Duality and Topological Phases.

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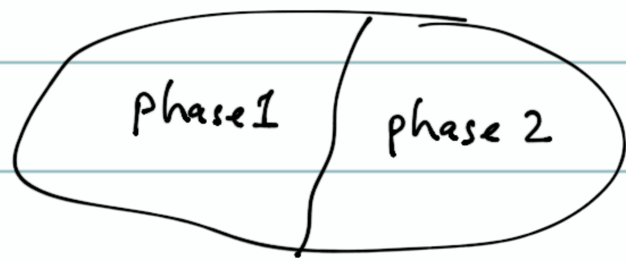
Plan for Lectures

1. QUANTUM PHASES of MANY BODY SYSTEMS ($T=0$, $N \to \infty$) Emergent property, qualitative properties differ between phases



2. Symmetry, topology. Locality plays a big role.

3. How to describe phases? Phase transition?

ENTIRELY NEW PROPERTIES → eg. insulator / conductor / superconductor. New "vacuua" that are possible.

A tabletop "multiverse". Remarkable answers
→ eg quantized Hall effect used to define fundamental constants.

4. MAIN message – find the right variables !

$\boxed{\text{Lecture 1.}}$ Look at duality and quantum phases in a 1+1D model. Exact solution possible.

Quantum Ising Model in $d = 1+1$

$$\underset{1}{o} - \underset{2}{o} - o - o - o - o - \underset{N}{o}$$

Chain of N sites (eventually $N \to \infty$)

On each site a "qbit". $|\uparrow\rangle$ or $|\downarrow\rangle$

2^N dim. Hilbert space.

$$\sigma^z_i \, |\uparrow,\downarrow\rangle = \pm \, |\uparrow,\downarrow\rangle$$

$$\sigma^x_i \, |\uparrow,\downarrow\rangle = |\downarrow,\uparrow\rangle$$

$$\sigma^x_i \, |\rightarrow\rangle = |\rightarrow\rangle \text{ etc.}$$

For an interesting phase diagram preserve a $\mathbb{Z}_2$ symmetry $G = \prod_{i=1}^{N} \sigma_i^x$; $G^2 = 1$.

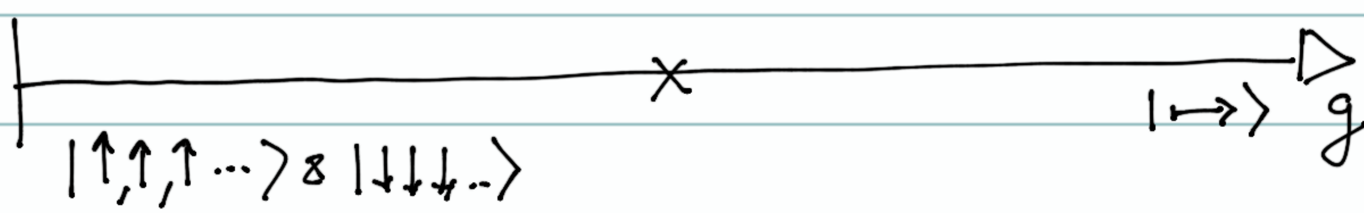
Hamiltonian invariant under this symmetry

$$[H, G] = 0.$$

can add σ^x terms and also even products of σ^z, σ^y. Simplest :-

$$H = -J \sum_i{}' \left(\sigma_i^z \sigma_{i+1}^z + g \sigma_i^x \right)$$

Two competing interactions.



Lecture 2 - Sep 3, 2013. Spontaneous Symmetry Breaking in the Quantum ising Model in D=1+1.

Phy 250 - Ashvin Vishwanath

PACS numbers:

I. OVERVIEW OF SYMMETRY

We will be studying the Hamiltonian:

$$H = -J\left[\sum_i \sigma_i^z \sigma_{i+1}^z + g\sum_i \sigma_i^x\right] \tag{1}$$

We note that ther eis a competition between the two terms, leading to interesting physics. The first ferromagnetic term wants adjacent pairs of spins to be both up or both down, while the second term wants to polarize the spins in the horizaontal direction. However there is more to that - consider for example the Hamiltonian: $H = -J\left[\sum_i \sigma_i^z + g\sum_i \sigma^x\right]$. Here too there is a competition of the two terms, but the evolution as a fiunction of g is smooth.

