Some modular features of string theory

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Feliz Cumpleaños !!

From your Cambridge colleagues

This talk will consider narrowly-focused aspects of maximally supersymmetric CLOSED STRING SCATTERING AMPLITUDES.

- SOME FEATURES OF STRING PERTURBATION THEORY

  Modular invariants of higher-genus Riemann surfaces

  Mathematical connections to multiple-zeta values and their elliptic generalisations

  Modular Forms; Automorphic forms for higher-rank groups; Multi-Zeta Values; ….

- [NON-PERTURBATIVE FEATURES – S-DUALITY]:

  Connects perturbative with non-perturbative effects

  Constraints imposed by SUSY, Duality, Unitarity

With: Eric D'Hoker; Pierre Vanhove; Stephen Miller; Carlos Mafra; Oliver Schlotterer; Boris Pioline; Jorge Russo; Rudolfo Russo; Don Zagier; ………..

New papers

1502.06698 1509.00363
1512.06779 1603.00839

Font, Ibanez, Lust, **Quevedo,** ***Strong-weak coupling duality and non-perturbative effects in string theory,***
Phys. Lett. B249 (1990) 35

We conjecture the existence of a new discrete symmetry of the modular type relating weak and strong coupling in string theory. The existence of this symmetry would strongly constrain the non-perturbative behaviour in string partition functions and introduces the notion of a maximal (minimal) coupling constant. An effective lagrangian analysis suggests that the dilaton vacuum expectation value is dynamically fixed to be of order one. In supersymmetric heterotic strings, supersymmetry (as well as this modular symmetry itself) is generically spontaneously broken.

THE LOW ENERGY EXPANSION OF STRING THEORY

- LOWEST ORDER TERM reproduces the results of classical supergravity

EINSTEIN-HILBERT

$\alpha' = \ell_s^2$

ℓ_s - STRING LENGTH SCALE

$$\frac{1}{\alpha'^4}\int d^{10}x\sqrt{-\det G}\, e^{-2\phi}\, R + \dots$$ Interactions of other supergravity fields

METRIC – $G_{\mu\nu}$

compactify space-time to dimensions D < 10

SCALAR FIELD - DILATON

$e^{-\phi} = \frac{1}{g_s}$ ← STRING COUPLING CONSTANT

- HIGHER ORDER TERMS: (maximal supersymmetry)

$$\frac{1}{\alpha'}\int d^{10}x\,\sqrt{-\det G}\,\mathcal{F}(\phi,\dots)\,R^4 + \dots$$

COEFFICIENT depends on moduli (scalar fields)

- Expansion in powers of $\alpha' R\,,\quad \alpha' D^2\,,\quad \dots$

- Perturbative expansion in powers of $g = e^{-\phi}$

DOUBLE EXPANSION – PERTURBATION AND LOW ENERGY EXPANSION

SCALAR FIELDS (MODULI) AND S-DUALITY

SUPERGRAVITY (low energy limit of string theory):

Scalar fields parameterize a symmetric space $G(\mathbb{R})/K(\mathbb{R})$

groups in E_n series (real split forms) — (Cremmer, Julia)

Maximal compact subgroup

STRING THEORY:

Discrete identifications of scalar fields $G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$

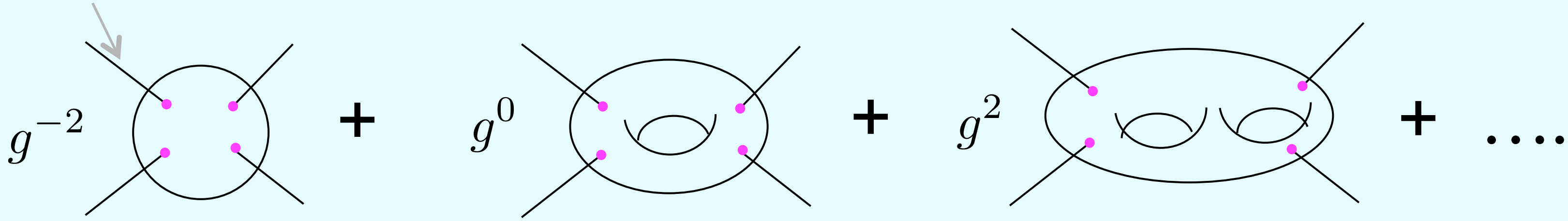
DUALITY GROUP $G(\mathbb{Z})$

Only a discrete arithmetic subgroup of $G(\mathbb{R})$ is symmetry of string theory

STRING PERTURBATION THEORY: Expansion around boundary of moduli space

e.g. in powers of $g = e^{\phi} \to 0$

Vertex operators



acts on complex structure of torus $\longrightarrow$ $SL(2,\mathbb{Z})$ $\quad Sp(4,\mathbb{Z})$ $\quad$ "$Sp(2h,\mathbb{Z})$"

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad a,\, b,\, c,\, d \in \mathbb{Z} \quad ad - bc = 1$$

WORLD-SHEET automorphic symmetries

e.g.

