Largest Eigenvalue.

(a

$\{\lambda_1, \dots, \lambda_N\}$: strongly correlated random variables.

Statistics of $\underline{\lambda_{max}}$?

<u>One step back</u> $\{X_1, \dots, X_N\}$ i.i.d. r.v. drawn from a common $p(x)$.

$$Q_N(x) = \left[\int_{-\infty}^{x} \cdots \int_{-\infty}^{x} dx_1 \dots dx_N \, p(x_1) \cdots p(x_N)\right] = \Big[\underbrace{\int_{-\infty}^{x} dy \, p(y)}_{\mathbb{P}(x)}\Big]^N .$$

$Q_N(x)$: $\mathbb{P}\mathrm{rob}[X_{max} \leq x]$

$\mathbb{P}(x)$: cumulative distribution function

If we now send $N \to \infty$, since $P(x)$ is $0 \leq \mathbb{P}(x) \leq 1$, the limit $[\mathbb{P}(x)]^N$ can only take values '0' or '1'.

To get a nontrivial limit, we need to send <u>both</u> $x \to \infty$ <u>and</u> $N \to \infty$, in such a way that the combination

$$z = \frac{x - a_N}{b_N} \quad \text{is kept constant}$$

(for suitable $a_N \in \mathbb{R}$ and $b_N > 0$).

The standard goal of EVS is: find $a_N, b_N, F(z)$ (b

such that

$$\lim_{N\to\infty} Q_N(a_N + b_N z) = F(z).$$

For iid variables, the celebrated Fisher-Tippett-Gnedenko thm. states that $F(z)$ can only be of three different types.

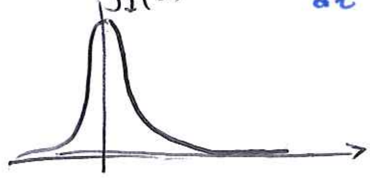
$$x^* = \sup(x : P(x) < 1)$$

$\to$ upper endpoint of the support of $p(x)$.

① If x^* is finite or infinite, and $p(x)$ falls off faster than any power for $x \to x^*$ (exponential and Gaussian case), then the limiting distribution is Gumbel

$$F_I(z) = e^{-e^{-z}}$$

$f_I(z) \to = \frac{dF_I(z)}{dz}$

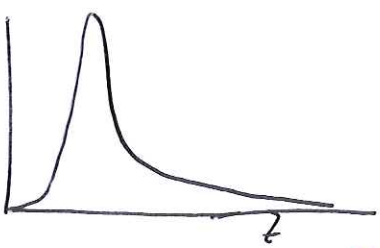


② If x^* is infinite, and $p(x)$ falls off as a power-la

$p(x) \sim x^{-(\gamma+1)}$, then the limiting distribution

is Fréchet $F_{II}(z) = e^{-1/z^{\gamma}}$, if $z>0$, and 0 otherw

(c)



(3) If x^* is finite and $p(x) \sim (x^* - x)^{\gamma - 1}$, then the limiting distribution $F(z)$ is Weibull.

$$F_{III}(z) = e^{-|z|^{\gamma}}, \quad z<0, \quad \text{and } 1 \text{ otherwise}$$

---

3 properties (maybe not so widely known)

(1) a_N, b_N can be found as follows.

is the CDF of each individual random variable

- Gumbel: $a_N = P^{-1}\left(1 - \frac{1}{N}\right)$ $\quad b_N = P^{-1}\left(1 - \frac{1}{Ne}\right) - a_N$
- Fréchet: $a_N = 0$ and $b_N = P^{-1}\left(1 - \frac{1}{N}\right)$.
- Weibull: $a_N = x^*$ and $b_N = x^* - P^{-1}\left(1 - \frac{1}{N}\right)$.

(2) How to predict the basin of attraction?

$$\lim_{\epsilon \to 0} \frac{P^{-1}(1-\epsilon) - P^{-1}(1-2\epsilon)}{P^{-1}(1-2\epsilon) - P^{-1}(1-4\epsilon)} = 2^c$$

if $c = 0, >0, <0$ $\quad$ G, F, W.

