ICTP Lectur 2.

Some odds and ends.

Mean Field Treatment of Ising Chain:

Let us begin in the large g limit.

$$|\psi\rangle = \prod_i |\rightarrow\rangle_i$$

Now, spins want to develop a "z" compoment. IF a magnetic field along z was applied

$$\begin{aligned} H &= -(\sigma^x + \epsilon\sigma^z) \\ &= -\sqrt{1+\epsilon^2}\left(\frac{1}{\sqrt{1+\epsilon^2}}\sigma^x + \frac{\epsilon}{\sqrt{1+\epsilon^2}}\sigma^z\right) \end{aligned}$$

where $\cos\phi = \frac{1}{\sqrt{1+\epsilon^2}}$; $\sin\phi = \frac{\epsilon}{\sqrt{1+\epsilon^2}}$

then the ground state is:

$$|\psi\rangle_i = \left(\cos\frac{\phi}{2}\,|\rightarrow\rangle + \sin\frac{\phi}{2}\,|\leftarrow\rangle\right)_i$$

However we do not have a field along "z".
Rather, the system must spontaneously choose $\pm\hat{z}$. Let us use this as a variational state and optimize energy wrt .

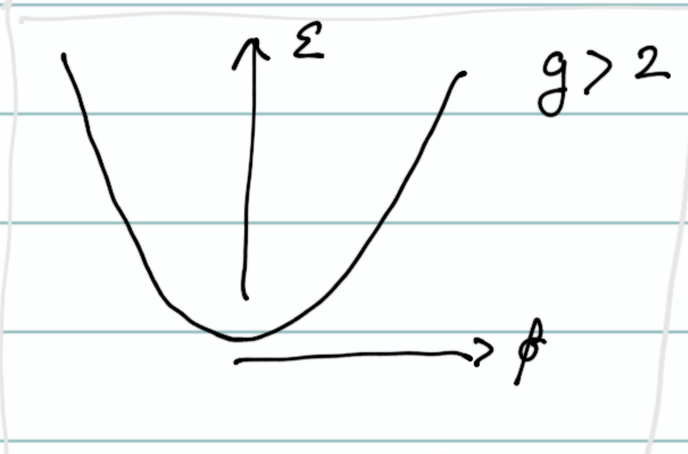
$$\langle\sigma^z\rangle = \sin\phi \qquad \langle\sigma^x\rangle = \cos\phi$$

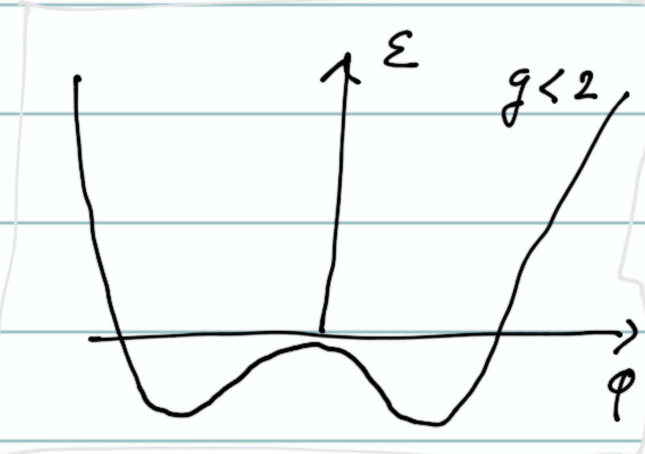
$$H = -JN\left(\sin^2\phi + g\cos\phi\right)$$

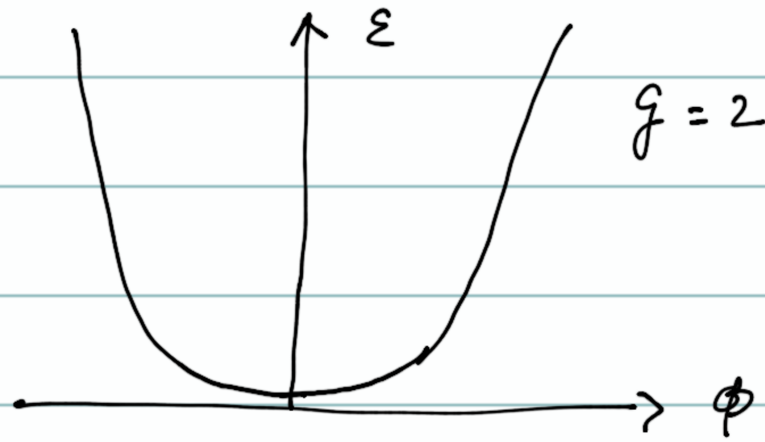
Consider small ϕ to study instability from $\phi = 0$ ground state on lowering "g".