Four-Graviton Scattering in Type II String Theory

$$A^{(4)}(\epsilon_r, k_r; \Omega) = \mathcal{R}^4\, T(s,t,u;\Omega)$$

$s = -2\,k_1 \cdot k_2$
$t = -2\,k_1 \cdot k_4$
$u = -2\;k_1 \cdot k_3$

$\mathcal{R}$ linearized curvature $\sim k_\mu\, k_\nu\, \epsilon_{\rho\sigma}$

Symmetric function of Mandelstam invariants s, t, u (with $s+t+u=0$).
Has an expansion in power series of $\sigma_2 = s^2+t^2+u^2$ and $\sigma_3 = s^3+t^3+u^3$.

(non-analytic pieces are essential, but will be ignored for now)

$$T(s,t,u;\Omega) = \sum_{p,q} \mathcal{E}_{(p,q)}(\Omega)\, \sigma_2^p\, \sigma_3^q$$

$\sim s^{2p+3q} + \ldots$

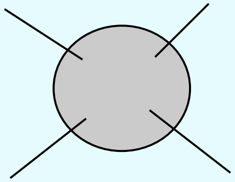
Coefficients are S-duality-invariant functions of scalar fields (moduli).

To what extent can we determine these coefficients?

Boundary data: String Perturbation Theory

$\Omega_2 \to \infty \quad (g \to 0)$

TREE-LEVEL (VIRASORO AMPLITUDE)



$$A_0^{(4)}(\epsilon_r, k_r) = g_s^{-2}\,\mathcal{R}^4\, T_0(s,t,u)$$

$$T_0^{(4)} = \frac{4}{stu}\,\frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)\Gamma(1-\alpha' u)}{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)\Gamma(1+\alpha' u)}$$

$s^k R^4 \sim d^{2k} R^4$

Tree-level SUPERGRAVITY

R^4 $\quad d^4 R^4$ $\quad d^6 R^4$ $\quad d^8 R^4$

$$= \frac{3}{\sigma_3} + 2\zeta(3)\,\alpha'^3 + \zeta(5)\,\alpha'^5 \sigma_2 + \frac{2\,\zeta(3)^2}{3}\,\alpha'^6\,\sigma_3 + \frac{\zeta(7)}{2}\,\alpha'^7\,\sigma_2^2$$

$$+ \frac{2\,\zeta(3)\,\zeta(5)}{3}\,\alpha'^8\,\sigma_2\,\sigma_3 + \frac{\zeta(9)}{4}\,\alpha'^8\,\sigma_2^3 + \frac{2}{27}(2\,\zeta(3)^2 + \zeta(9))\,\alpha'^9\,\sigma_3^2 + \ldots$$

$d^{10} R^4$ $\quad d^{12} R^4$

$\sigma_2 = s^2 + t^2 + u^2$

$\sigma_3 = s^3 + t^3 + u^3$

INFINITE SERIES of $d^{2k} R^4$ terms. Coefficients are powers of Riemann ζ values with rational coefficients

Generalises to N-particle scattering

ZETA VALUES AND MULTIPLE-ZETA VALUES

VERY BRIEF REVIEW

ZETA VALUES:

- Special values of POLYLOGARITHMS $Li_a(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^a}$

Riemann zeta $\zeta(a) = Li_a(1)$

Bernoulli numbers

Even zeta values $\zeta(2n) = -\frac{(2i)^{2n} B_{2n}}{2(2n)!} \pi^{2n}$ (powers of pi) e.g. $\zeta(2) = \frac{\pi^2}{6}$

Odd zeta values $\zeta(2n+1)$ transcendental? Independent?

