x} (F - yF_y) \, dx + \left\{ \frac{\phi_o}{2} \tau_{on} Y_o \int_{-\infty}^{\infty} X \, dx \right\}. \qquad (58)$$

Equations (58) differ from the "classical" Sverdrup balance by the presence of the terms in curly brackets. These terms describe a geostrophic x-independent jet. Note that the jet is x-independent regardless of the shape of the wind patch X(x)! This interesting current did not appear in Sverdrup's original solution because he chose a boundary condition of no normal flow at an eastern ocean body, rather than the radiation conditions, (56), adopted here.

Boundaries. Here I show how to include the effects of simple meridional barriers at x_o, and/or x_1. (The ocean will still lack northern and southern boundaries.) It is not necessary to discard the interior solution we have just found. Rather it is sufficient to add to the solution appropriate free waves that prevent flow into the boundaries.

The appropriate boundary conditions for an ocean with both meridional barriers are

$$u_n = 0 \quad @ \quad x = x_o, \; x_1 . \qquad (59a)$$

The ocean is still unbounded in the y-direction. What are the correct boundary conditions to impose at $y = \pm\infty$? Clearly we wish the only source of energy in the solution to be the interior wind patch. So, the correct boundary condition is

There are no sources of energy at high latitudes, that is, at $y = \pm\infty$. (59b)

When $A = 0$, but $\sigma \neq 0$, condition (59b) has a simple physical interpretation. In this case, at high latitudes the boundary solutions look like coastally trapped Kelvin waves. Condition (59b), then, requires that no energy is brought into the ocean by a high-latitude western boundary source of Kelvin waves.

If there is just a single western (eastern) boundary then (59a) is replaced by

$$u_n = 0 \quad @ \quad x = x_0 \; (x_1), \tag{59a'}$$

and (59b) becomes

There are no sources of energy at high latitudes, or at $x = +\infty$ $(x = -\infty)$. (59b')

Equation (59b') requires that western boundary waves all have eastward group velocity or decay to the east; that is, they can involve only the wavenumber k_2^m. These waves, designated by a prime, are

$$u_m' = A_m \left[\phi_{m+1} - \frac{ck_2^m + \omega}{ck_2^m - \omega} \sqrt{\frac{m}{m+1}} \; \phi_{m-1} \right] e^{ik_2^m (x - x_0)},$$

$$v_m' = \frac{i}{c\alpha_0} \, \frac{ck_2^m + \omega}{\sqrt{\frac{m}{2}}} \, A_m \, \phi_m \, e^{ik_2^m (x - x_0)}, \tag{60a}$$

$$p_m' = c A_m \left[\phi_{m+1} + \frac{ck_2^m + \omega}{ck_2^m - \omega} \sqrt{\frac{m}{m+1}} \; \phi_{m-1} \right] e^{ik_2^m (x - x_0)},$$

where $m = 0, 1, 2 \ldots$, and A_m is (for the moment) an unspecified constant. The wave with $m = 0$ is called the Yanai wave. There is another free solution of (40a) for which the v-field is identically zero. For convenience I assign this solution the label $m = -1$. Then

$$u_{-1}' = A_{-1} \, \phi_0 \, e^{-i \frac{\omega}{c}(x - x_0)}, \qquad p_{-1}' = c u_{-1}, \qquad v_{-1}' \equiv 0, \tag{60b}$$

and A_{-1} is (for the moment) an arbitrary constant. This wave is the equatorially trapped Kelvin wave.

Waves appropriate at an eastern ocean boundary all have westward group velocity or decay to the west, and so only involve the wavenumber, k_1^m. These waves, designated by a double prime, are

$$u_m'' = B_m \left[\frac{ck_1^m - \omega}{ck_1^m + \omega} \sqrt{\frac{m+1}{m}}\, \phi_{m+1} - \phi_{m-1} \right] e^{ik_1^m(x - x_1)} ,$$

$$v_m'' = \frac{i}{c\alpha_0} \frac{ck_1^m - \omega}{\sqrt{\frac{m}{2}}} B_m \phi_m e^{ik_1^m(x - x_1)} , \qquad (61)$$

$$p_m'' = c B_m \left[\frac{ck_1^m - \omega}{ck_1^m + \omega} \sqrt{\frac{m+1}{m}}\, \phi_{m+1} + \phi_{m-1} \right] e^{ik_1^m(x - x_1)} ,$$

where m = 1, 2, 3, ..., and B_m is (for the moment) arbitrary.

PROBLEM #9: Show that (60) and (61) are the homogeneous solutions of the complete equations of motion, (40a). One way to do this is by plugging them back into the homogeneous form of (40a). Also show that the m = 0 eastern boundary wave is not well defined (because $ck_1^0 + \omega = 0$). Therefore, it is not a physically realistic solution and is not included in (61).

Now suppose I want to bring the u-field to zero all along the western boundary. I require the u-field to vanish for each Hermite component. The contributions of zonal velocity proportional to $\phi_{m+1}(\eta)$ at the western boundary ($x = x_0$) are

$$\left. \begin{array}{c} A_m \\ -A_{m+2} \dfrac{ck_2^{m+2} + \omega}{ck_2^{m+2} - \omega} \sqrt{\dfrac{m+2}{m+3}} \end{array} \right\} , \quad \text{from (60)},$$

$$\left. \begin{array}{c} B_m \dfrac{ck_1^m - \omega}{ck_1^m + \omega} \sqrt{\dfrac{m+1}{m}}\, e^{-ik_1^m d} \\ -B_{m+2}\, e^{-ik_1^{m+2} d} \end{array} \right\} , \quad \text{from (61)},$$

$u_{m+1}(x_0)$, from the unbounded solution (57a);

