Heterotic String Compactifications

Geometry and Moduli

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Introduction

Let X be a 6 dimensional Riemannian manifold and V be a vector bundle on X.

Interested in geometry of pairs (X, V) relevant to supersymmetric compactifications of heterotic strings.

We have the following mathematical objects on these theories:

- Riemannian metric g_{mn} on X
- a scalar ϕ (the dilaton)
- Gauge fields A for the gauge group $G \subseteq E_8 \times E_8$.
  So, there is a vector bundle V on X with structure group contained in $E_8 \times E_8$.
- 3-form H, the flux, defined by

$$H = \mathrm{d}B + \frac{\alpha'}{4}(\mathcal{CS}[A] - \mathcal{CS}[\Theta])\,, \quad \mathcal{CS}[A] = \mathrm{tr}(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A)\,.$$

Want: The geometry and moduli of these compactifications.
But eventually we want to construct the effective field theory.

Outline

- Geometry of (X, V)
  (The class of manifolds X and bundles V which can occur)

  Review of A Strominger 1986 and C Hull 1987

- The tangent space of the moduli space of heterotic compactifications
  (XD and E Svanes, 1402.1725 and 1509.08724; XD, E Svanes and E Hardy 1409.7539; Anderson, Gray and Sharpe 1402.1532)
  - Moduli Space of X
  - Moduli Space of vector bundles over X and the Atiyah map
  - Moduli Space of heterotic compactifications

- Questions and open problems

The Geometry of X

Supersymmetry requires that on X there exist

- ω a non-degenerate well defined real 2 form
- Ψ is a locally decomposable no-where vanishing well defined complex 3-form
- $\Psi \wedge \omega = 0\ , \qquad \frac{1}{||\Psi||^2}\, \mathrm{i}\, \Psi \wedge \bar{\Psi} = \frac{1}{6}\, \omega \wedge \omega \wedge \omega = \mathrm{d}vol_6\ .$

The Geometry of X

The forms ω and Ψ must also satisfy differential equations

- $\mathrm{d}(e^{-2\phi}\, \omega \wedge \omega) = 0$ $\quad$ X is conformally balanced.
- $\mathrm{d}(e^{-2\phi}\, \Psi) = 0$.

  $\Omega = e^{-2\phi}\, \Psi$ is a holomorphic $(3, 0)$-form (unique up to a constant).
- $\Psi \wedge \omega = 0$, $\quad \frac{1}{||\Psi||^2}\, i\, \Psi \wedge \bar{\Psi} = \frac{1}{6}\, \omega \wedge \omega \wedge \omega = \mathrm{d}vol_6$.

Moreover, the flux must satisfy

- $H = J(\mathrm{d}\omega)$ $\quad$ [Recall that $H = \mathrm{d}B + \frac{\alpha'}{4}(\mathcal{CS}[A] - \mathcal{CS}[\Theta])$]

Remark:
If $H = 0$, then $\mathrm{d}\omega = 0$, so X is Kähler and therefore Calabi-Yau.

The Geometry of X

Summary

X is a complex hermitian manifold which admits a conformally balanced metric and admits a holomorphic $(3,0)$ form (and thus has vanishing first class).

The flux H on X must satisfy

$$H = J(\mathrm{d}\omega) = \mathrm{d}B + \frac{\alpha'}{4}(\mathcal{CS}[A] - \mathcal{CS}[\Theta])$$

The manifolds with this structure will be called manifolds with a Heterotic Structure.

The Geometry of X

Examples

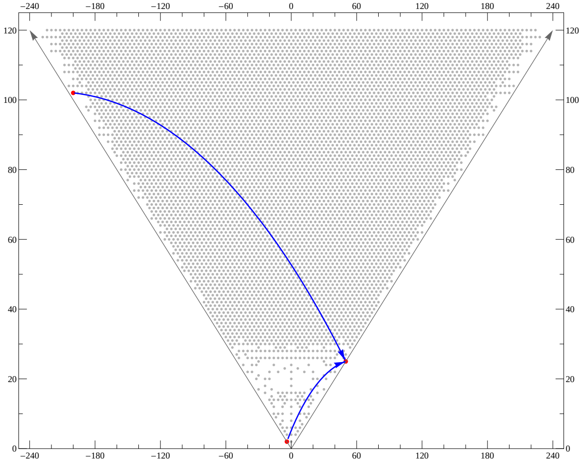
- Calabi-Yau manifolds
- ${}^{k}\#S^3 \times S^3$, $k \geq 2$ Li and Tian; Fu, Li,and Yau
- Non-Kähler elliptic fibrations over Kähler surfaces (like T^4, $K3$, $\mathbb{P}^2$, ...).