MULTI-ZETA VALUES (MZV's):

- Special values of MULTIPLE POLYLOGARITHMS $Li_{a_1,\ldots,a_r}(z_1,\ldots,z_r) = \sum_{0<k_1<\cdots<k_r} \prod_{\ell=1}^{r} \left(\frac{z_\ell}{k_\ell}\right)^{a_\ell}$

$$\zeta(a_1,\ldots,a_r) = Li_{a_1,\ldots,a_r}(1,\ldots,1)) = \sum_{0<k_1<\cdots<k_r} \prod_{\ell=1}^{r} k_\ell^{-a_\ell}$$

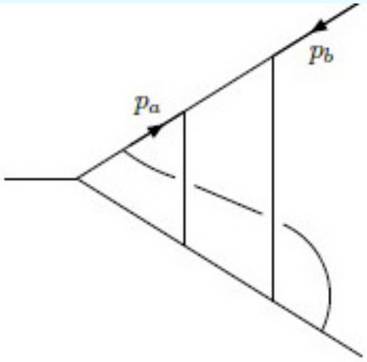
"weight" $w = \sum_{\ell=1}^{r} a_\ell$ "depth" r

- MZV's are numbers with fascinating algebraic properties inherited from the algebraic properties of multiple polylogarithms, e.g.

$$\zeta(5,2) = 5\zeta(2)\zeta(5) + 2\zeta(3)\zeta(4) - 11\zeta(7)$$

- First non-trivial (irreducible) case is $\zeta(5,3) + \ldots$ weight $w = 8$
- Arise in dimensional regularisation of renormalisable quantum field theories.

e.g. 3-loop contribution to form factor in N=4 supersymmetric Yang-Mills



F_5

$$F_5 = S_\Gamma^3 \left[-q^2 - i\eta\right]^{-2-3\epsilon} \cdot F_5^{\exp},$$

$$F_5^{\exp} = +\frac{1}{12\epsilon^6} + \frac{\pi^2}{27\epsilon^4} + \frac{17\zeta_3}{9\epsilon^3} + \frac{71\pi^4}{540\epsilon^2} + \frac{1}{\epsilon}\left(\frac{71\pi^2\zeta_3}{54} + \frac{13\zeta_5}{3}\right) - \frac{679\zeta_3^2}{6} + \frac{3991\pi^6}{136080} + \epsilon\left(-\frac{2837\pi^4\zeta_3}{540} + \frac{205\pi^2\zeta_5}{9} - \frac{25135\zeta_7}{24}\right) + \epsilon^2\left(\frac{4006}{3}\zeta_{5,3} - 59\zeta_3\zeta_5 - \frac{10\pi^2\zeta_3^2}{27} - \frac{14156063\pi^8}{16329600}\right) + \mathcal{O}(\epsilon^3).$$

$d = 4 - \epsilon$

Gehrmann, Henn, Huber

Note the irreducible MULTIZETA VALUE at weight 8.

Focus of mathematical interest within algebraic geometry and number theory

(e.g. Deligne, Zagier, Bloch, Brown, Kontsevich, Goncherov, Schnetz, ….)

However: Quite generally Feynman diagrams are known to lead to more general expressions, such as elliptic multiple polylogarithms and corresponding generalizations of multiple zeta values, as well as more exotic objects.

These even arise at quite low orders in perturbation theory in QCD and in N=4 supersymmetric Yang-Mills – contrary to some conjectures.

STRING THEORY provides a very efficient way of generating MZV's and these generalizations. There appears to be a strong analogy between the ϵ expansion of Feynman loop expansion and the α' of string theory - at least at tree level.

N-PARTICLE TREE AMPLITUDES

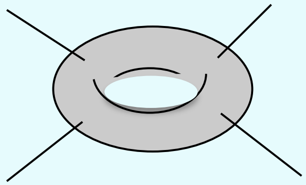
OPEN-STRING TREES: For $N > 4$ coefficients of higher derivative interactions of order
(Yang-Mills) α'^n are MZV's with weight n (Mafra, Schlotterer, Stieberger)

CLOSED-STRING TREES: For $N > 4$ coefficients are *SINGLE-VALUED* MZV's (svMZV's) (Brown)
(gravity)

- Special values OF *SINGLE-VALUED* MULTIPLE POLYLOGARITHMS – these have NO MONODROMIES (generalisations of BLOCH-WIGNER DILOGARITHM $\mathrm{Im}\left(\mathrm{Li}_2(z) + \log(1-z)\,\log|z|\right)$)
- No even s.v. zeta values $\zeta_{sv}(2n) = 0$ also $\zeta_{sv}(2n+1) = 2\zeta(2n+1)$
- First non-trivial case is $\zeta_{sv}(3,5,3) = 2\zeta(3,5,3) - 2\zeta(3)\zeta(3,5) - 10\zeta(3)^2\zeta(5)$
  weight $w = 11$

HOW DOES THIS GENERALIZE TO HIGHER GENUS ??