$d = x_1 - x_0$ is the separation between boundaries. These contributions must all add to zero, and yield the equation

$$A_m = \sqrt{\frac{m+2}{m+3}}\,\frac{ck_2^{m+2}+\omega}{ck_2^{m+2}-\omega}\,A_{m+2} - \sqrt{\frac{m+1}{m}}\,\frac{ck_1^m-\omega}{ck_1^m+\omega}\,e^{-ik_1^m d}\,B_m$$
$$+\; e^{-ik_1^{m+2}d}\,B_{m+2} - u_{m+1}(x_0)\,, \tag{62a}$$

where $m = -1, 0, 1, \ldots$, and $B_0 = B_{-1} \equiv 0$. Similarly, cancellation of the component of zonal velocity proportional to $\phi_{m-1}(\eta)$ at the eastern boundary ($x = x_1$) requires that

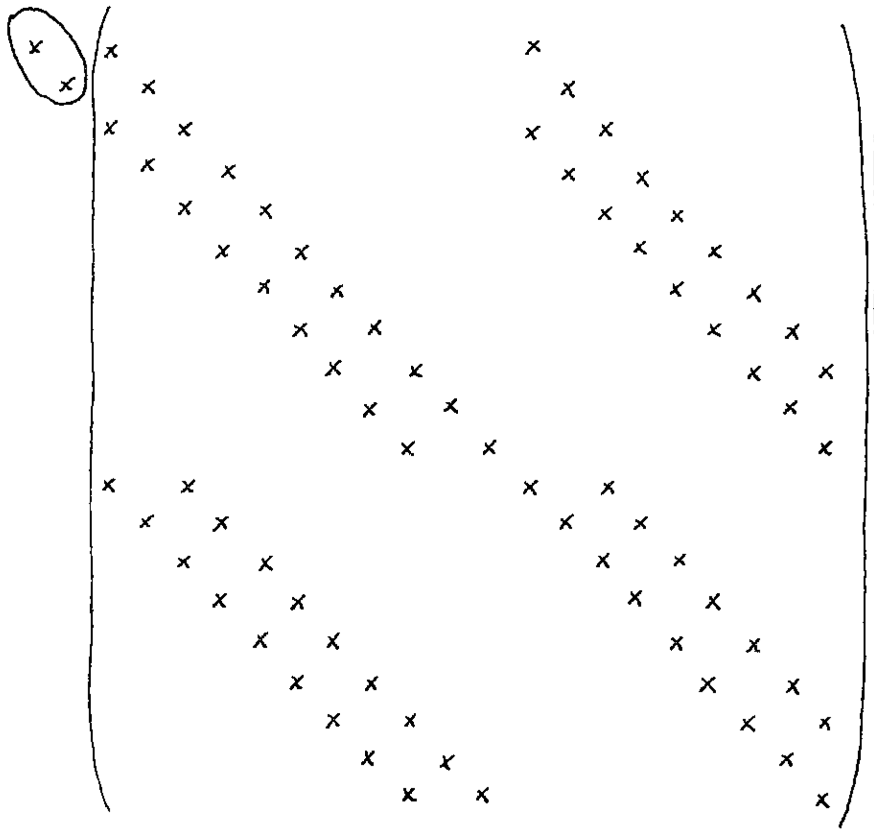
$$B_m = \sqrt{\frac{m-1}{m-2}}\,\frac{ck_1^{m-2}-\omega}{ck_1^{m-2}+\omega}\,B_{m-2} - \sqrt{\frac{m}{m+1}}\,\frac{ck_2^m+\omega}{ck_2^m-\omega}\,e^{ik_2^m d}\,A_m$$
$$+\; e^{ik_2^{m-2}d}\,A_{m-2} + u_{m-1}(x_1)\,, \tag{62b}$$

where $m = 1, 2, 3, \ldots$, and again $B_0 = B_{-1} \equiv 0$.

In principle an infinite number of boundary waves are needed at both boundaries. In practice the series must be truncated at some upper limit, M. In that case equations (62) can almost be written down concisely in a square matrix form

$$\mathbb{A}\,x = F. \tag{63}$$

The figure below shows (63) written out in component form for the case $M = 9$. Only the terms indicated by x's are non-zero.



$$
\begin{pmatrix} A_M \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ A_1 \\ A_0 \\ A_{-1} \\ B_M \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ B_2 \\ B_1 \end{pmatrix}
=
\begin{pmatrix} -u_{M+1}(x_0) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ -u_2(x_0) \\ -u_1(x_0) \\ -u_0(x_0) \\ u_{M-1}(x_1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_1(x_1) \\ u_0(x_1) \end{pmatrix}
$$

$$\mathbb{A} \qquad x = F$$

The circled terms in the upper left-hand corner arise from western boundary waves with amplitudes A_{M+2} and A_{M+1}, and prevent the matrix from being square. I now set both of these terms to zero. The neglect of these terms insures that the boundary conditions (59b) are satisfied. As a result, (63) now does have the form of a single square matrix equation. The equation is sparse and so can be solved economically on a computer.

PROBLEM #10: To solve (63) directly by Gaussian elimination requires a large number of steps of the order M^3. Because the matrix is sparse it is possible to solve the equation much more efficiently. Devise a scheme that will solve the system in a number of steps of the order M^2.

Having determined the coefficients A_m and B_m, we can finally write down the solutions for the ocean response in an ocean basin with two zonal boundaries. Let q be any of the fields u, v or p. Then

$$q_n = \sum_{m=0}^{M} q_m \phi_m(\eta) + \sum_{m=-1}^{M} q_m' + \sum_{m=1}^{M} q_m'', \tag{64}$$

where the coefficients q_m are given in (55) and (57), q_m' are from (60), and q_m'' are from (61). This procedure can be easily modified to describe the model response if the ocean basin has a single western or eastern boundary. Simply set the terms B_m or A_m to zero, respectively, and repeat the procedure indicated above.

