Free Probability

A, B are $N \times N$ Hermitian matrices.
We are interested in the eigenvalues of the sum $A+B$.
Not sufficient to know eig. of A and B separately, one needs also the relative position of the eigenvectors of A and B: for example the decomposition of B-eigenvectors in the eigenbasis of A.

Define a non-standard addition of matrices that depends only on the eigenvalues of individual matrices in the sum and not on their eigenvectors

$$A \boxplus B = U^{\dagger} A U + V^{\dagger} B V$$

where U, V are independent unitary matrices, distributed according to the uniform measure on the unitary group.

$$\Downarrow$$

$A \boxplus B$ is a <u>random</u> matrix
(even if A and B are deterministic/fixed)

The transformation
$$A \to U^\dagger A U$$
$$B \to V^\dagger B V$$
preserve the eigenvalues of the individual terms, but uniformly randomize their eigenvectors.

<u>Observation</u>
$$U^\dagger A U + V^\dagger B V = U^\dagger (A + W^\dagger B \underset{\substack{\downarrow \\ VU^{-1}}}{W}) U$$

VU^{-1} also Haar unitary

$\downarrow$

It is sufficient to randomize the eigenvectors of one matrix against the other.

<u>More general setting</u> A, B are independent random matrices themselves.

Can one determine the eigenvalue density $\rho_{A \boxplus B}(\lambda)$ from the eigenvalue densities of A and B separately ($\rho_A(\lambda)$ and $\rho_B(\lambda)$)?

The answer is affirmative for $N \to \infty$.

FREENESS ~~FREE ADDITION~~ : $N \to \infty$
(generalization of statistical independence to non-commutative objects)

A and B statistically independent

Eigenvectors are in generic position.

Theorem (Sufficient condition for freeness)

Let $N \times N$ random matrices A_N and B_N such that:

- A_N, B_N have an asymptotic eigenvalue density for $N \to \infty$.
- A_N, B_N are independent
- B_N is a unitarily invariant ensemble

Then A_N and B_N are asymptotically free.

$$A + B \iff A \boxplus B$$

if B is rotationally invariant.

Classical (commutative) probability

$X_1, \dots, X_L$ independent r.v. drawn from $p_{X_i}(x)$.

$$S = X_1 + X_2 + \dots + X_L \; .$$

$$\varphi_X(t) = \langle e^{itx} \rangle = \int_{-\infty}^{+\infty} dx \, p_X(x) \, e^{itx} \; .$$

$$\boxed{\varphi_S(t) = \varphi_{X_1}(t) \, \varphi_{X_2}(t) \dots \varphi_{X_L}(t)} \; .$$

$\varphi_X(t)$ is the generating function of moments $m_X^{(n)} = \int_{-\infty}^{+\infty} dx \, p_X(x) x^n$

via

$$\varphi_X(t) = \int_{-\infty}^{+\infty} dx \, p_X(x) e^{itx} = \int_{-\infty}^{+\infty} dx \, p_X(x) \sum_{n=0}^{\infty} \frac{(it)^n}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} m_X^{(n)} \implies m_X^{(n)} = \frac{1}{i^n} \frac{d^n}{dt^n} \varphi_X(t) \Big|_{t=0} .$$

If we take the log of $\varphi_X(t)$

$$g_{X_i}(t) \doteq \log \varphi_{X_i}(t)$$

$$g_S(t) = g_{X_1}(t) + g_{X_2}(t) + \ldots + g_{X_L}(t) .$$

$g_X(t)$ is the generating function of <u>cumulants</u>.

$$\Downarrow$$

$$\boxed{K_S^{(n)} = K_{X_1}^{(n)} + K_{X_2}^{(n)} + \ldots + K_{X_L}^{(n)}} .$$

What is the analogue for matrices?

$$G_A(z) = \left\langle \frac{1}{N} \mathrm{Tr} \frac{1}{z - A} \right\rangle = \int \frac{\rho_A(\lambda) \, d\lambda}{z - \lambda} .$$

Resolvent or Green's function

A. Zee, "Law of addition in random matrix theory"
Nucl. Phys. B 474, 726 (1996)

[36-37 Handout]

"For the sake of convenience, we may, with due respect to Green, somewhat fancifully define a "Blue's function" ...

$$G_A(B_A(z)) = B_A(G_A(z)) = z$$

Functional inverse of the Green's function.

