

Lecture 3: Dualities in 2+1 D.

References :

Bose - Bose Duality :

Peskin '78. Dasgupta & Halperin '81.

Fermi - Fermi Duality :

Son ; Metlitski - A.V. ; Wang & Senthil '15

Bose - Fermi Duality :

Polyakov '88 ; Jain 89 ; Chen - Fisher - Wu '93

Barkeshli - McGreevy '14 ; Karch - Tong '16 ;

Seiberg - Senthil - Wang - Witten '16 .

Lessons From 1+1 Dimension:

1+1 Dim.

2+1 Dim.

Bose - Bose Duality

\mathbb{Z}_2 charge \leftrightarrow Domain Wall

$U(1)$ charge \leftrightarrow Vortices

Free Fermion?

Free Majorana Fermion

$$\mathcal{L}_{1+1}^M = \bar{\eta} \not{D} \eta$$

Free Dirac Fermion

$$\mathcal{L}_{2+1} = \bar{\psi} \not{D} \psi$$

$$\leftrightarrow \mathcal{L}_{WF} = (\partial\phi)^2 + \lambda\phi^4$$

Fermi - Bose Duality

Fermi - Fermi Duality

Topology changing Transition

Wilson - Fisher

$$\mathcal{L}_{WF} \leftrightarrow \mathcal{L}_{1+1}^M$$

Bose - Fermi Duality

Bose \leftrightarrow Bose Duality

Basic Facts about vortices

$U(1)$ charged boson (global symmetry) Ψ .

$$G_\theta \Psi^+ = e^{i\theta} \Psi^+$$

In the ordered phase $\langle \Psi \rangle \neq 0$.

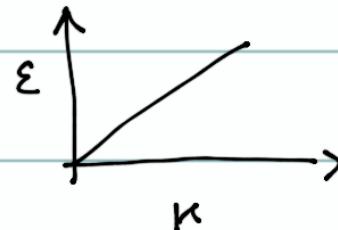
Degenerate ground states $|\Psi_0| e^{i\varphi} \rightarrow$ leads to Goldstone Modes.

$$\mathcal{L} = \frac{1}{2} |\partial_\mu \Psi|^2 - r |\Psi|^2 + \lambda |\Psi|^4$$

Solve for $|\Psi_0|$ & then allow $|\Psi_0| e^{i\varphi(r, t)}$

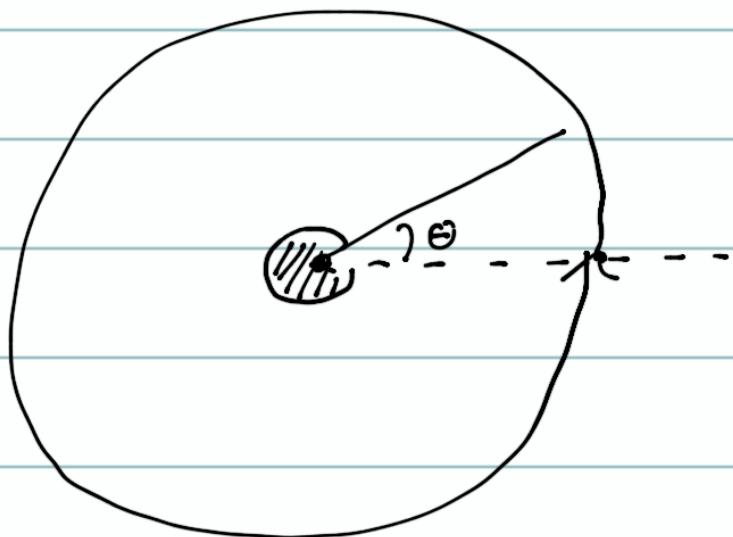
$$L_{\text{eff}} = \frac{1}{2} |\Psi_0|^2 |\partial_\mu \varphi|^2 = \frac{S_s}{2} |\partial_\mu \varphi|^2$$

Goldstone mode



Also topological defects - vortices. Locally ground states (low energy configurations)

$$\begin{aligned}\psi(r, \theta) \\ = |\psi| e^{i\Theta} \\ (r > r_c)\end{aligned}$$



unit vortex.

$$\Theta \equiv \Theta + 2\pi n \quad (\text{only } e^{i\Theta} \text{ physical})$$

$$\oint \nabla \Theta = 2\pi n \text{ vortices. } \pi_1(S^1) = \mathbb{Z}.$$

Usually $\nabla \times (\nabla \Theta) = 0$ but here not single valued in terms of Θ . So

$$\nabla \times (\nabla \Theta) = 2\pi n \delta(r - r_v)$$

Energy of a vortex

$$E = \frac{1}{2} |\nabla \psi|^2$$

"Elastic Energy" only from gradients

$$E = \int \frac{\delta_s}{2} (\nabla \phi)^2 \Rightarrow \frac{\delta_s}{2} \int \left(\frac{1}{r}\right)^2 dr = \frac{\delta_s}{2} 2\pi \int_{r_c}^L \frac{dr}{r}$$

$$E = \pi \delta_s \log \frac{L}{r_c}$$

Here L is the separation to the antivortex.
Interaction is logarithmic like charges in 2D.