Transient forcing: inertial oscillations

It is well known that at mid-latitudes, abrupt changes in the wind field excite inertial oscillations with a frequency set by the local value of the Coriolis parameter. A similar phenomenon exists at the equator even though $f \to 0$ there. To illustrate the nature of equatorial inertial oscillations, I choose an x-independent wind field of the form

$$\tau^x = \tau_0 Y(y) H(t), \quad F_n = \tau_{0n} Y(y) H(t), \tag{65}$$

where H(t) is the step function defined after (32). I solve (41a) by taking the Laplace transform of the equation, and expanding into Hermite functions as in (47). Since the forcing is x-independent so is the solution, and all x-derivatives can be ignored. The $\hat{v}_m$-field is

$$-\left[c^2\alpha_m^2 + \left(s + \frac{A}{c^2}\right)^2\right]\hat{v}_m = \tau_{on}\alpha_o c\,[\eta Y]_m \frac{1}{s}$$

$$\Rightarrow \hat{v}_m = -\frac{\tau_{on}\alpha_o c\,[\eta Y]_m}{s\left[\left(s + \frac{A}{c^2}\right)^2 + c^2\alpha_m^2\right]}. \tag{66}$$

The Laplace transform pair appropriate here is

$$\frac{1}{s\left[(s+a)^2 + b^2\right]} \longleftrightarrow \frac{1}{a^2+b^2} - e^{-at}\left(\frac{\cos bt + \frac{a}{b}\sin bt}{a^2+b^2}\right).$$

It follows that

$$\begin{aligned} v_n = \sum_{m=0}^{\infty} v_m \phi_m &= -\sum_{m=0}^{\infty} \frac{\tau_{on}\omega_o[\eta Y]_m}{c^2\alpha_m^2 + A^2/c^4}\phi_m H(t) \\ &+ e^{-\frac{A}{c^2}t}\sum_{m=0}^{\infty}\frac{\tau_{on}\omega_o[\eta Y]_m}{c^2\alpha_m^2 + A^2/c^4}\Big(\cos\sqrt{2m+1}\,\omega_o t + \\ &+ \frac{A}{c^3\alpha_m}\sin\sqrt{2m+1}\,\omega_o t\Big)\phi_m H(t). \end{aligned} \tag{67}$$

The response has a steady component that is in Yoshida balance. A number of terms oscillate at the discrete frequencies, $\sqrt{2m+1}\,\omega_0$, and decay with an e-folding time scale, c^2/A. These terms describe the equatorial inertial oscillations.

Moore (Moore and Philander, 1978) studied this x-independent spin-up problem with $A \equiv 0$, and described the response in detail. In contrast to the situation at mid-latitudes, the pressure field associated with equatorial inertial oscillations is not small. As a result, these oscillations can strongly affect sea level and thermocline depth. One component of the zonal velocity field is an accelerating jet strongly confined to the equator. This flow is commonly referred to as the Yoshida jet.

Transient forcing: wind patch

Here I consider the response of the equatorial ocean to a wind stress patch of the form (32), but now find solutions only to the approximate equations (40b). To solve (41b) I Fourier transform the equation in x, Laplace transform it in time, and expand into Hermite functions as in (47). The result is

$$\hat{\tilde{v}}_m = -\frac{\alpha_o}{\alpha_m^2}\tau_{on}\frac{(s + A/c^2)\frac{[\eta Y]_m}{c} - ik[Y_\eta]_m}{s\left(s + A/c^2 - ik\beta/\alpha_m^2\right)}\tilde{X}, \tag{68}$$

or, more conveniently,

$$\hat{\tilde{v}}_m = - \tau_{on} \frac{\alpha_0}{\beta} [Y_\eta]_m \frac{\tilde{X}}{s} - \frac{\tau_{on}\alpha_0}{\beta}\left(\frac{[\eta Y]_m}{2m+1} - [Y_\eta]_m\right) \frac{s + A/c^2}{s\left(s + A/c^2 - ik\beta/\alpha_m^2\right)} \tilde{X}. \tag{69}$$

Equation (69) is very similar to (33). The necessary Fourier and Laplace inversions go through just as they did in that case. In analogy with (37), then, the solution is

$$\begin{aligned} v_n = &- \tau_{on} \frac{Y_y}{\beta} X(x) H(t) \\ &- \tau_{on} \frac{\alpha_0}{\beta} \sum_{m=0}^{\infty} \left(\frac{[\eta Y]_m}{2m+1} - [Y_\eta]_m\right) X\left(x + \frac{\beta}{\alpha_m^2} t\right) e^{-\frac{A}{c^2}t} H(t) \phi_m \\ &+ \frac{A}{c^2} \tau_{on} \frac{\alpha_0}{\beta} \sum_{m=0}^{\infty} \frac{\alpha_m^2}{\beta}\left(\frac{[\eta Y]_m}{2m+1} - [Y_\eta]_m\right) \times \\ &\qquad \times e^{\frac{A}{c^2}\frac{\alpha_m^2}{\beta}x} \int_{\infty}^{x} e^{-\frac{A}{c^2}\frac{\alpha_m^2}{\beta}x'} X(x')\,dx'\; H(t)\, \phi_m \\ &- \frac{A}{c^2} \tau_{on} \frac{\alpha_0}{\beta} \sum_{m=0}^{\infty} \frac{\alpha_m^2}{\beta}\left(\frac{[\eta Y]_m}{2m+1} - [Y_\eta]_m\right) \times \\ &\qquad \times e^{\frac{A}{c^2}\frac{\alpha_m^2}{\beta}x} \int_{+\infty}^{x} e^{-\frac{A}{c^2}\frac{\alpha_m^2}{\beta}x'} X\left(x' + \frac{\beta}{\alpha_m^2}t\right) dx'\; e^{-\frac{A}{c^2}t} H(t)\, \phi_m . \end{aligned} \tag{70}$$

Lighthill (1969) first found this solution for the case A = 0. Note that there are no inertial oscillations in this solution. So one effect of studying equations (40b) is that inertial oscillations are filtered out. The solution involves only the radiation of packets of non-dispersive Rossby waves to the west. Thus, another effect of the approximate set is that eastward-propagating dispersive Rossby waves are also filtered out of the model.