(d)

(3) The constants $\{a_N, b_N\}$ are not unique, but can be replaced by $\{a_N', b_N'\}$ provided that

$$\begin{cases} \lim\limits_{N\to\infty} \dfrac{b_N'}{b_N} = 1 . \\ \lim\limits_{N\to\infty} \dfrac{a_N - a_N'}{b_N} = 0 . \end{cases}$$

Example. $p(x) = \mu e^{-\mu x}$, $x \geq 0$.

$$Q_N(x) = \left[\mu \int_0^x e^{-\mu y}\, dy\right]^N = \left[1 - e^{-\mu x}\right]^N$$

$$= e^{N \log\left[1 - e^{-\mu x}\right]}$$

As $x \to \infty$, expanding the logarithm

$$Q_N(x) \approx e^{-N e^{-\mu x}} = e^{-e^{-(\mu x - \ln N)}} \equiv F_I(z)$$

$$z = \mu x - \ln N$$

$$\hookrightarrow \quad \frac{1}{b_N} = \mu \implies \boxed{b_N = \frac{1}{\mu}}$$

$$-\frac{a_N}{b_N} = -\ln N \implies \boxed{a_N = \frac{\ln N}{\mu}}$$

recover it from previous formulae

$$P(x) = \mu \int_0^x \exp(-\mu y)\, dy = 1 - e^{-\mu x},$$

hence $P^{-1}(x) = -\frac{1}{\mu} \ln(1-x)$, for $0 \leq x < 1$.

* $$\lim_{\epsilon \to 0} \frac{-\cancel{\frac{1}{\mu}} \ln \epsilon + \cancel{\frac{1}{\mu}} \ln(2\epsilon)}{-\cancel{\frac{1}{\mu}} \ln(2\epsilon) + \cancel{\frac{1}{\mu}} \ln(4\epsilon)} = \frac{\ln 2}{\ln 2} = 2^0.$$

$c = 0 \Rightarrow G.$

* $$a_N = -\frac{1}{\mu} \ln\left(\frac{1}{N}\right) = \frac{\ln N}{\mu} \quad \checkmark$$

$$b_N = -\frac{1}{\mu} \ln\left(\frac{1}{Ne}\right) - \frac{\ln N}{\mu} = \frac{1}{\mu} \quad \checkmark.$$

What happens now for eigenvalues of random matrices? (f)

Lord May, Nature 238, 413 (1972) 24-25

"Will a large complex system be stable?".

+ALLESINA

Ecosystem: $\rho_i \rightarrow$ population density of the i-th species. 26-27-28-?

Completely <u>non-interacting</u> and <u>stable</u>

$$\downarrow$$

$$x_i = \rho_i - \rho_i^*$$

deviation from equilibrium density

$$\frac{dx_i}{dt} = -x_i$$

(exponentially fast, the density goes back to its eq. value).

Imagine now we switch on <u>random</u> interactions (pairwise)

$$\frac{dx_i}{dt} = -x_i + \alpha \sum_j A_{ij} x_j$$

strength of the interactions.

Are all the x_i's going to '0' as well as $t \to \infty$?

Take A as a Gaussian (symmetric) matrix (variance $\sigma^2 \sim \frac{1}{N}$).

Probability that the system remains stable?

$$\boxed{\frac{d}{dt}\vec{x} = (\alpha A - \mathbb{1})\vec{x}}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

$$A = S \Lambda S^{-1}$$

Make an orthogonal transformation $\vec{y} = S^{-1}\vec{x}$

$$\frac{d}{dt}\vec{y} = (\alpha \Lambda - \mathbb{1})\vec{y} \qquad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$$