Remarkably, the Blue's function satisfies the following addition rule

$$\boxed{B_{H_1+\dots+H_L}(z) = B_{H_1}(z) + B_{H_2}(z) + \dots + B_{H_L}(z) + \frac{1-L}{z}}$$

for free Hermitian matrices $\{H_1, \dots, H_L\}$.

$\downarrow$

It becomes additive if we define the R-transform

$$R_A(z) = B_A(z) - \frac{1}{z}.$$

$\Downarrow$

$$\boxed{R_{H_1+\dots+H_L}(z) = R_{H_1}(z) + \dots + R_{H_L}(z)}$$

<u>Free Prob.</u>	<u>Class. Prob.</u>
R-transform	g (cumulant generating function).

<u>Remarks</u>

(1) ~~[illegible]~~

$$B_A(G_A(z)) = z \Longrightarrow R_A(G_A(z)) + \frac{1}{G_A(z)} = z$$

$$\Downarrow$$

$$\boxed{G_A(z) = \frac{1}{z - \underbrace{R_A(G_A(z))}}}.$$

J Dyson-Schwinger eq.
$R_A(G_A(z))$ is the self-energy $\Sigma(z)$ of physicists....

(2) R-transform is the generating function of <u>free cumulants</u>

$$R(z) = \sum_{n=1}^{\infty} \kappa_n z^{n-1}.$$

(Trace) Moments : $\quad m_n(A) = \left\langle \frac{1}{N} \operatorname{Tr} A^n \right\rangle$

free cum.	moments
$K_1 =$	$m_1(A)$
$K_2 =$	$m_2(A) - [m_1(A)]^2$
$K_3 =$	$m_3(A) - 3m_2(A)m_1(A) + 2[m_1(A)]^3$
K_4	$= \ldots$

K_1, K_2, K_3: as in classical probability

K_4 and higher: different.

Example:

In classical probability

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

has all cumulants $= 0$ except $K_2 = 1$.

What is the corresponding RM eigenvalue density that has all free cumulants $= 0$ except $K_2 = 1$?

$$R(z) = \sum_{n=1}^{\infty} K_n z^{n-1} = z.$$

So from $G_A(z) = \dfrac{1}{z - R_A(G_A(z))} = \dfrac{1}{z - G_A(z)}$

$$\Downarrow$$

$$G(z) = \frac{1}{2}\left(z - \sqrt{z^2-4}\right) \Rightarrow \underbrace{\rho(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}}_{\text{semicircle}}$$

Semicircle law is stable with respect to free addition in the same way as the Gaussian law is stable with respect to addition in classical probability.

Examples.

① Addition of a deterministic matrix and a random matrix

$$H = H_d + M \qquad \mathbb{P}(M) = \frac{1}{Z} e^{-N \operatorname{Tr} \underbrace{V[M]}_{\frac{1}{2}M^2}}.$$

$$\Downarrow$$

$$G(z) = \frac{1}{z - G(z)}$$

$$\Downarrow$$

$$z = \frac{1}{B(z) - z}$$

$$\Downarrow$$

$$B(z) - z = \frac{1}{z}$$

$$\Downarrow$$

$$\boxed{B(z) = z + \frac{1}{z}}$$

The Blue's function has a simpler form than $G(z)$ itself.

H_d

$$\cancel{c\,\mathbb{1} \longrightarrow} \; G_{H_d}(z) = \frac{1}{N} \cancel{\mathrm{Tr}\,\frac{1}{(z-c)\,\mathbb{1}}} = \cancel{\frac{1}{z-c}} \Rightarrow z = \cancel{\frac{1}{B(z)-c}}$$

$$\Downarrow$$

$$\boxed{\cancel{B(z) = c + \frac{1}{z}}}$$

$$\mathrm{diag}(\epsilon_1, \dots, \epsilon_N) \longrightarrow G_{H_d}(z) = \frac{1}{N} \mathrm{Tr} \frac{1}{z\,\mathbb{1} - \mathrm{diag}(\epsilon_1, \dots, \epsilon_N)}$$

$$= \frac{1}{N} \sum_{i=1}^{N} \frac{1}{z - \epsilon_i} \, .$$

$$\Downarrow$$

$$\boxed{z = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{B(z) - \epsilon_i}}$$ implicitly.

$$\boxed{B_H(z) = B_{H_d}(z) + B_M(z) - \frac{1}{z}}$$ law of addition.