In the strong-drag limit, (70) reduces to

$$\lim_{A/c^2 \to \infty} v_n = - \tau_{on} \frac{\alpha_m^2}{\beta} \sum_{m=0}^{\infty} \frac{[\eta Y]_m}{2m+1} X(x) H(t) \phi_m .$$

This north-south flow occurs for a mode in pseudo-Yoshida balance. So, the high-order modes of the model come instantly into pseudo-Yoshida balance when the wind turns on, and thereafter remain in that balance.

In the weak-drag limit, (70) reduces to

$$\lim_{A/c^2\to 0} v_n = -\tau_{on}\frac{Y_y}{\beta}X(x)\,H(t) - \\ -\tau_{on}\frac{\alpha_o}{\beta}\sum_{m=0}^{\infty}\left(\frac{[\eta Y]_m}{2m+1} - [Y_\eta]_m\right)X\left(x+\frac{\beta}{\alpha_m^2}t\right)H(t)\,\phi_m .$$

Just after the wind is turned on this expression is just the pseudo-Yoshida balance. Because damping is weak, however, radiation subsequently affects the balance. The expression describes the set-up of the Sverdrup balance after a packet of non-dispersive Rossby waves has passed through the model.

PROBLEM #11: Find the u_n- and p_n-fields as expansions of Hermite functions by using similar techniques. Discuss the limits of these fields just as above. In particular, show that in the limits $A/c^2 \to 0$ and $t \to \infty$ (so that all transient wave packets have passed through the system) the solutions reduce to the form (58). The model describes the set-up of the x-independent equatorial jet via the radiation of wave packets of equatorially trapped Kelvin and Rossby waves. McCreary (1977, 1978) traced the adjustment of a single baroclinic mode to Sverdrup balance in this way. Lighthill (1969) missed the generation of the x-independent jet because he failed to include the equatorially trapped Kelvin wave in his solution.

HINT: For convenience, drop factors of τ_o, α_o, c, β. Find u_m in a similar way to that of problem #6. We have

$$\hat{\tilde{u}}_m = -i\sqrt{\frac{m+1}{2}}\;\frac{[\eta Y]_{m+1} - \frac{k}{\omega}[Y_\eta]_{m+1}}{(k-\omega)(k-k^{m+1})}\;\frac{\tilde{X}}{s} \\ + i\,\epsilon_{mo}\sqrt{\frac{m}{2}}\;\frac{[\eta Y]_{m-1} - \frac{k}{\omega}[Y_\eta]_{m-1}}{(k+\omega)(k-k^{m-1})}\;\frac{\tilde{X}}{s} \;+\; i\omega\,\frac{Y_m}{(k+\omega)(k-\omega)}\;\frac{\tilde{X}}{s},$$

where

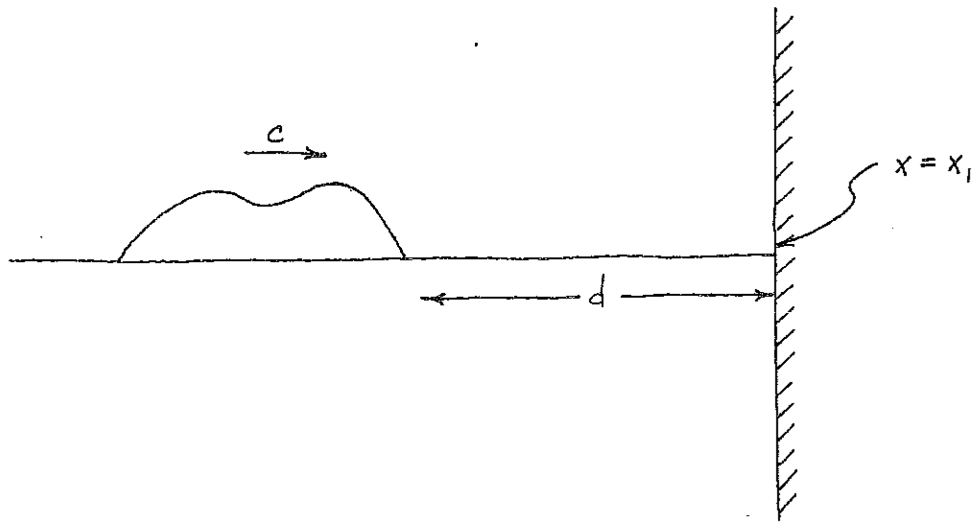
$$k^m = (2m+1)\,\omega, \qquad \omega = -i(s+A).$$

Expand these expressions into partial fractions and simplify. As in Problem #6, many cancellations occur.

Reflection of an equatorially trapped Kelvin wave from an eastern ocean boundary

Moore (1968) first showed how to reflect a sinusoidal Kelvin wave from an eastern boundary. Anderson and Rowlands (1976) extended Moore's work to consider the reflection of a Kelvin wave of step-function form. Here I show how to reflect a Kelvin wave of arbitrary shape, but find the reflection only for the simpler system of equations (40b).

Consider a wind patch that is turned on at time t = 0 in the ocean



interior. Subsequently, a packet of Kelvin waves leaves the locally forced region and travels toward the eastern ocean boundary. Such a packet is pictured in the diagram. Let that packet be described by

$$u_n(x,y,t) = U_0(x-ct)\ e^{-\frac{A}{c^2}t}\ \phi_0(\eta). \tag{71}$$

Then, evaluated at the coast ($x = x_1$), its amplitude is just a function of time, that is,

$$u_n(x_1,y,t) = e^{-\frac{A}{c^2}t}\ U_0(x_1-ct)\ \phi_0(\eta) \equiv T(t)\ \phi_0(\eta) \tag{72}$$

$$\Rightarrow \quad \hat{u}_n = \hat{T}(s)\ \phi_0(\eta). \tag{73}$$

In order to cancel this flow field, I add to the solution additional free solutions of equations (40b). Rossby waves allowed by this set have a particularly simple form because they satisfy the simple dispersion relation

$$k_1^m = \frac{\alpha_m^2}{\beta}\omega = \frac{\omega}{c}(2m+1). \tag{74}$$

There are no eastward-propagating dispersive Rossby waves in this system. The westward propagating waves, (61), reduce to