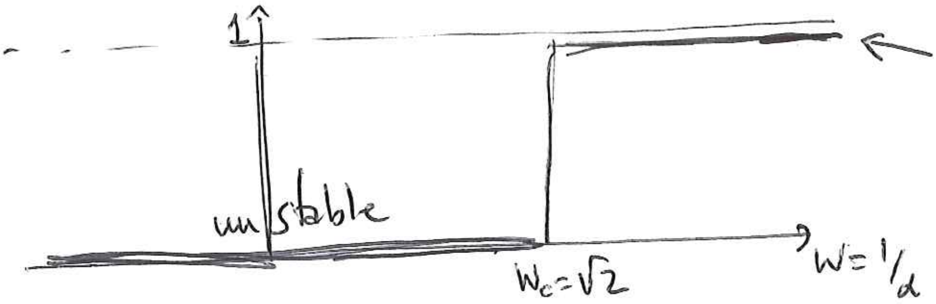
Stability criterion: $\alpha \lambda_i - 1 < 0 \quad \forall i$

$$\Downarrow$$

$$\boxed{\lambda_i < \frac{1}{\alpha} = w}$$

$$\mathrm{Prob}\left[\lambda_i < w, \forall i\right] = \mathrm{Prob}\left[\lambda_{max} < w\right].$$

(h

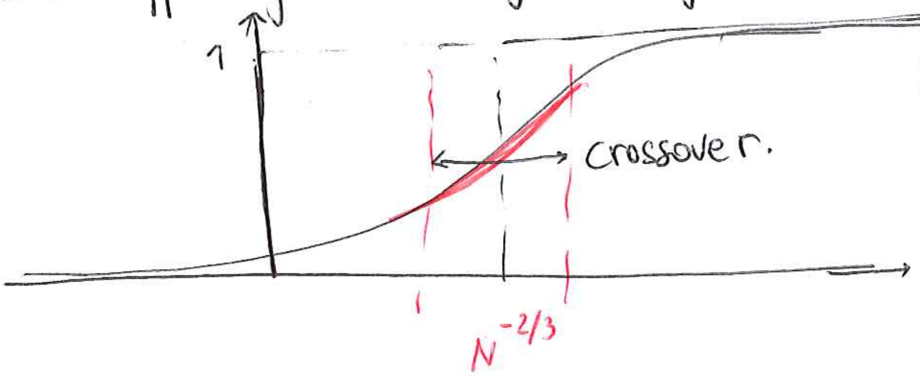


$N \to \infty$

$$\lim_{N\to\infty} P_{stable}(\alpha, N) = \begin{cases} 1 & \alpha < \alpha_c \\ 0 & \alpha > \alpha_c . \end{cases}$$

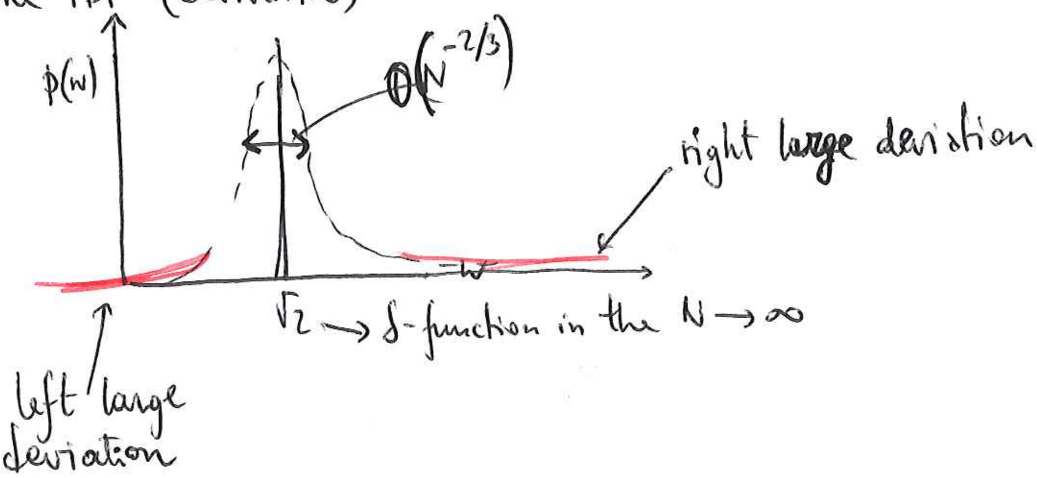
Phase transition.

What happens for N large but finite?



"Too rich a web connectance (too large a C) or too large an average interaction strength (too large an α) leads to instability. The larger the number of species, the more pronounced the effect."