$$= B_{H_d}(z) + z + \cancel{\frac{1}{z}} - \cancel{\frac{1}{z}} \qquad (*)$$

To find $G_H(z)$, we use $B_H(G_H(z)) = z$, or using $(*)$

$$B_H(\mathcal{G}_H(z)) = B_{H_d}(\mathcal{G}_H(z)) + \mathcal{G}_H(z)$$

$$\downarrow$$

$$B_{H_d}(\mathcal{G}_H(z)) = z - \mathcal{G}_H(z) \, .$$

Applying $\mathcal{G}_{H_d}(\cdot)$ to both sides :

$$\boxed{\begin{aligned} \mathcal{G}_H(z) &= \mathcal{G}_{H_d}(z - \mathcal{G}_H(z)) \, . \\ &= \frac{1}{N}\sum_{i=1}^{N} \frac{1}{z - \epsilon_i - \mathcal{G}_H(z)} \, . \end{aligned}}$$

$$\Big\downarrow N \to \infty$$

$$\boxed{\mathcal{G}_H(z) = \int d\varepsilon \, \frac{\rho(\varepsilon)}{z - \varepsilon - \mathcal{G}_H(z)}}$$

"Pastur equation" (derived by other means in L. Pastur, Theor. Math. Phys. 10, 67 (1972) see HO 38-39)

<u>Check</u> : if $\rho(\varepsilon) = \delta(0) \Leftarrow H_d$ is not there

$$\mathcal{G}_H(z) = \frac{1}{z - \mathcal{G}_H(z)}$$ standard Gaussian equation

Sum of GOE and Wishart random matrices.

$$W = \frac{1}{M} X \cdot X^T$$

$$X = N \overbrace{\left[\qquad\qquad\qquad\qquad\right]}^{M} \qquad X_{ij} \sim \mathcal{N}(0,1)$$

$r = \frac{N}{M}$ fixed as $N, M \to \infty$.

$$P(\lambda_1, \dots, \lambda_N) \propto e^{-\frac{M}{2}\sum_{i=1}^{N}{}' \lambda_i} \prod_{i=1}^{N} \lambda_i^{\frac{1}{2}(M-N-1)} \prod_{j<k} |\lambda_j - \lambda_k| .$$

$\{\lambda_i\}$ are all positive.

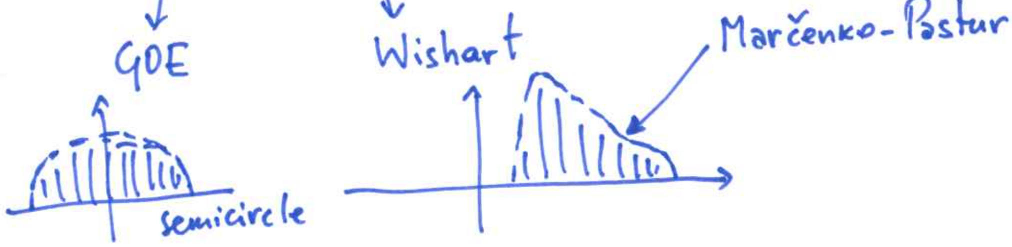
Using the resolvent method (careful!) one obtains the following equation for $G_W(z)$

$$\frac{1}{2r} G_W(z) + \left(\frac{1}{2} - \frac{1}{2r}\right)\frac{1}{z}\left[\frac{1}{1-r} + G_W(z)\right] = \frac{1}{2} G_W^2(z),$$

from which the Blue function

$$B_W(z) = \frac{1}{z} + \frac{1}{1-rz} .$$

$$S = a\,G + b\,W$$



$$B_G(z) = z + \frac{1}{z}$$

$$B_W(z) = \frac{1}{z} + \frac{1}{1-rz}.$$

$$\Downarrow$$

$$R_G(z) = z$$

$$R_W(z) = \frac{1}{1-rz}.$$

Using the rescaling property of the R-transform

$$R_{cH}(z) = c\,R_H(cz).$$

$$R_S(z) = a\,R_G(az) + b\,R_W(bz)$$

$$= a^2 z + \frac{b}{1-rbz}.$$

$$B_S(z) = R_S(z) + \frac{1}{z}$$

$$= \frac{1}{z} + a^2 z + \frac{b}{1 - rbz}$$

$$\Downarrow$$

The Green's function for the sum satisfies:

$$\boxed{z = \frac{1}{G_S(z)} + a^2 G_S(z) + \frac{b}{1 - rb\, G_S(z)}.}$$

p. 26-27-28 Codes.

[Application to random supergravity 45-46-47].