$$\begin{aligned} u_m'' &= \hat{B}_m\left[\sqrt{\frac{m}{m+1}}\ \phi_{m+1} - \phi_{m-1}\right] e^{\left(s+\frac{A}{c^2}\right)(2m+1)(x-x_1)/c}, \\ v_m'' &= \hat{B}_m\, 2\sqrt{2}\sqrt{m}\ \frac{s}{\omega_0}\ \phi_m\ e^{\left(s+\frac{A}{c^2}\right)(2m+1)(x-x_1)/c}, \\ p_m'' &= c\,\hat{B}_m\left[\sqrt{\frac{m}{m+1}}\ \phi_{m+1} + \phi_{m-1}\right] e^{\left(s+\frac{A}{c^2}\right)(2m+1)(x-x_1)/c}, \end{aligned} \tag{75}$$

where I have made the replacement, $i\sigma \to s$. The analog of (62b) in Laplace transform space is therefore

$$\begin{aligned} \hat{B}_m &= \sqrt{\frac{m-2}{m-1}}\ \hat{B}_{m-2} + \delta_{m-1,0}\ \hat{T}, \qquad m = 1, 2, 3, \cdots, \\ B_0 &= B_{-1} = 0. \end{aligned} \tag{76}$$

A matrix formulation is not necessary in order to see how to solve this set of equations. By inspection

$$\hat{B}_1 = \hat{T}, \qquad \hat{B}_3 = \sqrt{\frac{1}{2}}\ \hat{T}, \qquad B_5 = \sqrt{\frac{1}{2}}\sqrt{\frac{3}{4}}\ \hat{T},$$

$$\hat{B}_2 = 0, \quad \hat{B}_4 = 0, \quad \hat{B}_6 = 0, \quad \ldots$$

To summarize the solution I introduce the quantity:

$$R_m = \begin{cases} 1 & , \quad m = 1 \\ \sqrt{\frac{1}{2}}\sqrt{\frac{3}{4}}\cdots\sqrt{\frac{m-2}{m-1}} & , \quad m = 3, 5, 7, \cdots, \\ 0 & , \quad m = 2, 4, 6, \cdots, \end{cases}$$

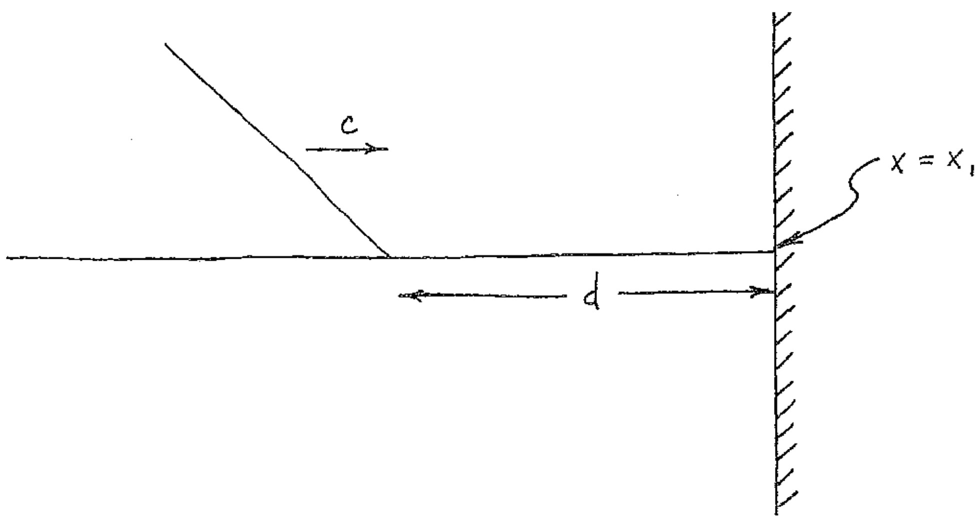
so that

$$\hat{B}_m = R_m \hat{T}, \quad m = 1, 2, 3, \cdots \tag{77}$$

The boundary waves are now uniquely specified. To find them I plug (77) back into (75) and find the Laplace inversion of each term. The inversions can be accomplished easily with the convolution theorem for Laplace transforms.

It follows from (77) and (75) that at the coast the s-dependence of each boundary wave is just $\hat{T}(s)$. Therefore they all have the same time dependence. Physically this property means that there is instantaneous propagation of signals poleward along the coast, that is, the phase speed of coastally trapped Kelvin waves is infinite! This is a major, and unavoidable, distortion introduced by solving the simpler system of equations (40b).

EXAMPLE: Suppose that the shape of the u-field is the linear segment shown in the diagram.



For simplicity, I assume that the ocean is inviscid. Then

$$A \equiv 0, \quad u_n = [ct - (x - x_1 + d)] H [ct - (x - x_1 + d)] \; \phi_0(\eta).$$

At the coast this u-field is

$$u_n(x_1, y, t) = (ct - d) H(ct - d) \phi_0(\eta) \equiv T(t) \phi_0(\eta)$$

$$\Longrightarrow \hat{T} = \frac{c}{s^2} e^{-s\frac{d}{c}}.$$

To find the Laplace inversion of (75) I use the transform pair

$$\hat{T} e^{s(2m+1)(x - x_1)/c} = \frac{c}{s^2} e^{-s/c\,[d - (2m+1)(x - x_1)]}$$
$$\longleftrightarrow \; [ct - d + (2m+1)(x - x_1)] H [ct - d + (2m+1)(x - x_1)], \tag{78}$$

The individual boundary waves are therefore

$$u_m'' = R_m \left[\sqrt{\frac{m}{m+1}} \phi_{m+1} - \phi_{m-1} \right] \xi_m H(\xi_m),$$

$$v_m'' = 2\sqrt{2}\sqrt{m} \frac{1}{\alpha_0} R_m \phi_m H(\xi_m), \tag{79}$$

$$p_m'' = c R_m \left[\sqrt{\frac{m}{m+1}} \phi_{m+1} + \phi_{m-1} \right] \xi_m H(\xi_m),$$

where

$$\xi_m = ct - d + (2m+1)(x - x_1).$$

McCreary (1977) built up his eastern boundary solutions out of linear components like (79).