Plot the PDF (derivative)



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Longest increasing subsequence

$$\{0, 8, 4, 12, 2, 10, 6, 14, 1, 9, 5, 13, 3, 11, 7, 15\}$$

increasing subsequence $\{0, 6, 9, 13, 15\}$

longest increasing sub. $\{0, 2, 6, 9, 11, 15\}$

$\{0, 4, 6, 9, 11, 15\}$ $\quad \underline{\ell = 6}$.

$\{0, 4, 6, 9, 13, 15\}$ $\quad$ length

- S_N : group of permutations of $1, 2, \ldots, N$.

If $\pi \in S_N$

$$N=5 \qquad \begin{array}{ccccccc} & & 1 & 2 & 3 & 4 & 5 \\ \pi_1 & : & 5 & 1 & 3 & 2 & 4 \end{array} \rightarrow \boxed{\ell_{N=5}(\pi_1) = 3.}$$

- Equip S_N with uniform distribution, and define

$$\mathrm{Prob}\left[\ell_N \le n\right] = \frac{f_{N,n}}{N!}$$

$N!$: total number of permutations

$f_{N,n}$: n° of permutations with $\ell_N \le n$.

$N \to \infty$?

Problem was raised by Ulam $\rightarrow$ conjectured that $C = \lim\limits_{N\to\infty} \frac{\mathbb{E}[\ell_N]}{\sqrt{N}}$ exists.

"Ulam problem".

Various conjectures about the variance.

1993 Odlyzko and Rains $\lim\limits_{N\to\infty} \frac{\mathrm{Var}(\ell_N)}{N^{1/3}} \approx c_0$,

where $c_0 \approx 0.819\ldots$

$$\lim_{N\to\infty} \frac{\mathbb{E}[\ell_N] - 2\sqrt{N}}{N^{1/6}} \approx c_1$$

where $c_1 \approx -1.758\ldots$

<u>Thm.</u> [Baik, Deift, Johansson].

$$\chi_N = \frac{\ell_N - 2\sqrt{N}}{N^{1/6}} \xrightarrow{d} \chi$$

where $\mathrm{Prob}[\chi \leq s] = F_2(s)$ TW $\rightarrow \beta = 2$.

if you compute

$$\lim_{N\to\infty} \frac{\mathrm{Var}[\ell_N]}{N^{1/3}} = \int_{-\infty}^{+\infty} s^2 F(s)\,ds - \left(\int_{-\infty}^{+\infty} s F(s)\,ds\right)^2 \approx 0.8132\ldots$$

$$\lim_{N\to\infty} \frac{\mathbb{E}[\ell_N] - 2\sqrt{N}}{N^{1/6}} = \int_{-\infty}^{+\infty} s F(s)\,ds \approx -1.7711\ldots$$

Handout...

Large Deviations of the Largest Eigenvalue.

$$P(\lambda_1, \dots, \lambda_N) = B_N(\beta) \exp\left[-\beta\left(\frac{N}{2}\sum_{i=1}^{N}\lambda_i^2 - \frac{1}{2}\sum_{i\neq j}\ln|\lambda_i - \lambda_j|\right)\right].$$

$$F_N(w) = \mathbb{P}\mathrm{rob}\left[\lambda_{max} \leq w\right] = \mathbb{P}\mathrm{rob}\left[\lambda_1 \leq w, \lambda_2 \leq w, \dots, \lambda_N \leq w\right]$$

$$F_N(w) = \frac{Z_N(w)}{Z_N(w \to +\infty)}.$$

$$Z_N(w) = \int_{-\infty}^{w} d\lambda_1 \cdots \int_{-\infty}^{w} d\lambda_N \exp\left[-\frac{\beta}{2}\left(N\sum_{i=1}^{N}\lambda_i^2 - \sum_{i\neq j}\ln|\lambda_i - \lambda_j|\right)\right].$$

$$F_N(w) \approx \begin{cases} \exp\left[-\beta N^2 \Phi_-(w)\right] & w<\sqrt{2} \ \& \ |w-\sqrt{2}| \sim O(1) \\ \mathcal{F}_\beta\left(\sqrt{2}\, N^{2/3}(w-\sqrt{2})\right) & |w-\sqrt{2}| \sim O(N^{-2/3}). \\ 1 - \exp\left[-\beta N \Phi_+(w)\right] & w>\sqrt{2} \ \& \ |w-\sqrt{2}| \sim O(1). \end{cases}$$