Genus one



Genus one amplitude

$$\mathcal{A}_1^{(4)}(\epsilon_r, k_r) = \frac{\pi}{16}\, \mathcal{R}^4 \int_{\mathcal{M}_1} \frac{d\tau^2}{y^2}\, \mathcal{B}_1(s,t,u;\tau)$$

Integral over complex structure $\tau = x + iy$

$$\mathcal{B}_1(s,t,u;\tau) = \frac{1}{y^4} \int_{\Sigma^4} \prod_{i=1}^{4} d^2 z \, \exp\left(-\frac{\alpha'}{2} \sum_{i<j} k_i \cdot k_j\, G(z_i, z_j)\right)$$

Vertex operator Corr. function

Green function

Low energy expansion - integrate powers of the genus-one Green function over the torus and over the modulus of the torus – difficult!

(MBG, D'Hoker, Russo, Vanhove)

Expanding in a power series in momenta gives (with $\alpha' = 4$)

$$\frac{1}{w!}\,\frac{1}{y^4} \int_{\Sigma^4} \prod_{i=1}^{4} d^2 z_i \left(\sum_{0<i<j\leq 4} s_{ij} G(z_i - z_j)\right)^w = \sum_i \sigma_2^{p_i}\, \sigma_3^{q_i}\, j^{(p_i,q_i)}(\tau)$$

$\sim s^w$

$$\sum_i (2p_i + 3q_i) = w$$

Coefficients of higher derivative interactions

Modular invariants for surface

Feynman diagrams on toroidal world-sheet

Coefficients of higher derivative interactions:

$$\Xi^{(p,q)} = \int_{\mathcal{M}_1} \frac{d^2\tau}{y^2}\, j^{(p,q)}(\tau)$$

"MODULAR GRAPH FUNCTIONS"

(D'Hoker, MBG, Vanhove)

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$j^{(p,q)}(\tau)$ is sum of world-sheet Feynman diagrams.

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

Each of these is a modular function - invariant under $SL(2,\mathbb{Z})$

$$a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1$$

The Green function on a torus of complex structure : $\tau = x + iy$

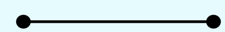
$$G(z) = -\ln\left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right|^2 - \frac{\pi}{2y}(z - \bar{z})^2 \qquad z = u + \tau\, v$$

doubly periodic function

$$= \sum_{(m,n)\neq(0,0)} \hat{G}(m,n) e^{2\pi i(mu - nv)} + 2\,\ln\left(2\pi\,|\eta(\tau)|^2\right)$$

MOMENTUM-SPACE PROPAGATOR: integer world-sheet momenta $m, n \in \mathbb{Z}$

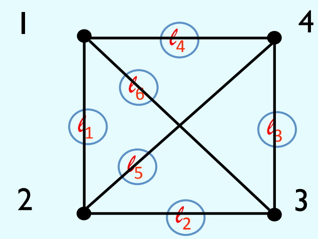
$$\hat{G}(m,n) = \frac{y}{|m\tau + n|^2}$$



General contribution to 4-particle amplitude: $i\,,\ j = 1, 2, 3, 4$

Modular function

$$D_{\ell_1,\ell_2,\ell_3,\ell_4;\ell_5,\ell_6} =$$



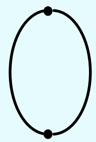
ℓ_S labels number of propagators on line S

"Weight" $w = \ell_1 + \ell_2 + \cdots + \ell_6$ contributes to $D^{2w}\,\mathcal{R}^4$

World-sheet Feynman diagrams

Multiple sums:

e.g. $d^4\mathcal{R}^4$



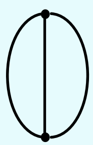
D_2

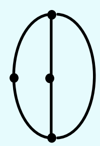
$$= \sum_{(m,n)\neq(0,0)} \frac{y^2}{|m\tau+n|^4} \equiv E_2(\tau)$$

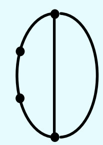
← Non-holomorphic SL(2) Eisenstein series

$$E_s(\tau) = \sum_{(m,n)\neq(0,0)} \frac{y^s}{|m\tau+n|^{2s}}$$

e.g. $C_{a,b,c}$ sequence $w = a+b+c$ $(w-1)$ vertices $d^{2w}\mathcal{R}^4$







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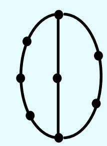