The key difference is one of symmetry - the first Hamiltonian posesses a Z_2 symmetry that is spontaneously broken at small values of g , giving rise to two phases. Intuitively, the symmetry is the fact that the Hamitlonain does not distinguish between spins being up or down.

To formalize this - the mathematical structure that describes symmetry is group theory. Symmetry is an invariance (you make a transformation to the system and ask if that brings it to a distinct but equivalent state). Clearly, a combination of two such invariance operations is also an invariance. This is the property of closure that group elements satisfy, where the group product is performing one action after another, and the group elements are the transformations that leave the system in an equivalanet state. Similarly the other properties of groups - associativity, and existence of an identity and inverse, can be redily checked.

What is the group here? Our symmetry takes up to down spins - this is implemented as $(\sigma_x,\, \sigma_y,\, \sigma_z) \rightarrow (\sigma_x,\, -\sigma_y,\, -\sigma_z)$. Obviously doing this twice is doeing nothing. So the group is Z_2, the elements are (1,U), where Uˆ2=1. The operator that implements this symmetry is $U = \prod_i \sigma_i^x$. Clearly $H = U^\dagger H U$ hence this is a symmetry.

II. QUANTUM DISORDERED STATE (SYMMETRY PRESERVED)

Analyze the limit $g \gg 1$. First we ignore the ferromagnetic term and just satisfy the second term. This gives a unique ground state with all spins along the 'x' direction which we represent as $|0\rangle = |\rightarrow\rightarrow\rightarrow\cdots\rightarrow\rangle$. Clearly this ground state respects the symmetry as can be seen by applying $U:\ U|0\rangle = |0\rangle$. The first excited states are gotten by reversing the spin at some point 'i': $|1\rangle = |\leftarrow\rightarrow\rightarrow\cdots\rightarrow\rangle$, $|i\rangle = |\rightarrow\rightarrow\rightarrow\cdots\leftarrow\cdots\rightarrow\rangle$. These excited states have energy $\Delta E = 2gJ$ above the ground state, whose energy we label as E_0. However there are 'N' of them for a length 'N' chain and are simply localized in this limit. However adding the ferromagnetic term as a perturbation gives them a dispersion. Restricting to the low energy space of a single spin flip, the ferromagnetic term induces a transition from$|i\rangle \rightarrow |i+1\rangle,\, |i-1\rangle$. So we can write the approximate Eigenvalue equation:

$$H|i\rangle = -J\left[|i+1\rangle + |i-1\rangle\right] + (E_0 + 2gJ)|i\rangle$$

It is simplest to consider the system on a circle, with 'N' sites, so the displaced indices $i \pm 1$ are interpreted with periodic boundary conditions at the edges. Then, this is readily solved by a Fourier transform. By writing in Fourier space: $|i\rangle = \frac{1}{\sqrt{N}}\sum_j e^{ikr_j}|k\rangle$, where the locations of the spins are at $r_j = ja$, we see that the momentum labels 'k' must satisfy $k = 2\pi\frac{m}{Na}$, in order to satisy periodic boundary conditions with integer 'm'. Furthemore, only $m \in 1,\, 2,\, \ldots,\, N$ are distinct. In the limit of $N \rightarrow \infty$, we get all values in the line $k \in [-\pi/a,\, \pi/a]$

In these variables we find:

$$[H - E_0]\,|k\rangle = [-2J\cos k + 2gJ]\,|k\rangle$$

thus, for a given momentum thereis a specific energy $\epsilon(k) = 2J[g - \cos k]$. This is a *particle* excitation - for a given momentum there is a fixed energy. Hence the lowest excitations above the symmetric state are gapped particles, with energy gap $\Delta = 2J(g-1)$ and a dispersion $\epsilon(k) \sim \Delta + Jk^2$, for small momenta. The operator that measures the number of particles is σ_i^x while the one that creates them is σ_i^z. Since these particles have a Z_2 character, the creation and annihilation operators are identical.

Naively, if we simply extend the calculation well beyond its regime of validity at $g \gg 1$, we anticipate a transition at $g = 1$, where the gap to these excitations close, and they 'condense'. Serendipiously this turns out to be the exact value.