$$\Phi_-(w) = \frac{1}{108}\left[36w^2 - w^4 - (15w + w^3)\sqrt{w^2+6} + 27\left(\ln 18 - 2\ln\left(-w + \sqrt{w^2+6}\right)\right)\right], \quad w<\sqrt{2}$$

$$\sim \frac{1}{6\sqrt{2}}(\sqrt{2}-w)^3, \quad w \to \sqrt{2}^-.$$

$$\Phi_+(w) = \frac{1}{2}w\sqrt{w^2-2} + \ln\left[\frac{w-\sqrt{w^2-2}}{\sqrt{2}}\right].$$

Phase transition at the critical point $w_c = \sqrt{2}$

(May's transition, with $\alpha_c = \frac{1}{w_c} = \frac{1}{\sqrt{2}}$).

$$\lim_{N\to\infty} -\frac{1}{N^2} \ln F_N(w) = \begin{cases} \Phi_-(w) & w < \sqrt{2} \\ 0 & w > 2 \end{cases}$$

3rd derivative of the free energy of the Coulomb gas at the critical point is discontinuous: <u>3rd order phase transition</u> (in the sense of Ehrenfest).

[See review by Majumdar & Schehr, JSTAT 2013].

<u>Derivation using Coulomb gas method</u>.

<u>Goal</u>

$$Z_N(w) = \int_{-\infty}^{w} \cdots \int_{-\infty}^{w} d\lambda_1 \ldots d\lambda_N \exp\left[-\frac{\beta N^2}{2}\left(\frac{1}{2N}\sum_{i=1}^{N} \lambda_i^2 - \frac{1}{2N^2}\sum_{i\neq j} \ln|\lambda_i - \lambda_j|\right)\right]$$

for $N \to \infty$.

(1) Introduce a "counting function"

$$n(x) = \frac{1}{N}\sum_{i=1}^{N} \delta(x - \lambda_i)$$

$\int_{\mathbb{R}} dx\, n(x) = 1$

$n(x) \geq 0$ everywhere

Assumptions: $N \to \infty$ $n(x)$ becomes a smooth function of 'x'.

(2) Instead of summing (integrating) over $\{\lambda_1, \dots, \lambda_N\}$ (microstates), we first fix a certain profile $n(x)$ (non-negative, smooth, normalized). Then, we sum over all <u>microstates</u> "compatible" with $n(x)$, and then we sum over all possible $n(x)$.

$$1 = \int \mathcal{D}[n(x)]\, \delta\left(n(x) - \frac{1}{N}\sum_{i=1}^{N} \delta(x-\lambda_i)\right).$$

$$Z_N(\omega) = \int \mathcal{D}[n(x)] \int_{-\infty}^{w} \cdots \int_{-\infty}^{w} d\lambda_1 \cdots d\lambda_N \exp\left[\,\cdot\; - - - - -\right] \times$$
$$\times\, \delta\left(n(x) - \frac{1}{N}\sum_{i=1}^{N} \delta(x-\lambda_i)\right).$$

(3) Use the identities

$$\sum_{i=1}^{N} f(\lambda_i) = N \int dx\, n(x) f(x)$$

$$\sum_{i,j=1}^{N} g(\lambda_i, \lambda_j) = N^2 \iint dx\, dx'\, n(x)\, n(x')\, g(x, x').$$

$$\frac{1}{2N}\sum_{i=1}^{N}\lambda_i^2 = \frac{1}{2N}\times N\int dx\, n(x)\, x^2 .$$

$$\frac{1}{2N^2}\sum_{i\neq j}{}' \ln|\lambda_i-\lambda_j| = \frac{1}{2N^2}\left[\sum_{i,j}{}' \ln|\lambda_i-\lambda_j| - \sum_i{}' \ln \Delta(\lambda_i)\right]$$

$$= \frac{1}{2N^2}\times N^2 \iint dx\,dx'\, n(x)\, n(x') \ln|x-x'| - \frac{1}{2N^2}\times N \int dx\, n(x) \ln \underbrace{\Delta(x)}_{\text{short-distance cutoff.}}$$