PROBLEM #12: Do a case with $A \neq 0$. Assume that the incoming Kelvin wave has the form

$$u_n = e^{-\frac{A}{c^2}t}\left[ct - (x - x_1 - d)\right] H\left[ct - (x - x_1 - d)\right] \phi_0 .$$

Describe the response of each eastern boundary wave.

EASTERN COASTAL SOLUTIONS

In this section I study a filtered version of (13) that neglects the terms u_{nt} and $(A/c^2)u_n$ in (13). This simplification allows solutions to be found without the use of Hermite functions, but the solutions are no longer valid near the equator. The equations are

$$
\begin{aligned}
-f v_n + p_{nx} &= 0, \\
i\omega v_n + f u_n + p_{ny} &= G_n, \\
i\omega \frac{p_n}{c_n^2} + u_{nx} + v_{ny} &= 0,
\end{aligned}
\tag{80}
$$

where $i\omega$ is the operator defined in (18b). Since it is well known that the coastal ocean responds most strongly to alongshore winds, zonal winds are subsequently ignored. It is not necessary to restrict solutions to the equatorial β-plane, so in the remainder of this section I take

$$
\theta = y/R, \quad f = 2\Omega \sin\theta, \quad \beta = \frac{2\Omega}{R}\cos\theta, \tag{81}
$$

where $\Omega = 2\pi/\text{day}$ and R is the radius of the earth. Again I neglect subscripts n for the sake of notational simplicity.

To see whether equations (80) are a reasonable approximation to the exact equations, I again compare v-equations. The exact v-equation is

$$
\begin{aligned}
i\omega\left[\overset{(b)}{v_{yy}} + \overset{(a)}{\frac{\omega^2}{c^2}v}\right] + i\omega v_{xx} - i\omega \overset{(c)}{\frac{f^2}{c^2}v} + \beta v_x \\
= \overset{(d)}{\left[\frac{\omega^2}{c^2}G\right]} + G_{xx},
\end{aligned}
\tag{82a}
$$

whereas (80) gives

$$
i\omega v_{xx} - i\omega\frac{f^2}{c^2}v + \beta v_x = G_{xx}. \tag{82b}
$$

The simplification deletes the terms in square brackets. These terms are negligible provided that terms ⓐ and ⓑ are << ⓒ, and that ⓓ is not a significant driving force. The approximation is valid provided that

$$\frac{1}{T} \ll f, \qquad \frac{A}{c^2} \ll f, \qquad L_y \gg \alpha^{-1}.$$

The first and third inequalities require that the wind field is low-frequency and has a large meridional length-scale. When the wind forcing is low-frequency the factor of ω^2/c^2 insures that term ⓓ is not significant. For significantly large modenumber the second inequality does not hold. This error is due to the fact that Ekman flow is replaced with pseudo-Ekman flow. In the ocean interior, well away from its eastern boundary, the third inequality need not always be satisfied [the discussion after (25) is applicable here]. There is no limitation on zonal length-scales, and so (80) can describe narrow coastal jets. However, solutions to (80), as well as to the exact system (13), can produce unrealistically narrow western boundary currents [see discussion following (87)]. So, we can expect that equations (80) can describe a wide variety of physically realistic problems that occur <u>near</u> north-south oriented eastern ocean boundaries.

The material in this section summarizes several other studies. The coastal response to steady alongshore winds is discussed in McCreary (1980b). The interested reader should also consult Yoshida (1967); although the algebra is needlessly complicated, there is a wealth of information in this paper. The coastal response to oscillatory winds is considered by McCreary (1977, 1978).

Oscillating winds

I show here how to find the response of the extra-equatorial eastern ocean to an oscillating band of wind of the form

$$\tau^y = \tau_0\, Y(y)\, e^{i\sigma t}, \qquad G_n = \tau_{0n}\, Y(y)\, e^{i\sigma t}, \tag{83}$$

where τ_{0n} is defined in (21). Since the wind field is x-independent, it has no curl. I assume that the band does not extend to the equator, but has an equatorward edge, y_0. I proceed by first rewriting equations (80) to get an equation in p_n alone,

$$i\omega p_{xx} - i\omega \frac{f^2}{c^2} p + \beta p_x = f G_x \equiv 0. \tag{84a}$$

The u_u- and v_n-fields, in terms of p_n, are

$$u = \frac{G}{f} - \frac{p_y}{f} - \frac{i\omega}{f} p_x ,$$
$$v = \frac{p_x}{f}, \qquad w = \frac{i\omega}{c^2} p , \tag{84b}$$

where now $i\omega$ is the complex number given in (22).

I solve (84) including the effects of an eastern ocean boundary at $x = x_1$. As in the equatorial solution, it is useful to split the total solution into two pieces: an unbounded piece that is valid in the absence of boundaries, and boundary reactions that satisfy the homogeneous form of (84). The boundary solutions must be chosen to satisfy (59b'), and the total solution must satisfy (59a').

Because there is no wind curl and the forcing is independent of x, we can ignore the operator ∂_x in (84). Then the interior solution, indicated by a prime, is

$$u' = \frac{G}{f}, \quad v' = 0, \quad p' = 0, \quad w' = 0. \tag{85}$$

This solution is nothing more than a drift to the right of the wind stress confined to the surface mixed layer. That is, it is just a pseudo-Ekman flow.