III. BROKEN SYMMETRY STATE

Consider now the weak coupling limit - $g \ll 1$. Actually, let us set $g = 0$, and then carefully discuss what happens at finite but small g. Now, since the first term is the only one it is minimized by states $|+\rangle = |\uparrow\uparrow\uparrow \ldots \uparrow\rangle$ and by $|-\rangle = |\downarrow\downarrow\downarrow \ldots \downarrow\rangle$ and by any combination of them. Clearly these ground states break symmetry so that $U|+\rangle = |-\rangle$. However, the linear combinations $(|+\rangle \pm |-\rangle)$ do respect the symmetry. If we are at any nonzero g, there is a finite matrix element for $|+\rangle$ to mix with $|-\rangle$. Thus they are not eigenstates. However, the matrix element is vanishingly small in the themrodynamic limit of $N \to \infty$. We will argue this has to do with the gapped excitations in these states. However, if we assume that for a moment, then we can see how spontaneous symmetry breaking appears. While at any finite system size the ground states actually respect the symmetry, if one prepares the system in say all spin up, it takes an extremely long time to evolve out of it and demonstrate that it is not an eigenstate. So for all practical purposes we can consider it to be an eigenstate, and one can make the approximation better and better by going to larger systems, keeping all parameters fixed. We will see later that

Let us begin with the state$|+\rangle = |\uparrow\uparrow\uparrow \ldots \uparrow\rangle$. What is the lowest energy excitation here? For simplicity consider an open chain. One may naively say this is again a spin flip - now the flip occurs from up to down, and costs $\Delta E = 4J$. However there is actually a lower energy excitation - a domain wall with energy 2J. On the periodic boundary conditions system we need to make a pair of domain walls, however these are independent excitations - i.e. the single spin flip is not a 'particle' it does not have a fixed energy momentum relation in general. A Given energy may be divided into various momenta. So we get a '2-particle continuum'. However there are particle like excitations - domain walls, which however are non local objects. You need to change the state of a macroscopic number of spins to create them.

As before, we see that the fist excited states are at energy 2J, and represent single domain walls, that are localized in the limit of g=0. Consider now the action of the perturbation $g\sum_i \sigma_i^x$. This will cause the Domain walls to move. The domain walls are most naturally represented as living on the bonds i.e. $|\bar{i} = i + 1/2\rangle$. Flipping a spin will move it either to the left or right. Hence, we have a very similar situation as above, which can be written as :

$$(H - E_0)|\bar{i}\rangle = -gJ\left[|\bar{i}+1\rangle + |\bar{i}-1\rangle\right] + 2J|\bar{i}\rangle$$

Again, by Fourier transforming we get the spectrum $\epsilon(k) = 2J\left[1 - g\cos k\right]$- for the domain walls. (We have glossed over the fact that we now have periodic boundary conditions and hence have an even number of domain walls, but this can be justified). Again one may expect a transition when the domain walls condense at g=1.

Now, one can see why the broken symmetry state is stable - one needs to create a pair of domain walls and make them move around the system and annihilate. However these are gapped excitations - the energy cost to do this is 2J, while the perturbation gJ is the one that moves it through the system. Thus, the action cost is $S \sim e^{-N\log\frac{1}{g}}$ which vanishes in the thermodynamic limit.

IV. DUALITY

There is a suggestive symmetry between dispersion of single domain wall and single spin flip - seem to require g\rightarrow 1/g. Indeed this duality between strong and weak coupling (and spin flips and defects) can be made completely rigorous in the quantum Ising model as we explain below.

First assume open boundaries. Then, the state of the spin system can be completely specified by a knowledge of the location of domain walls and one spin (say the first one). So we can rewrite the problem in terms of domain wall variables. We would like to know the operators that create a domain wall and measure it, like for the case of spin flips. By analogy we will call them $\tau_{\bar{i}}^z$ and $\tau_{\bar{i}}^x$. Clearly:

$$\tau^x_{\bar{i}} = \sigma^z_{\bar{i}+\frac{1}{2}}\sigma^z_{\bar{i}-\frac{1}{2}} \tag{2}$$

while the operator to insert a single domain wall is: $\tau^z_{\bar{i}} = \prod_{j>\bar{i}} \sigma^x_j$. Furthermore this can be inverted to give:

$$\sigma^x_i = \tau^z_{\bar{i}-\frac{1}{2}}\tau^z_{\bar{i}+\frac{1}{2}} \tag{3}$$

Clearly these may be represented as Pauli spin operators. Now, we may represent the Hamiltonian as:

$$H = -J\left[g\sum_{\bar{i}} \tau^z_{\bar{i}}\tau^z_{\bar{i}+1} + \sum_i \tau^x_{\bar{i}}\right] \tag{4}$$

Thus the Hamiltonian is self dual, which exchanges weak and strong coupling $g \to 1/g$. Thus if there is a single transiton it must occur at $g = 1$.