  Goldstein and Prokushkin; Fu and Yau; Becker, Becker, Yau, Sethi, Dasgupta, H Lee and XD; etc
- Let $\mathcal{C}$ be the class of all manifolds bimeromorphic to Kähler manifolds (this is the Fujiki class). These are all balanced

  Alessandrini and Bassanelli 1993.

  Wanted: those manifolds in $\mathcal{C}$ which have vanishing first Chern class?
- Conifold transitions from Calabi-Yau manifolds?

Conifold Transitions



Constraints on V

Supersymmetry imposes conditions on the curvature F of the Yang-Mills connection:

$$F^{(0,2)} = 0\,, \qquad \omega \lrcorner F = 0\,.$$

That is, we want a holomorphic bundle V with a connection which is an instanton.

There are many examples and constructions, but more needs to be done.

Anderson, Gray, Lukas, He, Ovrut, Palti; Bouchard, Donagi, Pantev; Li, Yau; Andreas, García-Fernandez, etc

A Further Constraint on (X, V)

There is a further constraint on (X, V) (anomaly cancelation condition)

$$\mathrm{d}H = \mathrm{d}(J(\mathrm{d}\omega)) = \frac{\alpha'}{4}\left(\mathrm{tr}\, F \wedge F - \mathrm{tr}\, R^l \wedge R^l\right) \ .$$

where R^l is the curvature 2-form of X with respect to a metric connection ∇^l on TX.
A solution of the supersymmetry conditions, which also satisfies the anomaly cancelation automatically satisfies the equations of motion iff ∇^l satisfies

$$R^l \wedge \Omega = 0 \ , \qquad \omega \lrcorner R^l = 0 \ .$$

∇^l must be an instanton. (Hull, Ivanov, Martelli and Sparks)

A Further Constraint on (X, V)

Trick: promote the connection ∇^I to a dynamical field.

This idea has also appeared in work by García-Fernández and Baraglia+Hekmati (they where attempting to construct generalised geometry for heterotic strings, see also Anderson+Gray+Sharpe.

We will need this to be able to include the anomaly cancelation condition.

One can prove that these correspond to field redefinitions (XD and E Svanes: ArXiv 1409.7539)

Summary: The Strominger/Hull system

A supersymmetric heterotic compactification is a pair (X, V), where

(X, ω, Ω) is a 3-fold which has a heterotic structure and

V and TX are a (poly)stable holomorphic vector bundles on X,

and which satisfies the anomaly cancelation

$$H = J(\mathrm{d}\omega) = \mathrm{d}B + \frac{\alpha'}{4}(\mathcal{CS}[A] - \mathcal{CS}[\Theta]) .$$

NEXT: The Strominger/Hull system on (X, V) is equivalent to the holomorphic structure on $\bar{D}$ on an extension bundle $\mathcal{Q}$ on X.

Moduli Space of X

Becker, Becker, Tseng, Yau, Sethi, Sharpe, Donagi, Melnikov, Katz, etc

Consider a one parameter family of manifolds with a Heterotic $SU(3)$ structure $(X_t, \omega_t, \Omega_t)$ with $(X_0, \omega_0, \Omega_0) = (X, \omega, \Omega)$

Deformations of the complex structure of X:

$$J_t = J + t\Delta_t$$

To first order

$$\Delta_t \in H_{\bar{\partial}}^{(0,1)}(X, TX) \cong H_{\bar{\partial}}^{(2,1)}(X)$$

Remark: these are unobstructed if the $\partial\bar{\partial}$-lemma is satisfied.

Moduli Space of X

The deformations of the hermitian form ω are complicated.

Let $$\rho = \frac{1}{2}\, e^{-2\phi}\, \omega \wedge \omega$$

The conformally balanced condition then reads:

$$\mathrm{d}\rho = 0$$

Deformations must preserve this: $\mathrm{d}\partial_t \rho = 0$

Modding out diffeomorphisms

$$\mathcal{L}_v \rho = \mathrm{d}(v \lrcorner \rho) = -\mathrm{d}(e^{-2\phi}\, J(v) \wedge \omega)$$

Space of deformations is not necessarily finite.

M. Becker, Tseng and Yau, 2006

This problem fixed when including the anomaly cancelation condition.

Deformations of the holomorphic structure on (X, V)

Consider a family

$$(X_t, V_t)$$

of heterotic compactifications with $(X, V) = (X_0, V_0)$.