$$\frac{\mathcal{E}(\phi)}{JN} = -\left[\left(\phi - \frac{\phi^3}{6}\right)^2 + g\left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!}\cdots\right)\right]$$

$$= \left(\frac{g}{2} - 1\right)\phi^2 + \left(\frac{1}{3} - \frac{g}{24}\right)\phi^4 + \dots$$







Matrix Product States

For better qantitative agreement:

$$\psi = \prod_i (u|\rightarrow\rangle + v|\leftarrow\rangle)_i$$

No entanglement. Introduce entanglement by making u, v matrices $\chi \times \chi$ u_{ij}, v_{ij}.

$$\psi = \mathrm{Tr}\left\{\prod (\hat{u}|\rightarrow\rangle + \hat{v}|\leftarrow\rangle)_i\right\}$$

eg two sites $u = \hat{Z}$; $v = \hat{X}$ Paulis Not SPIN operators.

$|\rightarrow\rangle = |0\rangle$ $|\leftarrow\rangle = |1\rangle$

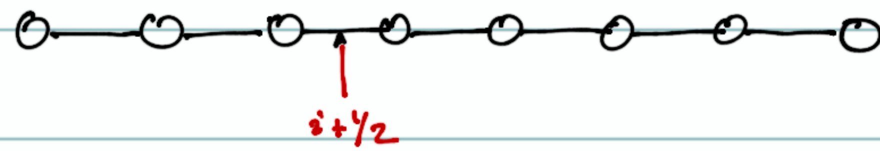
$|00\rangle = \mathrm{Tr}\, ZZ$ $|11\rangle = \mathrm{Tr}\, XX$

$|01\rangle$ & $|10\rangle = \mathrm{Tr}\, ZX = 0.$

$|\psi\rangle = |00\rangle + |11\rangle.$ entangled!

Fermions:

Previously we defined domain wall creation operators



$$\mu^z_{i+1/2} = \prod_{j>i} \sigma^x_j \qquad \mu^x_{i+1/2} = \sigma^z_i \, \sigma^z_{i+1}$$

We define fermion χ_i, $\tilde{\chi}_i$ at each site

$$\chi_i = \sigma^z_i \, \mu^z_{i+1/2}$$

$$\tilde{\chi}_i = \sigma^y_i \, \mu^z_{i+1/2}$$

verify $\chi^\dagger = \chi$ (Majorana Fermions)

$\chi\tilde{\chi} = -\tilde{\chi}\chi$ and $\chi^2 = \tilde{\chi}^2 = 1$.

and $\chi_i \chi_j = -\chi_j \chi_i$ $(i \neq j)$

$c_i = \dfrac{(\chi_i - i\tilde{\chi}_i)}{2}$ complex fermions

Hamiltonian in terms of Majorana Fermions

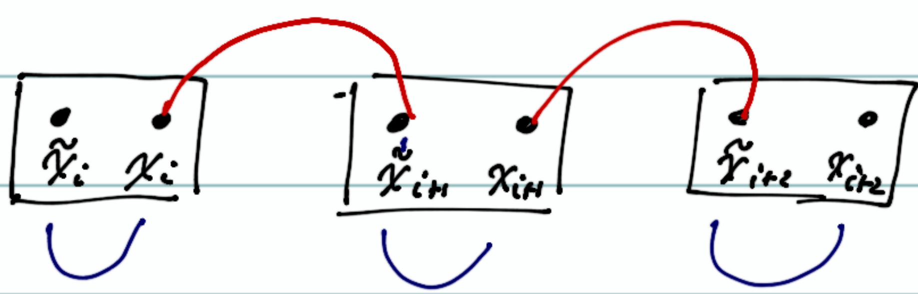
Note $\quad i\chi_j \tilde{\chi}_j = i Z_j Y_j = X_j$ $\quad$ (shorthand $\sigma_x = X$ etc.)

neighboring sites $\quad i\chi_j \tilde{\chi}_{j+1} = i(Z_j X_{j+1}) Y_{j+1}$.

$$= -Z_j Z_{j+1}$$

Thus

$$\frac{H}{J} = i\sum_j \left(\underline{\underline{\chi_j \tilde{\chi}_{j+1}}} + \underline{\underline{g\, \tilde{\chi}_j \chi_j}} \right)$$



Two phases $\quad g < 1 \quad \& \quad g > 1$.