The boundary reactions must be composed of the homogeneous solutions to (84) since the interior solution is already in balance with the wind forcing. I indicate the boundary field by a double prime, and assume

$$p'' = P(y)\, e^{ik(x-x_1)} , \tag{86a}$$

so that

$$u'' = -\frac{1}{f}\left(\partial_y - \frac{\omega k}{f}\right) P\, e^{ik(x-x_1)} ,$$
$$v'' = \frac{ik}{f} P\, e^{ik(x-x_1)} . \tag{86b}$$

The value of k is specified by (84a) to be either one of the roots of

$$-i\omega k^2 - i\omega \frac{f^2}{c^2} + ik\beta = 0,$$

and the boundary condition (59b') requires that k be the root

$$k = \frac{\beta}{2\omega}\left[1 - \sqrt{1 - 4\frac{\omega^2}{\beta^2}\alpha^2}\,\right]. \tag{87}$$

Compare (87) to (51). There is an obvious close relationship between these mid-latitude waves and the equatorially trapped ones.

It is now easy to see why western boundary solutions can be unrealistically narrow. At a western ocean boundary waves must reflect energy eastward, and the appropriate root has a plus sign in front of the radical in (87). This root can become very large. For example, assume that the wind field is steady so that $\omega = -iA/c^2$. A typical value for A is $10^{-4}\,\mathrm{cm}^2/\mathrm{sec}^3$ (McCreary, 1980a); a typical value of c for the n = 1 vertical mode is 200 cm/sec. Therefore, $|k|^{-1} \approx \omega/\beta = 25$ km; that is, the width of the western boundary current of this model is only 25 km.

There can be no flow through the eastern boundary. I adjust the amplitude, P(y), so that this condition is met. That is,

$$u = u' + u'' = \frac{G}{f} - \frac{1}{f}\left(\partial_y - \frac{\omega k}{f}\right)P e^{ik(x-x_1)} = 0 \qquad @\ x = x_1$$

$$\Longrightarrow \quad \left(\partial_y - \frac{\omega k}{f}\right)P = G. \tag{88}$$

Equation (88) is easily solved by introducing the integration factor e^{Δ}, where

$$\Delta = \int_{y_c}^{y} \frac{\omega k}{f}\,dy. \tag{89}$$

The choice of lower limit, y_c, in (89) is arbitrary. The solution to (88) is

$$e^{\Delta}\left(e^{-\Delta}P\right) = G \quad \Longrightarrow \quad P = e^{\Delta}\int_{y_0}^{y} e^{-\Delta}G\,dy. \tag{90}$$

The choice of the lower limit in (90) is crucial, and amounts to specifying a radiation condition for the problem. There are no free waves at an eastern ocean boundary that can carry signals equatorward. Choosing the lower limit to be y_0, the equatorward edge of the wind band, insures that no signals are ever found equatorward of the wind band.

Finally, the complete solutions are

$$
\begin{aligned}
p_n &= \tau_{on}\, e^{ik(x-x_1)+i\sigma t}\, e^{\Delta}\int_{y_0}^{y} e^{-\Delta}\, Y\, dy\,, \\
v_n &= \tau_{on}\, \frac{ik}{f}\, e^{ik(x-x_1)+i\sigma t}\, e^{\Delta}\int_{y_0}^{y} e^{-\Delta}\, Y\, dy\,, \\
w_n &= \tau_{on}\, \frac{i\omega}{c^2}\, e^{ik(x-x_1)+i\sigma t}\, e^{\Delta}\int_{y_0}^{y} e^{-\Delta}\, Y\, dy\,.
\end{aligned}
\tag{91a}
$$

and with the aid of (91a) and (84b) the u-field is

$$
\begin{aligned}
u_n &= \frac{G}{f} - \frac{1}{f}\left(\partial_y - \frac{\omega k}{f}\right)\left[e^{ik(x-x_1)}\, e^{\Delta}\int_{y_0}^{y} e^{-\Delta}\, G\, dy \right] \\
&= \frac{G}{f} - i\,\frac{x-x_1}{f}\, k_y\, p_n - \frac{1}{f}\, e^{ik(x-x_1)}\left[\left(\partial_y - \frac{\omega k}{f}\right) e^{\Delta}\int_{y_0}^{y} e^{-\Delta}\, G\, dy\right],
\end{aligned}
$$

or

$$
u_n = \tau_{on}\,\frac{Y}{f}\left[1 - e^{ik(x-x_1)}\right] e^{i\sigma t} - i\,\frac{x-x_1}{f}\, k_y\, p_n\,. \tag{91b}
$$

Evidence of radiation. Here I assume that friction is weak, so that $\omega = \sigma$. Then, since $\alpha^2 = f^2/c^2$ is zero at the equator and β is zero at the poles, the radical of (87) must become zero at some intermediate latitude, Θ_{cr}, the critical latitude. According to (87), Θ_{cr} is given by

$$
1 = 4\alpha^2 \frac{\omega^2}{\beta^2} \quad\Longrightarrow\quad \theta_{cr} = \tan^{-1} \frac{c^2}{2R\sigma}\,. \tag{92}
$$

Note that Θ_{cr} is a strong function of both n (through its dependence on c) and σ.

At the annual frequency Θ_{cr} occurs at mid-latitudes for the first vertical mode. A typical value of c for this mode is 200 cm/sec; it follows that $\Theta_{cr} = 38^o$ (the latitude of San Francisco). A typical value of c for the second vertical mode is 100 cm/sec, and for this mode $\Theta_{cr} = 11^o$. For modes with $n > 2$ values of Θ_{cr} are even closer to the equator.

Θ_{cr} splits the boundary response into two dynamically different regions. Boundary waves possible in each region are quite different. Poleward of Θ_{cr}, where $\Theta >> \Theta_{cr}$, (87) shows that

$$\lim_{\theta >> \theta_{cr}} k = \frac{\beta}{\sigma} - i\alpha . \qquad (93)$$

I choose y_c so that $y_c/R >> \Theta_{cr}$. Then, with the aid of (93),

$$\lim_{\theta >> \theta_{cr}} \Delta = \int_{y_c}^{y} \frac{\sigma}{f}\left(\frac{\beta}{\sigma} - i\alpha\right) dy = \int_{y_c}^{y} \frac{1}{R} \cot\theta \, dy - i\frac{\sigma}{c}(y-y_c).$$

So for $\Theta >> \Theta_{cr}$ boundary waves have the form

$$D(y)\, e^{\alpha(x-x_1)}\, e^{i\frac{\beta}{\sigma}x}\, e^{i\sigma(t-y/c)} , \qquad (94)$$

where D(y) is a slowly varying amplitude. Equation (94) describes β-plane Kelvin waves [first discussed by Moore (1968)]. These waves propagate poleward at speed c and decay offshore in α^{-1}. In contrast to f-plane Kelvin waves they also propagate offshore at the speed α^2/β.