Keep ω fixed and ignore the anomaly cancelation condition (we will relax these conditions later).

Thus, the generic parameter t refers to

- z^a are complex structure parameters
- w^i are bundle moduli

Deformations of the holomorphic structure on (X, V)

Want:

- deformations of the complex structure J of X which preserve the integrability of the complex structure

$$J + t\Delta_t \ , \quad \Delta \in H_{\bar{\partial}}^{(0,1)}(X, TX)$$

- deformations of the bundle V for fixed J:

$$A + t\partial_t A$$

such that simultaneous deformations of J and A preserve the holomorphic structure of the bundle V.

This question was answered by M Atiyah.

Deformations of the holomorphic structure on (X, V)

Let F be the curvature of the bundle V

$$F = \mathrm{d}A + A \wedge A$$

Let $\mathcal{A} = A^{(0,1)}$. A holomorphic structure on V is determined by a holomorphic connection $\bar{\partial}_{\mathcal{A}}$

$$\bar{\partial}_{\mathcal{A}}\beta = \bar{\partial}\beta + \mathcal{A} \wedge \beta + (-1)^q \, \beta \wedge \mathcal{A}, \quad \forall \beta \in \Omega^{(*,*)}(X)$$

which squares to zero

$$\bar{\partial}^2_{\mathcal{A}} = 0$$

(It is easy to prove that $\bar{\partial}^2_{\mathcal{A}} = 0$ equivalent to $F^{(0,2)} = 0$.)

A deformation of J changes what we mean by a (p, q) form.

Deformations of the holomorphic structure on (X, V)

We find

$$\bar{\partial}_{\mathcal{A}}\alpha_t = \Delta_t^m \wedge F_{mn}\,\mathrm{d}x^n$$

where $\alpha_t = (\partial_t A)^{(0,1)}$. If we vary the bundle keeping J fixed

$$\bar{\partial}_{\mathcal{A}}\alpha_i = 0 \quad \Longrightarrow \quad \alpha_i \in H_{\bar{\partial}_{\mathcal{A}}}{}^{(0,1)}(X, \mathrm{End}(V))$$

However, varying J so that V stays holomorphic poses a constraint on the elements $\Delta_a \in H_{\bar{\partial}}^{(0,1)}(X, TX)$:

$$\bar{\partial}_{\mathcal{A}}\alpha_a = \Delta_a^m \wedge F_{mn}\,\mathrm{d}x^n$$

that is, $\Delta_a^m \wedge F_{mn}$ must be $\bar{\partial}_{\mathcal{A}}$-exact.

Deformations of the holomorphic structure on (X, V)

We want to refrase this constraint.

Let $\mathcal{F}$ be the map (this is the Atiyah map)

$$\begin{aligned} \mathcal{F} : \Omega^{(0,q)}(X, TX) &\longrightarrow \Omega^{(0,q+1)}(X, \mathrm{End}(V)) \\ \Delta \qquad &\longmapsto \mathcal{F}(\Delta) = (-1)^q \, \Delta^m \wedge F_{mn} \, \mathrm{d}x^n \end{aligned}$$

$\mathcal{F}$ is in fact a map between cohomologies

$$\mathcal{F}(\bar{\partial}\Delta) + \bar{\partial}_{\mathcal{A}} \mathcal{F}(\Delta) = 0$$

This follows from the Bianchi identity: $\bar{\partial}_{\mathcal{A}} F = 0.$

It follows that the constraint on Δ is

$$\Delta \in \ker \mathcal{F} \subseteq H_{\bar{\partial}}{}^{(0,1)}(X, TX)$$

Therefore

$$\mathcal{TM}_1 = H_{\bar{\partial}_{\mathcal{A}}}{}^{(0,1)}(X, \mathrm{End}(V)) \oplus \ker \mathcal{F} \, , \quad \ker \mathcal{F} \subseteq H_{\bar{\partial}}{}^{(0,1)}(X, TX)$$

Deformations of the holomorphic structure on (X, V)