1 Critical point :- $g = 1$.

1 All couplings uniform.

Let us write then
$$\left.\begin{aligned} \eta_{2i-1} &= \tilde{\chi}_i \\ \eta_{2i} &= \chi_i \end{aligned}\right\}$$

$$\frac{H}{J} = i \sum_j \eta_{2j}\,\eta_{2j+1} + \eta_{2j-1}\,\eta_{2j}$$

$$\frac{H}{J} = i \sum_r \eta_r\,\eta_{r+1}.$$

$E_k = \sin k.$

Low energy theory: $\eta_r \simeq \eta_1(r) + (-1)^r \eta_2(r)$

η_1, η_2 slow functions of r for low E.

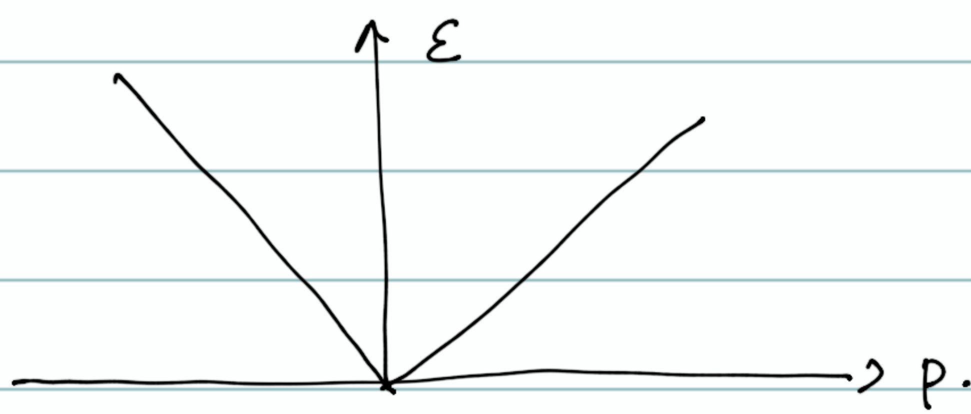
$$\frac{H}{J} \cong i \sum_r \eta_1(r)\, \eta_1(r+1) - \eta_2(r)\, \eta_2(r+1)$$

$$H \simeq i v_F \int dr \left[\eta_1(r)\, \partial_r \eta_1(r) - \eta_2 \partial_r \eta_2(r)\right]$$

define spinor $\eta(r) = \begin{pmatrix} \eta_1(r) \\ \eta_2(r) \end{pmatrix}$

$$\mathcal{H}(r) \cong i v_F \eta^T(r)\, \sigma^z \partial_r \eta(r).$$

(Nonchiral) Majorana fermion -> massless



Now what does deviation from $g=1$ do?

Say $|g-1| \ll 1$ then additional term

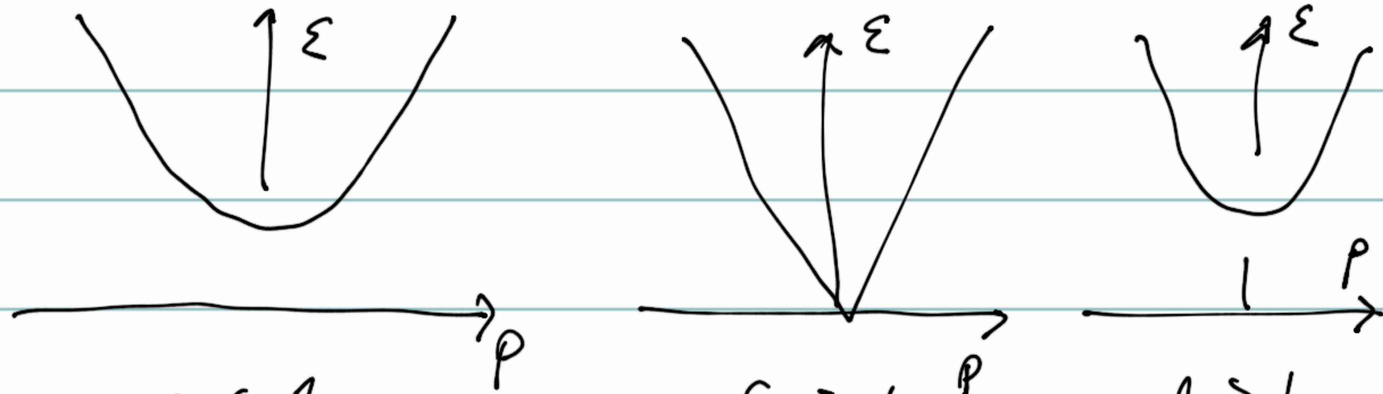
$$\simeq \frac{(g-1)}{2} \sum_r (-1)^r i \eta_r \eta_{r+1}$$