Also, important to have short distance cutoff r_c to make energy finite. Naive continuum limit \rightarrow no vortices.

Charge quantized - important.

Topological conservation of vortices \rightarrow
 reminiscent of charges in a gauge theory.

Use electromagnetic fields to model vortex?

$$\vec{\nabla} \cdot \vec{e} = S_v ?$$

Works if we set: $e_i = \epsilon_{ij} \partial_j \varphi / 2\pi$

$$\partial_i e_i = \frac{\nabla \times (\nabla \varphi)}{2\pi} = \frac{2\pi}{2\pi} \sum_i n_j^v \delta(r - r_j^v)$$

$$= S_v.$$

Vortex creation operator ψ_v is gauge charged.

Guess: $\mathcal{L}_{\text{dual}} = \mathcal{L}_\psi [\psi_v, a] + \mathcal{L}_{\text{Maxwell}} [a]$

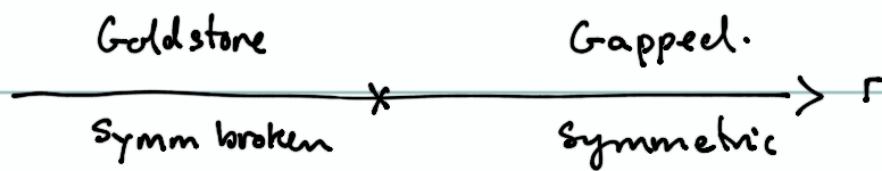
$$\mathcal{L}_{\psi_\nu} = |(\partial_\mu - i\omega_\mu) \psi_\nu|^2 + r' |\psi_\nu|^2 + \lambda' |\psi_\nu|^4$$

$$\mathcal{L}_{\text{Maxwell}} = \frac{1}{2K} f_{\mu\nu} f^{\mu\nu}$$

$$f_{\mu\nu} = (\partial_\mu a_\nu - \partial_\nu a_\mu)$$

$$f_{12} = b; \quad f_{ij} = \epsilon_{ij} e_j$$

Test 1 Phase Diagram:

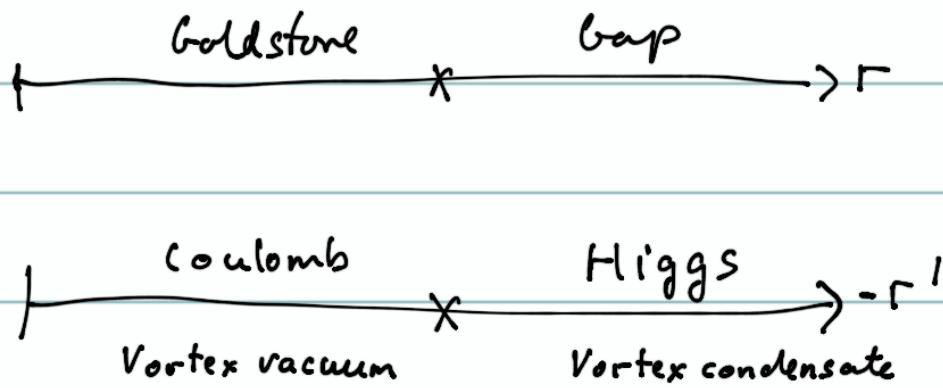


$$\mathcal{L}_0 = |\partial_\mu \psi|^2 + r |\psi|^2 + \lambda |\psi|^4$$

How does this work in the vortex theory?

ψ_ν condensed \rightarrow Higgs field. Gap to photon

ψ_ν gapped \rightarrow gapless photon (Goldstone Mode)



Photon \leftrightarrow Goldstone mode

$$L_{GM} = \frac{f}{2} |\partial_\mu \varphi|^2$$

$$L_{\text{photon}} = \frac{1}{2K} f_{\mu\nu} f^{\mu\nu}$$

but we said $f^{\mu\nu} = \epsilon^{\mu\nu\rho} \partial_\rho \varphi$

(e.g. $f^{0i} = e_i = \epsilon^{ij} \partial_j \varphi$ as we assumed before)

Does not work in 1+3D \sim photon \neq Goldstones

Test 2: Track global $U(1)$ using A_{ext}

$$\left| (\partial_\mu - i A_\mu^{\text{ext}}) \psi \right|^2 + V(|\psi|^2) = L_{\text{boson}}$$

in the vortex theory where is $U(1)$ conserved chg.?