Equatorward of Θ_{cr}, where $\Theta << \Theta_{cr}$, (87) gives

$$\lim_{\theta << \theta_{cr}} k = \frac{\sigma}{\beta}\alpha^2 .$$

Sor for $\Theta << \Theta_{cr}$ boundary waves have the form

$$D(y)\, e^{i\sigma\left[t + \frac{\alpha^2}{\beta}(x-x_1)\right]} , \qquad (95)$$

where D(y) again is a slowly varying amplitude. Equation (95) describes non-dispersive Rossby waves. The boundary response below the critical latitude propagates rapidly offshore back into the ocean interior.

These properties have important implications for the modelling of the annual cycle in eastern oceans. Off Oregon all baroclinic boundary effects remain trapped to the coast, since $\Theta > \Theta_{cr}$ for all values of n. Off Baha California the baroclinic effects associated with the first baroclinic mode can radiate offshore as Rossby waves, since $\Theta < \Theta_{cr}$ for $n = 1$. All other modes remain trapped to the coast, since $\Theta > \Theta_{cr}$ for $n > 1$.

Dynamics. Just as in the models previously discussed, the character of flow associated with each mode changes markedly with increasing mode-number because drag plays an increasingly important role in the dynamics. Low-order modes essentially have an inviscid dynamics and are inherently non-local in nature. High-order modes are dominated by drag and are local along the coast. It is this change in character that allows the model to develop a realistic flow field. For example, the model generates a coastal undercurrent in good agreement with observations (McCreary, 1980b).

The solutions simplify considerably in the weak-drag steady limit, where $\omega \to 0$. According to (87) the value of k reduces to

$$\lim_{\omega \to 0} k = \frac{\omega}{\beta} \alpha^2 \longrightarrow 0 .$$

Consequently,

$$\lim_{\omega \to 0} \Delta \longrightarrow 0, \qquad \lim_{\omega \to 0} e^{\Delta} \longrightarrow 1 .$$

So for the low-order modes the solutions (91) become

$$p_n = \tau_{on} \int_{y_0}^{y} Y(y)\, dy ,$$

$$u_n \longrightarrow 0, \qquad v_n \longrightarrow 0, \qquad w_n \longrightarrow 0 . \tag{96}$$

There is no flow associated with this balance; rather a pressure gradient exists everywhere that exactly balances the wind. This oceanic state is just the Sverdrup balance for this wind field since there is no wind curl.

The solutions again simplify in the strong-drag steady limit where $\omega \to \infty$. According to (87) the value of k reduces to

$$4 \frac{\omega^2}{\beta^2} \alpha^2 \gg 1 \quad \Rightarrow \quad \lim_{\omega \to \infty} k \longrightarrow -i\alpha. \tag{97}$$

I choose $y_c = 0$, and in that case Δ becomes

$$\lim_{\omega \to \infty} \Delta = -\int_0^y \frac{A}{c^2} \frac{\alpha}{f} dy = -\frac{A}{c^3} y \longrightarrow \infty,$$

and

$$e^{\Delta} \int_{y_0}^{y} e^{-\Delta} Y dy \longrightarrow \frac{c^3}{A} Y. \tag{98}$$

So for the high-order modes (91) becomes

$$\begin{aligned} p_n &= \tau_{on} \frac{c^3}{A} Y(y)\, e^{\alpha(x-x_1)}, \\ v_n &= \frac{p_n}{c}, \qquad w_n = \frac{A}{c^4} p_n, \\ u_n &= \tau_{on} \frac{Y}{f} \left[1 - e^{\alpha(x-x_1)}\right] - \frac{x-x_1}{f} \beta \frac{p_n}{c}. \end{aligned} \tag{99}$$

The flow field in this limit is strongly trapped to the coast, except for the zonal velocity field. The zonal flow vanishes there and converges to pseudo-Ekman flow back in the ocean interior.

CONCLUSIONS

It is essential to understand the limitations of any ocean model. The list of assumptions in Section 2 of these notes points out the limitations of the present one. It is equally essential to understand clearly the usefulness of a model. To conclude these notes, then, I list the advantages of the present one.

1. Solutions are found analytically, rather than numerically. As in similar inviscid models, they are expanded into sums over vertical normal modes. This expansion allows an insightful way of discussing the dynamics. Rather than describing the dominant balances which govern the total flow field, it is possible instead to describe those balances which occur for individual modes. In this way the three-dimensional dynamics of equations (1) can be understood by using concepts appropriate to the simpler two-dimensional dynamics of equations (13).

2. Vertical friction affects the dynamics of each vertical mode simply by introducing a linear drag. The <u>drag coefficient is mode dependent</u>, and increases roughly like the square of the modenumber. This property is perhaps the single most important aspect of the model dynamics. As a result, high-order and low-order modes respond quite differently.

3. For the low-order modes drag is weak. The effects of radiation are apparent. For example, if the wind stress is switched-on the locally forced region radiates patches of Rossby and Kelvin waves, and eventually the mode adjusts to Sverdrup balance.

4. For the high-order modes the drag dominates the dynamics. Waves are strongly damped before they can propagate out of the region of the winds, and so the dynamics is local. The high-order modes always sum to create the Ekman drift component to the flow field.

5. Finally, the model has been applied with success to a wide variety of problems. For example, the model generates a realistic Equatorial Undercurrent, and provides a possible explanation for the poleward-flowing Coastal Undercurrent found along most eastern ocean boundaries. It is currently being used to investigate the annual cycle of equatorial and coastal currents (McCreary, 1980c), and also the effects of wind curl in the tropical ocean.

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