4

$$Z_N(w) = \int \mathcal{D}[n(x)]\, \exp\left[-\beta N^2\left(\frac{1}{2}\int dx\, n(x) x^2 - \frac{1}{2}\iint dx\, dx'\, n(x)\, n(x') \ln|x-x'| + \frac{1}{2N}\int dx\, n(x) \ln\Delta(x)\right)\right] \times$$

$$\times \underbrace{\int_{-\infty}^{w} d\lambda_1 \cdots \int_{-\infty}^{w} d\lambda_N\, \delta\left(n(x) - \frac{1}{N}\sum_{i=1}^{N}{}' \delta(x-\lambda_i)\right)}_{I_N[n(x);\, w] \sim \exp\left[-N\int dx\, n(x) \ln n(x)\right]} .$$

$-N\int dx\, n(x) \ln n(x)$: "entropic" term (counts how many microstates are there compatible with $n(x)$)

Exercise: see Dean & Majumdar

To leading order in 'N'

$$Z_N(\omega) \simeq \int \mathcal{D}[n(x)] \exp\left[-\beta N^2 \mathcal{E}_\omega[n(x)] + O(N)\right]$$

where

$$\mathcal{E}_\omega[n(x)] = \frac{1}{2}\int_{-\infty}^{W} dx\, n(x)\, x^2 - \frac{1}{2}\iint_{-\infty}^{W} dx\, dx'\, n(x)\, n(x') \ln|x-x'| .$$

⑤ Saddle-point evaluation.

$$Z_N(\omega) \simeq \exp\left[-\beta N^2 \mathcal{E}_\omega[n^*(x)]\right],$$

where $n^*(x)$ is such that:

normalization of the density

$$\left.\frac{\delta \mathcal{E}_\omega[n(x)]}{\delta n(x)}\right|_{n(x)=n^*(x)} = \frac{1}{2}x^2 - \int_{-\infty}^{W} dx'\, n^*_\omega(x') \ln|x-x'| + C = 0$$

for $x \in \mathrm{supp}\left[n_\omega(x)\right]$

Claim $n^*_\omega(x)$ cannot have unbounded support.

For large $|x|$ LHS $\sim |x|^2$

RHS $\sim \ln|x|$

so they can never balance each other.

(6) Differentiate the integral equation wrt 'x'.

$$\frac{1}{2}x^2 = \int_{-\infty}^{\omega} dx' \, n_\omega^*(x') \ln|x-x'| + C$$

Problem: $\ln|x-x'|$ is not differentiable $\longrightarrow$ Notion of 'weak derivative'.

Let u be a function in $\mathcal{L}^1([a,b])$. We say that $v \in \mathcal{L}^1([a,b])$ is a weak derivative of u if

$$\int_a^b dx \, u(x) \varphi'(x) = -\int_a^b dx \, v(x) \varphi(x) .$$

for all infinitely differentiable functions φ with $\varphi(a) = \varphi(b) = 0$.

[The notion of weak derivative extends the standard (strong) derivative to functions that are not <u>differentiable</u>, but <u>integrable</u>].

Setting $u(x) = \int dx' \, n_\omega^*(x') \ln|x-x'|$, we can write

$$\int dx \, \varphi'(x) \left[\int dx' \, n_\omega^*(x') \ln|x-x'| \right] = \frac{1}{2} \lim_{\varepsilon \to 0} \int dx \, \varphi'(x) \left[\int dx' \, n^*(x') \ln\left[(x-x')^2 + \varepsilon^2\right] \right]$$

$$= -\frac{1}{2} \lim_{\varepsilon \to 0} \int dx \, \varphi(x) \left[\int dx' \, \frac{2(x-x')}{(x-x')^2 + \varepsilon^2} \right] = -\int dx \, \varphi(x) \left[\Pr \int dx' \, \frac{n^*(x')}{x-x'} \right]$$