Equivalently: let the bundle $\mathcal{Q}_1$ be defined by

$$0 \to \mathrm{End}(V) \to \mathcal{Q}_1 \to TX \to 0 \ ,$$

with extension class $\mathcal{F}$. There is a holomorphic structure $\bar{\partial}_1$ on $\mathcal{Q}_1$ defined by the exterior derivative

$$\bar{\partial}_1 \ = \left[\begin{array}{cc} \bar{\partial}_{\mathcal{A}} & \mathcal{F} \\ 0 & \bar{\partial} \end{array} \right],$$

which acts on $\Omega^{(0,q)}(X, \mathcal{Q}_1)$. Note that

$$\bar{\partial}_1^2 = 0 \quad \Longleftrightarrow \quad \mathcal{F}(\bar{\partial}\Delta) + \bar{\partial}_{\mathcal{A}}\mathcal{F}(\Delta) = 0 \quad \Longleftrightarrow \quad \bar{\partial}_{\mathcal{A}} F = 0$$

Deformations of the holomorphic structure on (X, V)

The infinitesimal moduli space of the holomorphic structure $\bar{\partial}_1$ on the extension bundle $\mathcal{Q}_1$, is given by

$$
\begin{aligned}
\mathcal{T}\mathcal{M}_1 &= H^{(0,1)}_{\bar{\partial}_1}(X, \mathcal{Q}_1) \\
&= H_{\bar{\partial}_{\mathcal{A}}}{}^{(0,1)}(X, \mathrm{End}(V)) \oplus \ker \mathcal{F}
\end{aligned}
$$

where

$$\ker \mathcal{F} \subseteq H_{\bar{\partial}}{}^{(0,1)}(X, TX)$$

and gives the simultaneous deformations of $\bar{\partial}_{\mathcal{A}}$ and $\bar{\partial}$ which preserve the holomorphic structures.

Deformations of the holomorphic tangent bundle

We extend the holomorphic structure on $\mathcal{Q}_1$ to include the variations on the holomorphic structure ∇^I on TX.

Let E be the bundle defined by

$$0 \to \mathrm{End}(TX) \to E \to \mathcal{Q}_1 \to 0$$

There is a holomorphic structure on E defined by the holomorphic connection $\bar{\partial}_E$

$$\bar{\partial}_E \;=\; \begin{bmatrix} \bar{\partial}_{\vartheta^I} & 0 & \mathcal{R} \\ 0 & \bar{\partial}_{\mathcal{A}} & \mathcal{F} \\ 0 & 0 & \bar{\partial} \end{bmatrix}$$

which acts on $\Omega^{(0,q)}(E)$ and squares to zero $\bar{\partial}_E^2 = 0$.

In fact: $\quad \bar{\partial}_E^2 = 0 \quad \Longleftrightarrow \quad \bar{\partial}_{\mathcal{A}} F = 0 \quad \text{and} \quad \bar{\partial}_{\vartheta^I} R = 0.$

Deformations of the holomorphic tangent bundle

The infinitesimal moduli space of the holomorphic structure $\bar{\partial}_E$ on the extension bundle E

$$\mathcal{TM}_E = H^{(0,1)}_{\bar{\partial}_E}(X, E) \ ,$$

gives the simultaneous deformations of $\bar{\partial}_{\vartheta^I}$, $\bar{\partial}_{\mathcal{A}}$ and $\bar{\partial}$ which preserve the holomorphic structures. We obtain

$$\mathcal{TM}_E = H^{(0,1)}_{\bar{\partial}_{\vartheta^I}}(X, \mathrm{End}(TX)) \oplus H^{(0,1)}_{\bar{\partial}_{\mathcal{A}}}(X, \mathrm{End}(V)) \oplus (\ker \mathcal{F} \cap \ker \mathcal{R})$$

$$\ker \mathcal{F} \cap \ker \mathcal{R} \subseteq H_{\bar{\partial}}^{(0,1)}(X, TX)$$

Remark: The “moduli" in $H^{(0,1)}_{\bar{\partial}_{\vartheta^I}}(X, \mathrm{End}(TX))$ are not physical. They correspond to field redefinitions. XD and E Svanes 1409.7539

Anomaly cancelation condition and the moduli space

Now we let ω vary and try to include the anomaly cancelation condition.

Recall the Bianchi identity for the anomaly cancelation condition

$$\mathrm{d}H = \mathrm{d}(J(\mathrm{d}\omega)) = \frac{\alpha'}{4}\left(\mathrm{tr}(F \wedge F) - \mathrm{tr}(R \wedge R)\right)$$

Is there an extension bundle $\mathcal{Q}$ of E with a holomorphic structure $\bar{D}$ which enforces the anomaly?