note $\nabla \phi \cdot A$ couplings \Rightarrow flux $f_{\mu\nu}$ related to conserved $U(1)$ current. $\frac{b}{2\pi} \rightarrow$ charge density

$$\mathcal{L}_{\text{vortex}} + \mathcal{L}_{\text{Maxwell}} - \frac{\epsilon^{\mu\nu\lambda}}{2\pi} \partial_\mu a_\nu A_\lambda = \mathcal{L}_{\text{vortex}}$$

As a check imagine inserting one unit of magnetic flux from A_{ext} so $\int B_{\text{ext}} = 2\pi$. This should induce $\int S_V = \frac{1}{2\pi} \int B_{\text{ext}} = 1$. Using $\frac{\delta \mathcal{L}}{\delta a_0} = -j^0 = S_V$

we see $S_V = \frac{1}{2\pi} \epsilon^{0ij} \partial_i A_j = \frac{1}{2\pi} B$ as required.

Lets check we get the same effective action for A_{ext} .

$$\mathcal{L}_{\text{boson}} = \begin{cases} r > 0 & \Rightarrow \mathcal{L}_{\text{eff}}[A_{\text{ext}}] = \frac{\kappa}{2} F_{\mu\nu} F^{\mu\nu} \\ r < 0 & \mathcal{L}_{\text{eff}}[A_{\text{ext}}] = \frac{\lambda}{2} |A^\perp|^2 \end{cases}$$

Argue that for $r > 0$ analytic function of gauge invariant $F_{\mu\nu}^{\text{ext}}$ and derivatives.

$$\frac{\kappa}{2} F F = \frac{\kappa}{2} (\epsilon^2 - B^2) \quad \text{renormalized dielectric function}$$

INSULATOR / DIELECTRIC.

For $r < 0$ superfluid. Now we have

$$L_{\text{eff}} = \frac{s_s}{2} (\nabla \phi - A^{\text{ext}})^2 = \frac{s_s}{2} (A_{\perp}^{\text{ext}})^2$$

where $\partial^\mu A_\mu^{\perp} = 0$. This is a superfluid.

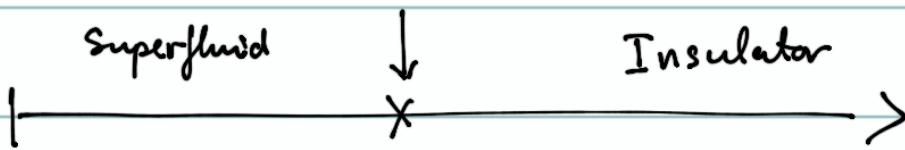
$$\frac{\delta L}{\delta A_\mu} = - J^\mu = s_s A_{\perp}^{\text{ext}\mu};$$

Meissner Effect. $\nabla \times J = - s_s B^{\text{ext}}$ London Eqn.

$$\text{But } \nabla \times B^{\text{ext}} = \mu J \quad \nabla \times (\nabla \times B) = - s_s \mu B^{\text{ext}}$$

$$\nabla^2 B^{\text{ext}} = s_s \mu B^{\text{ext}} = \text{decays}$$

$$\text{Metal. } \sigma = q^2/h^*$$



Show this works out in the dual theory

$$\text{eg Higgs phase } \mathcal{L}_r = a_\perp^2 - \frac{1}{2\pi} A \wedge dA$$

$$\mathcal{L}_{\text{ef}} = (dA)^2 \quad \text{Insulator.}$$

$$\text{Gapped vortex } \mathcal{L}_v = f^2 - \frac{1}{2\pi} A \wedge da.$$

$$\mathcal{L}_{\text{eff}} = A_\perp^2 \quad \text{superfluid.}$$

Application 1: Superfluid Insulator Trans.

$$\mathcal{L}_{WF} = |\partial_\mu \psi|^2 + \lambda/4|^4 \leftrightarrow \mathcal{L}_v = |(\partial_\mu - i\alpha)\psi_\nu|^2 + \lambda/4_\nu|^4 + \frac{1}{2\pi} f^2$$

LHS good & expansion. Not for RHS, first order near 3+1D, for $N_f < 365$!

But continuous in 2+1D! Many applications to superconductors in 3D critical behavior etc.

Application 2: New phases that are hard to describe using \mathbb{Z} bosons. Eg fractionalized phase with charge $1/2$ particles coupled to a gauge field (Gauge group = \mathbb{Z}_2)

Pair condensate of vortices $\phi_v = \psi_v^2$
 $\langle \phi_v \rangle \neq 0$ but $\langle \psi_v \rangle = 0$.

Excitations \rightarrow "vortices" strength $\frac{1}{2} \cdot 2\pi$
 (Half strength). But these just correspond to charges since $\frac{\nabla \times \alpha}{2\pi} = S_{\text{boson}}$. $S = 1/2$

Also verify \mathbb{Z}_2 gauge structure. Unpaired vortex ψ_v is a gauge charged excitation
 "e" & "m" particles of Toric Code topological order.