Note, there is not a perfect symmetry between the two sides of the phase diagram - we have glossed over a few details while doing this duality and we will come back and fix it in the problem sets.

For example, if a field along ‘z’ is appled, the domain walls are confined. These are analogs of quarks bound into a meson and show resonances. Later we will discuss an experimental observation of these resonances.

A. References:

Subir Sachdev. ‘Quantum Phase Transitions’ 1st edition. Page 39-49

Lecture 4 - Sep 10, 2013. Solution to the Quantum Ising Chain Using Fermions

PACS numbers:

I. HEURISTIC CONSTRUCTION OF FERMIONIC OPERATORS

We have seen that the Hamiltonian:

$$H = -J\left[\sum_i \sigma_i^z\sigma_{i+1}^z + g\sum_i \sigma_i^x\right] \tag{1}$$

Has two phases separated by a transition. In the large g disordered (symmetry respecting) phase the excitations are spin flips, created by the operator σ_i^z. One may even adopt a free particle viewpoint for the spin flips, to a first approximation, and add interactions between them as a perturbation. This gives an accurate description of the excitations in the entire phase. On the other hand, in terms of domain wall variables if

$$\mu_{\bar{i}}^z = \prod_{j>\bar{i}} \sigma_j^x \tag{2}$$

creates a domain wall, then the disordered phase may be considered a condensate of domain wall creation operator, i.e. $\langle\mu_{\bar{i}}^z\rangle \neq 0$. On the other hand the ordered phase has the roles of these two operators reversed. The the operator μ_i^z creates excitations in this phase which are domain walls, while the ground state itself can be characterized by the order parameter: $\langle\sigma_{\bar{i}}^z\rangle \neq 0$.

We seek a set of variables that creates the particle like excitations across the entire phase diagram. These are the 'right' variables to describe the problem. Conside the combinations:

$$\chi_i \;=\; \sigma_i^z\mu_{i+1/2}^x \text{ and } \bar{\chi}_i = \;\; \sigma_i^y\mu_{i+1/2}^x \tag{3}$$

Clear, χ_i reduces to the relevant operators that create excitations in the two phases. For example in the ordered phase where we can replace σ^z by its expectation value, it turns into the domain wall creation operator. Similarly in the disordered phase where μ^z has a finite expectation value it reduces to the spin flip creation operator.

A. The χ operators are fermions:

We note that the algebra satisified by the χ fields is reminescent of fermions. That is, if we take the χ operators at two different sites they anti-commute. That is: $X_iY_j = -Y_jX_i$ where the $X,\, Y \in \{\chi,\, \bar{\chi}\}$.Essentially this is because the spin flip changes sign on moving through a domain wall. A less familar feature is that $\chi_i^2 = \bar{\chi}_i^2 = 1$. Also, the hermitian conjugate of these operators is themselves, from Equation 3. That is they are 'real' or Majorana fermions. However, these can be combined into complex fermions via the formula:

$$c_j^\dagger = \frac{\chi_j + i\,\bar{\chi}_j}{2} \tag{4}$$

Now, the fermions $c^\dagger$ and its hermitian conjugate $c = (\chi - i\bar{\chi})/2$ satisfy the usual femrionic anticommutation rules:

$$\left\{c_i^\dagger,\, c_j\right\} = \delta_{ij} \text{ and } \{c_i,\, c_j\} = 0 \tag{5}$$

where $\{a,\, b\} = ab + ba$.

In two dimensions we will see that the analogous proceedure is to attach charge to vortices, which leads to a description of quantum Hall states. In 3D, an analogy is to attach gauge charge to monopoles to create dyons, whose uses are currently being explored.

FIG. 1: Original Hamiltonian has interaction Jg onsite and J between sites. Dual Hamiltonian groups the Majorana operators on bond centers, such that for the new 'dual sites', the onsite interactions are J and inter 'site' interactions are Jg. This is the required duality. Note however, for an open chain there are two Majorana operators left over at the ends in the dual site representation. When the interaction between dual sites Jg is weak, this leads to degenerate end states - which indicates a topological phase of fermions.