Anomaly cancelation condition and the moduli space

Let $\mathcal{Q}$ be the extension bundle

$$0 \to T^*X \to \mathcal{Q} \to E \to 0$$

We now define a holomorphic structure on $\mathcal{Q}$ by defining an operator $\bar{D}$ on $\mathcal{Q}$

$$\bar{D} = \begin{bmatrix} \bar{\partial} & \mathcal{H} \\ 0 & \bar{\partial}_E \end{bmatrix}$$

where we construct the map $\mathcal{H}$ such that the condition $\bar{D}^2 = 0$ is equivalent to the Bianchi identity.

Anomaly cancelation condition and the moduli space

We find that $\mathcal{H}$ is given by

$$\mathcal{H} \ : \ \Omega^{(0,q)}(X, E) \longrightarrow \Omega^{(0,q+1)}(X, T^{*(1,0)}X)$$

$$\mathcal{H}(x) = i(-1)^q \, \Delta^p \wedge (\partial\omega)_{pmn} \, \mathrm{d}x^m \wedge \mathrm{d}x^n - \frac{\alpha'}{4}(\mathrm{tr}(\alpha \wedge F) - \mathrm{tr}(\kappa \wedge R))$$

where

$$x = \begin{pmatrix} \kappa \\ \alpha \\ \Delta \end{pmatrix} \in \Omega^{(0,q)}(X, E)$$

and Δ is valued in $T^{(1,0)}X$, α is valued in $\mathrm{End}(V)$, and κ is valued in $\mathrm{End}(TX)$.

Anomaly cancelation condition and the moduli space

We have

$$\mathcal{H}(x) = i(-1)^q \, \Delta^p \wedge (\partial\omega)_{pmn} \, \mathrm{d}x^m \wedge \mathrm{d}x^n - \frac{\alpha'}{4}(\mathrm{tr}(\alpha \wedge F) - \mathrm{tr}(\kappa \wedge R^l))$$

The map $\mathcal{H}$ is a well defined map between cohomologies:

$$\bar{\partial}(\mathcal{H}(x)) + \mathcal{H}(\bar{\partial}_E(x)) = 0 \; , \quad \forall x \in \Omega^{(0,q)}(X, E)$$

This condition is satisfied iff the Bianchi identity is satisfied.

The Strominger/Hull system on (X, V) is equivalent to the holomorphic structure on $\bar{D}$ on $\mathcal{Q}$.

Anomaly cancelation condition and the moduli space

The infinitesimal moduli space of the Strominger/Hull system corresponds to the deformations of the holomorphic structure defined by $\bar{D}$ on the extension bundle $\mathcal{Q}$

$$H^1_{\bar{D}}(X,\mathcal{Q}) \cong \left[H^1_{\bar{\partial}}(X, T^*X) \Big/ \mathrm{Im}(\mathcal{H}_0) \right] \oplus \ker(\mathcal{H})$$

- Deformations of the holomorphic structure on E

$$\ker(\mathcal{H}) \subseteq H^{(0,1)}_{\bar{\partial}_E}(X, E) \ ,$$

- Moduli of the (complexified) hermitian structure

$$\mathcal{M}_{HS} = \left[H^1_{\bar{\partial}}(X, T^*X) \Big/ \mathrm{Im}(\mathcal{H}_0) \right]$$

The elements are

$$\mathcal{Z}_t = \mathcal{B}_t + i\partial_t\omega$$

where $\mathcal{B}_t = \partial_t B - \frac{\alpha'}{4} \left(\mathrm{tr}(A \wedge \partial_t A) - \mathrm{tr}(\Theta^I \wedge \partial_t \Theta^I) \right)$

Outlook

Main result:

Strominger system on X $\longleftrightarrow$ holomorphic structure $\bar{D}$ on $\mathcal{Q}$

Infinitesimal moduli of the Strominger/Hull system $\longleftrightarrow$ infinitesimal deformations of the holomorphic structure on $\mathcal{Q}$

In terms of the low energy effective field theory:

$$\text{Massless fields} \longleftrightarrow \text{Moduli} \longleftrightarrow H^{(0,1)}_{\bar{D}}$$

Main message: there are many open questions.

Open Problems

- Better understanding of the moduli space of (X, V)

  kinetic terms in the action $\longleftrightarrow$ Metric on moduli space

  Supersymmetry predicts that this metric should be Kähler.

  (to appear: with Candelas and McOrist)

- Domain Wall compactifications: X has an integrable G_2 structure and V has an instanton connection

  (with M Larfors and E Svanes, in progress)

- Quantum corrections? We have world sheet instanton corrections and NS5branes. What are the corrections to the classical geometry?
- Concept of mirror symmetry? Dualities?
- Transgressions: generalize the idea of geometric transitions to heterotic string theories

Happy Birthday Fernando!