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$C_{1,1,1} \equiv D_3$ $d^6\mathcal{R}^4$

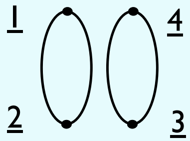
$C_{2,2,1} \equiv D_{1,1,1,1;1}$ $d^{10}\mathcal{R}^4$

$C_{3,1,1} \equiv D_{2,1,1,1}$ $d^{10}\mathcal{R}^4$

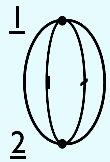
$C_{4,3,2}$ $d^{18}\mathcal{R}^4$

$$C_{a,b,c}(\tau) = \sum_{\substack{(m_r,n_r)\neq(0,0)\\ \sum_i m_i=0=\sum_j n_j}} \frac{y^{a+b+c}}{|m_1\tau+n_1|^{2a}|m_2\tau+n_2|^{2b}|m_3\tau+n_3|^{2c}}$$

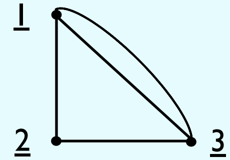
CONTRIBUTIONS TO $d^8\mathcal{R}^4$ (WEIGHT-4)



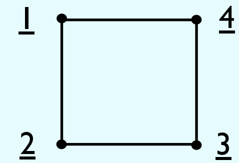
$D_2^2 = E_2^2$



D_4

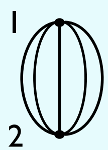


$D_{2,1,1} \equiv C_{2,1,1}$

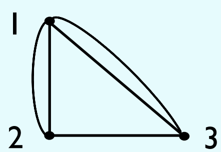


$D_{1,1,1,1}$

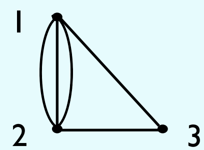
CONTRIBUTIONS TO $D^{10}\,\mathcal{R}^4$ (WEIGHT-5)



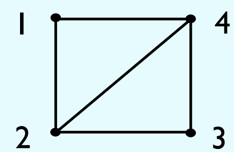
D_5



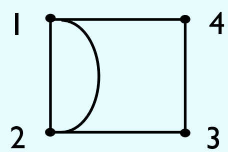
$D_{2,2,1}$



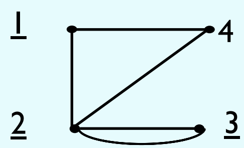
$D_{3,1,1}$



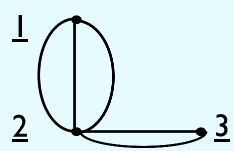
$D_{1,1,1,1;1} \equiv C_{2,2,1}$



$D_{2,1,1,1} \equiv C_{3,1,1}$



$D_{1,1,1}\,D_2$



$D_3\,D_2$

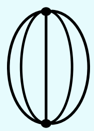
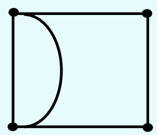
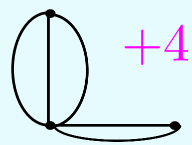
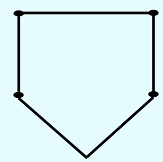
- Direct analysis looks forbidding. But $C_{a,b,c}$ functions satisfy simple Laplace equations with Laplacian $\Delta = y^2\,(\partial_x^2 + \partial_y^2)$

- Modular graph functions of arbitrary weight are special values of

  SINGLE-VALUED ELLIPTIC MULTIPLE POLYLOGARITHMS (D'Hoker, MBG, Gurdogan, Vanhove)

- As with MZV's, these elliptic functions satisfy a fascinating set of polynomial relationships – we have found just a few of these.

e.g. weight 5 $\mathcal{F} \equiv \quad D_5 - 60\,C_{3,1,1} - 10\,E_2\,C_{1,1,1} + 48\,E_5 - 16\,\zeta(5) = 0$

$=$  -60  -10  $+48$  $-16\,\zeta(5)$

polynomial of weight 5 in functions of different depth (no. of loops)

- Such identities relate Feynman diagrams with different numbers of loops.

Proof: Show (with great difficulty!) by manipulating multiple sums that $\nabla^4\mathcal{F} = 0$

where $\nabla = \tau_2^2\,\dfrac{\partial}{\partial\tau}$

CONJECTURE:

MODULAR GRAPH FUNCTIONS SATISFY POLYNOMIAL RELATIONS WITH RATIONAL COEFFICIENTS

Elliptic generalisation of the rational polynomial relations between multiple polylogarithms and single-valued MZV's

QUESTION:

WHAT IS THE BASIS OF MODULAR GRAPH FUNCTIONS?