Therefore, differentiating

$$x = \mathrm{Pr}\int_{a_1}^{a_2} dx' \, \frac{n_\omega^*(x')}{x-x'}$$

. Singular integral equation

Tricomi formula.

$$g(\lambda) = \mathrm{Pr}\int_{a_1}^{a_2} dx' \, \frac{\rho(x')}{x-x'}$$

$$\hookrightarrow \quad \rho(\lambda) = \frac{1}{\pi\sqrt{(a_2-\lambda)(\lambda-a_1)}}\left[C_0 - \mathrm{Pr}\int_{a_1}^{a_2}\frac{dt}{\pi}\,\frac{\sqrt{(a_2-t)(t-a_1)}}{\lambda-t}\,g(t)\right]$$

where $C_0 = \int_{a_1}^{a_2}\rho(\lambda)\,d\lambda$ is a constant. In our case, $g(\lambda)=\lambda$ and $C_0 = 1$ due to normalization.

$$I(\lambda) = \mathrm{Pr}\int_{a_1}^{a_2} \frac{dt}{\pi}\, \frac{\sqrt{(a_2-t)(t-a_1)}}{\lambda - t}\cdot t$$

$$\frac{t-a_1}{a_2-a_1} = \xi$$

$$I(\lambda) = \frac{(a_2-a_1)^2}{\pi} \int_0^1 d\xi \left[(a_2-a_1)\xi + a_1\right] \frac{\sqrt{\xi(1-\xi)}}{\lambda - (a_2-a_1)\xi - a_1}$$

$$= \frac{(a_2-a_1)^2}{\pi} \underbrace{\int_0^1 d\xi\, \xi \frac{\sqrt{\xi(1-\xi)}}{\frac{\lambda-a_1}{a_2-a_1} - \xi}}_{\frac{\pi}{8}\left[8\left(\frac{\lambda-a_1}{a_2-a_1}\right)^2 - 4\left(\frac{\lambda-a_1}{a_2-a_1}\right) - 1\right]} + \frac{a_1(a_2-a_1)}{\pi} \underbrace{\int_0^1 d\xi \frac{\sqrt{\xi(1-\xi)}}{\frac{\lambda-a_1}{a_2-a_1} - \xi}}_{\pi\left[\frac{\lambda-a_1}{a_2-a_1} - \frac{1}{2}\right]}.$$

$$\Downarrow$$

$$h^*_\omega(\lambda) = \frac{1}{8\pi\sqrt{(a_2-\lambda)(\lambda-a_1)}}\left[8 + (a_2-a_1)^2 + 4(a_1+a_2)\lambda - 8\lambda^2\right]$$

Check that $\int_{a_1}^{a_2} d\lambda\, \rho(\lambda) = 1$.

$$\mathcal{E}_\omega[n^*_\omega(x)] = \frac{1}{2}\int_{-\infty}^{w} dx\, n^*_\omega(x)\, x^2 - \frac{1}{2}\iint_{-\infty}^{w} dx\, dx'\, n^*_\omega(x)\, n^*_\omega(x') \ln|x-x'| \,.$$

$\longrightarrow$ it will be a function of $[a_1, a_2)$

Saddle point equation:

$$\frac{1}{2}x^2 - \int_{-\infty}^{w} dx'\, n^*_\omega(x') \ln|x-x'| + C = 0 \,.$$

$$\underline{x=0} \longrightarrow C = \int_{-\infty}^{w} dx'\, n^*_\omega(x') \ln|x'| \,.$$

Multiplying by $n^*_\omega(x)$ and integrating:

$$\int_{-\infty}^{w} \frac{1}{2} x^2 n^*_\omega(x)\, dx - \iint_{-\infty}^{w} dx\, dx'\, n^*_\omega(x)\, n^*_\omega(x') \ln|x-x'| + C = 0$$

$$\downarrow$$

$$\iint_{-\infty}^{w} \cdots = \frac{1}{2}\int_{-\infty}^{w} x^2 n^*_\omega(x)\, dx + \int_{-\infty}^{w} dx^*\, n^*_\omega(x) \ln|x| \,.$$