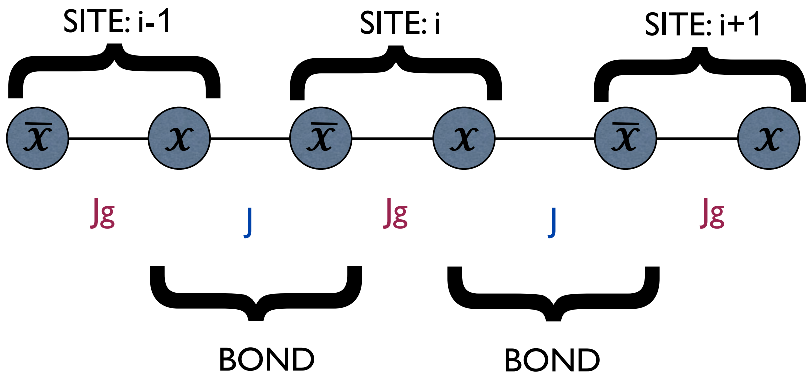


FIG. 2:

B. The Hamiltonian is noninteracting in fermion variables:

The Hamiltonian (1) is rewritten in terms of the fermion fields. Note, we can write $i\bar{\chi}_j\chi_j = \sigma^x_j$. So that takes care of the second term. The first term can be written as $i\bar{\chi}_{j+1}\bar{\chi}_j = \sigma^z_j\sigma^z_{j+1}$. Therefore, taken together we have:

$$H = -i\,J\sum_j\left(\bar{\chi}_{j+1}\chi_j + g\bar{\chi}_j\chi_j\right) \tag{6}$$

Although this is written in the unfamilar Majorana language, we can rewrite it in terms of conventional (or complex c) femrion operators. However, this is just a linear transformation of the fields. The important thing is that H is quadratic in the fields - hence it is a free fermion problem. Shortly we will see that we can map it to a BCS superconducting Hamiltonian, which can be readily solved by a change of basis.

C. Duality in terms of Fermions:

Consider defining new fermions on the bonds $\bar{j}$ that live between a pair of sites $\bar{j} \pm 1/2$. The pair of fermions $\tau_{\bar{j}}$and$\bar{\tau}_{\bar{j}}$ are defined by taking the fermions on the neighboring sites:

$$\chi_j = \bar{\tau}_{j+\frac{1}{2}} \text{ and } \bar{\chi}_{j+1} = -\tau_{j+\frac{1}{2}} \tag{7}$$

In these variables the Hamiltonian is:

$$H = -i\,J\sum_{\bar{j}}\left(\bar{\tau}_{\bar{j}}\tau_{\bar{j}} + g\bar{\tau}_{\bar{j}+1}\tau_{\bar{j}}\right) \tag{8}$$

This is the required duality.

D. Topological Phase of 1D fermions

The two phases in terms of fermions are not distinguished by symmetry, but by topology (see Ref [1] for more details). One can analyze the two limits $g \to \infty$ and $g \to 0$. In the first limit, one simply obtains a vacuum of fermions, ddefined via $c^\dagger_i = (\chi_i + i\bar{\chi}_i)/2$. On the other hand the opposite limit $g \to 0$, the dual fermions defined on the bonds ar eput in the vacuum state. However, this leaves a pair of Majorana fermions at the ends (see Figure 2). These can be combined into a pair of states, labelled by the occupation number of the complex fermions $a^\dagger = (\chi_N + i\bar{\chi}_1)/2$.

A spiltting between these two states is a term $\epsilon a^\dagger a = \epsilon i \bar{\chi}_1 \chi_N$. However, this involes modes tht live on opposite ends of the chain, which has an energy gap in the bulk. Hence the coefficient is expected to be exponentially small in the length $\epsilon \sim e^{-N/\xi}$. Thus there is a near degeneracy of levels which develops into a doubly degenerate ground state in the thermodynamic limit. This accounts for the two fold degeneracy of the ordered state of the spin model. Recent experiments have claimed to observe these Majorana zero modes at the end of a superconducting wire see Refs. [2].

References:

[1] Topological Phases and Quantum Computation, A. Kitaev and C. Laumann, http://xxx.lanl.gov/pdf/0904.2771.pdf

[2] New directions in the pursuit of Majorana fermions in solid state systems. Jason Alicea. http://xxx.lanl.gov/pdf/1202.1293v1.pdf