Analogous to interesting issues in multiple zeta vlues

Integration over fundamental domain

Genus-one coefficients $\Xi^{(p,q)} = \int_{\mathcal{M}_1} \frac{d^2\tau}{y^2}\, j^{(p,q)}(\tau)$

Genus-one expansion:

After ntegrating over τ (and dealing with divergences) gives the one-loop expansion:

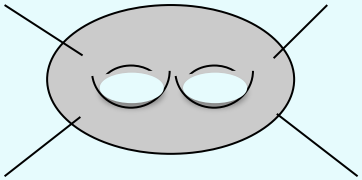
$$A_1^{(4)} = \frac{\pi}{3}\left(1 + 0\,\sigma_2 + \frac{\zeta(3)}{3}\,\sigma_3 + 0\,\sigma_2^2 + \frac{116\,\zeta(5)}{5}\,\sigma_2\,\sigma_3 \dots\right)\mathcal{R}^4$$

$\mathcal{R}^4$ $d^4\,\mathcal{R}^4$ $d^6\,\mathcal{R}^4$ $d^8\,\mathcal{R}^4$ $d^{10}\,\mathcal{R}^4$

These coefficients are analogous to the tree-level coefficients:

What is the connection between them??

Genus Two



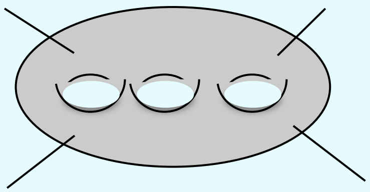
Amplitude is explicit but difficult to study. Low energy expansion:

(D'Hoker, MBG, Pioline, R. Russo)

Result:
$$A_2^{(4)} = g_s^2\Big(\frac{4}{3}\zeta(4)\,\sigma_2 R^4 \;+4\zeta(4)\sigma_3 R^4 + \dots\Big)$$

$d^4\,R^4$ $\qquad$ $d^6\,R^4$

Genus Three



Technical difficulties analysing 3-loops. Gomez and Mafra evaluated the leading low energy behaviour using PURE SPINOR FORMALISM, giving

$$A_3^{(4)} = g_s^4\left(\frac{4}{27}\zeta(6)\,\sigma_3 + \dots\right)\mathcal{R}^4$$

$d^6\,R^4$

Higher Orders

New problems - No explicit expressions

Non-perturbative extension

How powerful are the constraints imposed by (maximal) SUSY and Duality ??

Investigate the exact moduli dependence of low lying terms in the low energy expansion.

Duality relates different regions of moduli space –
Connects perturbative and non-perturbative features in a highly nontrivial manner.

Simple example:

10-dimensional Type IIB - maximal supersymmetry

One complex modulus $\Omega = \Omega_1 + i\Omega_2$

inverse string coupling constant $\longrightarrow$ $\Omega_2 = \frac{1}{g} = e^{-\phi}$

Duality group $SL(2,\mathbb{Z})$ $\Omega \to \frac{a\,\Omega + b}{c\Omega + d}$ $a, b, c, d \in \mathbb{Z}$, $ad - bc = 1$

Relates strong and weak coupling.

Non-perturbative extension

e.g. rank-d S-duality groups for M-theory on a d-torus

$E_d(\mathbb{Z})$ sequence on d-torus

	$SL(2,\mathbb{Z})$	$SL(2,\mathbb{Z})$	$SL(3,\mathbb{Z}) \times SL(2,\mathbb{Z})$	$SL(5,\mathbb{Z})$	$SO(5,5,\mathbb{Z})$	$E_{6(6)}(\mathbb{Z})$	$E_{7(7)}(\mathbb{Z})$	$E_{8(8)}(\mathbb{Z})$
D=11-d	10B	9	8	7	6	5	4	3

Using:

- Nonlinear maximal supersymmetry
- Duality between M-theory (quantum 11-dimensional supergravity) on d-torus and and string theory compactified on a (d-1)-torus

Leads to a precise expressions for the exact coefficients of "BPS" interactions

$$\mathcal{R}^4 \quad d^4\,\mathcal{R}^4 \quad d^6\,\mathcal{R}^4$$

These coefficients are automorphic functions of the S-duality groups:

Generalizations of Non-holomorphic Eisenstein series

The perturbative terms agree exactly with string perturbation theory at genus 0,1,2,3

But that is the subject of a separate lecture !!

Feliz Cumpleaños !!

WE WANT YOU BACK!!