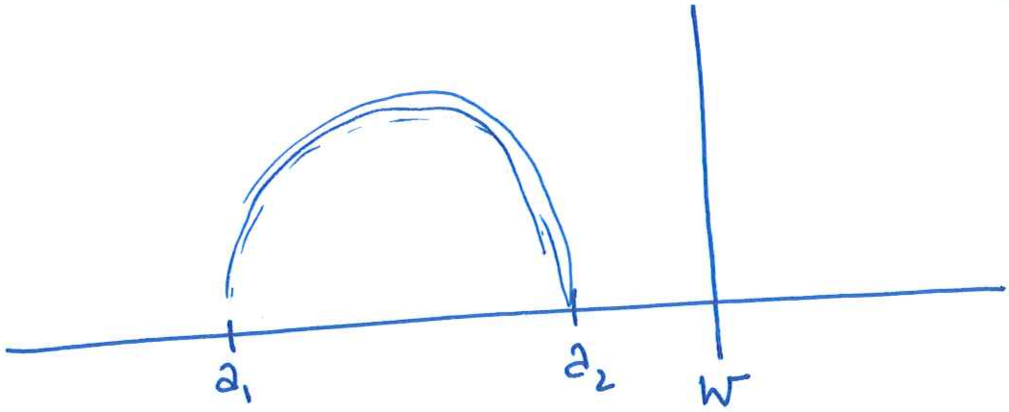
Plugging it into $\mathcal{E}_\omega[n^*(x)]$ we obtain

$$\mathcal{E}_\omega[n^*_\omega(x)] = \frac{1}{4}\int_{-\infty \to a_1}^{w} x^2 n^*_\omega(x)\, dx - \frac{1}{2}\int_{-\infty \to a_1}^{w} dx^*\, n^*_\omega(x) \ln|x| \,.$$

if $\underline{w > a_2}$

$$\mathcal{E}_\omega[n^*(x)] = \frac{1}{4}\int_{a_1}^{a_2} x^2 n^*(x)\,dx - \frac{1}{2}\int_{a_1}^{a_2} dx\, n^*(x) \ln|x| .$$

does not depend on 'w'.



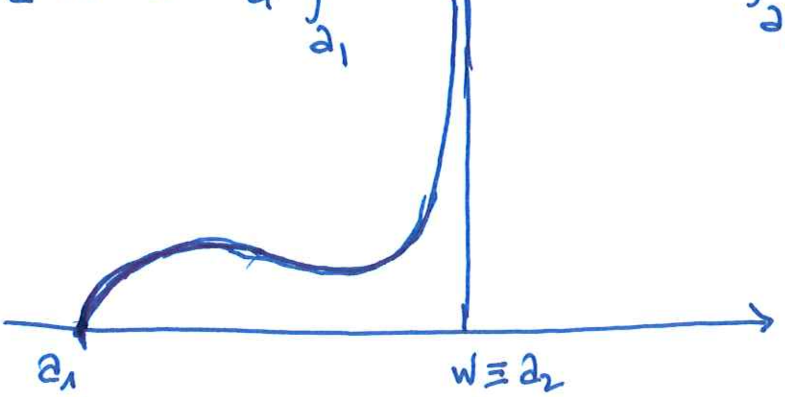
$\to F(a_1, a_2)$ to be minimized w.r.t. a_1 and a_2.

the result yields $a_1 = -a_2 = -\sqrt{2}$.

$$\boxed{n^*(\lambda) = \frac{1}{\pi}\sqrt{2-\lambda^2}} .$$

if $\underline{w < a_2}$

$$\mathcal{E}_\omega[n^*_\omega(x)] = \frac{1}{4}\int_{a_1}^{w} x^2 n^*_\omega(x)\,dx - \frac{1}{2}\int_{a_1}^{w} dx\, n^*_\omega(x) \ln|x| .$$



$$n_w^*(\lambda) = \frac{\sqrt{\lambda + L(\omega)}}{2\pi\sqrt{\omega - \lambda}}\left[\omega + L(\omega) - 2\lambda\right], \text{ with}$$

$-L(\omega) \leq \lambda \leq \omega$

for $\omega < \sqrt{2}$.

where $L(\omega) = \dfrac{2\sqrt{\omega^2+6} - \omega}{3}$.