$$= \frac{m}{2} \int dr\, i \eta_1(r) \eta_2(r)$$

$$= m \int \bar{\eta} \eta \, dr$$

where $\bar{\eta} = \eta^T \sigma_y$

This is just a mass term and the two phases just correspond to



$g < 1$ $\qquad$ $g = 1$ $\qquad$ $g > 1$

Duality

$$\mathcal{L}_f = \bar{\eta}\, \gamma^\mu \partial_\mu \eta + m \bar{\eta} \eta$$

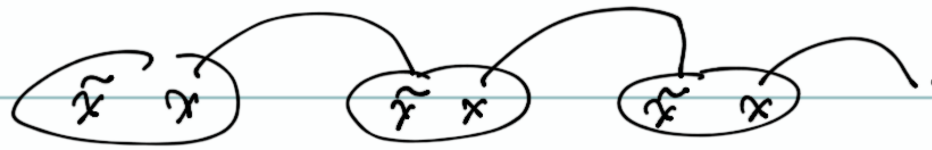
$$\longleftrightarrow \quad \mathcal{L}_\varphi = (\partial_\mu \phi)^2 + r \varphi^2 + \lambda \varphi^4$$

$$m \longleftrightarrow r$$

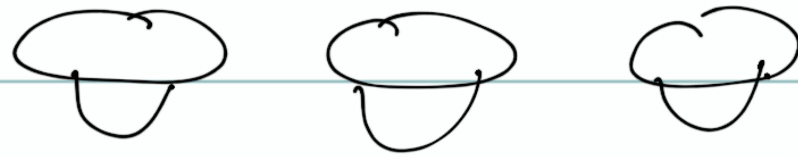
correlation length exponent $\xi \simeq m^{-1} = \frac{1}{r}$

$\nu = 1$.

What is the distinction between these phases?



vs



First has a zero energy mode (topological)

Locality important → for real electrons always protected.

But emergent fermions rely on $\mathbb{Z}_2$ symmetry $\quad G = (-1)^F$ fermion parity.

So broken $\mathbb{Z}_2$ allows us to add χ_0 or χ_2 and break degeneracy.

One question that came up in the lecture:-

The critical point has chiral symmetry

$$\mathcal{L} = \bar{\eta}\gamma^{\mu}\partial_{\mu}\eta$$

or $\mathcal{L} = \eta_1(\partial_t + \partial_x)\eta_1 + \eta_2(\partial_t - \partial_x)\eta_2$

$\eta_1 \to \eta_1 \quad \eta_2 \to -\eta_2$ symmetry.

This is broken by the mass term on moving away from $g = 1$.

$$\Delta\mathcal{L} = (g-1)\bar{\eta}\eta = m\eta_1\eta_2 \qquad m \xrightarrow{\text{Chirality}} -m$$

What does this chiral symmetry correspond to?

Actually, since it is only present at the critical point it must be the <u>duality</u> transformation which sends $m \to -m$.

Chiral Symmetry Breaking.

Lets say we sit <u>at</u> self dual point and add interactions that promote spontaneous chiral symmetry breaking.

$$\mathcal{L} = \bar{\eta}\, \gamma^{\mu} \partial_{\mu} \eta - g(\bar{\eta}\eta)^2$$

Wants $\langle \bar{\eta}\eta \rangle \neq 0$ breaks Chiral Symm. for large "g".

Theory of transition? $\mathcal{L}_{\eta} + \mathcal{L}_{\phi} + \phi\, \bar{\eta}\eta = \mathcal{L}_{tot}$.

where $\mathcal{L}_{\varphi} = (\partial\varphi)^2 + r' \phi^2 + \lambda' \phi^4$.

Emergent SUSY.

What about bosonic description?

$$\mathcal{L}_\phi = (\partial\phi)^2 + r\phi^2 + \lambda\phi^4 + \tilde{\lambda}\phi^6$$

self duality $r=0$;
further tune $\lambda = 0$ (tricritical Ising)

Friedan - Shenker : Tricritical Ising
emergent SUSY.