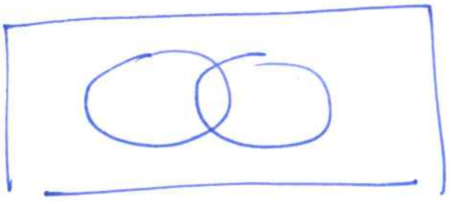
Edwards-Jones formula



valid (in principle) everywhere

Goal: connect the joint probability density of matrix entries $\underbrace{\mathbb{P}(x_{11},\dots,x_{NN})}_{\text{INPUT}}$ with the average spectral density $\underbrace{\rho_N(\lambda)}_{\text{OUTPUT}}$.

Normally, one would like to establish the following chain

$$\mathbb{P}(\underbrace{x_{11},\dots,x_{NN}}_{\sim O(N^2)}) \longrightarrow \mathbb{P}(\underbrace{\lambda_1,\dots,\lambda_N}_{\sim O(N)}) \longrightarrow \rho_N(\lambda),$$

as $\rho_N(\lambda)$ is the marginal of the joint pdf of eigenvalues (and therefore it can be – at least in principle – computed from it).

However, we know that the first step of the chain

$$\mathbb{P}(x_{11},\dots,x_{NN}) \longrightarrow \mathbb{P}(\lambda_1,\dots,\lambda_N)$$

is not always possible (the integration over eigenvectors degrees of freedom may not be carried out in closed form).

The Edwards-Jones formula connects <u>directly</u> the jpdf of entries and the average spectral density

$$\mathbb{P}(x_{11},\dots,x_{NN}) \longrightarrow \rho_N(\lambda),$$

bypassing completely the jpdf of eigenvalues. As such, it is particularly welcome for cases where the jpdf of eigenvalues is not known (e.g. independent entries...)

The formula reads

$$\rho_N(\lambda) = \frac{-2}{\pi N} \lim_{\varepsilon\to 0^+} \mathrm{Im} \frac{\partial}{\partial\lambda} \left\langle \mathrm{Log}\, Z(\lambda) \right\rangle,$$

where

$$Z(\lambda) = \int_{\mathbb{R}^N} d\mathbf{y}\ \exp\left[-\frac{i}{2}\mathbf{y}^T(\lambda_\varepsilon \mathbb{1} - X)\mathbf{y}\right].$$

the entries of X (real symmetric) appear here....

$$\lambda_\varepsilon = \lambda - i\varepsilon.$$

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_N \end{pmatrix}$$

The average $\langle \ldots \rangle$ is taken "over the disorder", i.e.

$$\langle (\ldots) \rangle = \int dx_{11} \ldots dx_{NN} \underbrace{P(x_{11}, \ldots, x_{NN})}_{\text{this is the only required input}} (\ldots)$$

Remark: while the formula is valid for finite N, in practice the calculations can be carried out until the end only in the limit $N \to \infty$, where several simplifications take place.

The proof

$$\rho_N(\lambda) = \left\langle \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i) \right\rangle, \text{ by definition.}$$

Recall the Sokhotski-Plemelj identity

$$\frac{1}{x \pm i\varepsilon} \underset{\text{as } \varepsilon \to 0^+}{\Rightarrow} \Pr\left(\frac{1}{x}\right) \mp i\pi\delta(x),$$

which offers an interesting representation of the Dirac delta.

$$\rho_N(\lambda) = \frac{1}{\pi N} \lim_{\varepsilon \to 0^+} \operatorname{Im} \left\langle \sum_{i=1}^{N} \frac{1}{\lambda - i\varepsilon - \lambda_i} \right\rangle$$

$$= \frac{-1}{\pi N} \lim_{\varepsilon \to 0^+} \operatorname{Im} \left\langle \sum_{i=1}^{N} \frac{1}{\lambda_i + i\varepsilon - \lambda} \right\rangle.$$

Now, we write the denominator as the derivative of a logarithm. Careful! The denominator is a complex number, and complex logarithms are not nearly as "nice" as their real counterparts....

We can denote by Log the principal branch of the complex logarithm (recall that logarithms in the complex plane are multivalued functions)

↓

the logarithm of a complex number z is a complex number w such that $e^w = z$.

Problem: the complex exponential is not injective (it does not map distinct values to distinct values), since

$$e^{w+2\pi i} = e^w \text{ for any } w.$$

Solution: restrict the domain of the exponential function to a region that does not contain any two numbers differing by an integer multiple of $2\pi i$ ⟶ "branches" of the logarithm.

For each nonzero complex number $z = x + iy$, the "principal value" Log z is the logarithm whose imaginary part lies in the interval $(-\pi, \pi]$.

$z = re^{i\theta} \Longrightarrow \operatorname{Log} z = \ln r + i\theta$

$\theta \in (-\pi, \pi]$

- Not all the properties we are familiar with for the real logarithm carry over the complex logarithm.

$\operatorname{Log} e^{z}$ may be different from z

$\operatorname{Log}(z_1 z_2)$ may be different from $\operatorname{Log} z_1 + \operatorname{Log} z_2$.

<u>Exercise</u> $\operatorname{Log}\left((-1)i\right) = -\frac{\pi i}{2}$.

$\operatorname{Log}(-1) + \operatorname{Log} i = \frac{3\pi i}{2}$.

Anyway, we can still write

$$\sum_{i=1}^{N} \frac{1}{\lambda_i + i\varepsilon - \lambda} = \frac{\partial}{\partial \lambda} \sum_{i=1}^{N} \operatorname{Log}(\lambda_i + i\varepsilon - \lambda) .$$

Now, we would like to use the familiar "Tr log = log det" identity (which for real logarithms is simply a translation of $\sum_i \ln(\ldots) = \ln \prod_i (\ldots)$). However, we'll need to me more careful here, due to the complex nature of the logarithms involved.

We can use the identity:

$$Z(\lambda) = \underbrace{\int_{\mathbb{R}^N} d\mathbf{y}\, \exp\left[-\frac{i}{2}\mathbf{y}^T \overbrace{(\lambda_\varepsilon \mathbb{1} - X)}^{\text{complex symmetric matrix}} \mathbf{y}\right]}_{\text{Multidimensional Fresnel integral.}} = (2\pi)^{N/2} \exp\left[-\frac{1}{2}\underbrace{\sum_{i=1}^{N} \mathrm{Log}(\lambda_i + i\varepsilon - \lambda)}_{\text{this is the chunk we need...}} + i\frac{N\pi}{4}\right].$$

We would need to extract the term $\sum_{i=1}^{N} \mathrm{Log}(\lambda_i + i\varepsilon - \lambda)$ by taking the Log on both sides - but recall that $\mathrm{Log}\exp(z)$ may not be just equal to z !

However, we can still write

$$\sum_{i=1}^{N} \mathrm{Log}(\lambda_i + i\varepsilon - \lambda) = -2\,\mathrm{Log}\, Z(\lambda) + \text{terms that are killed by } \frac{\partial}{\partial\lambda}.$$

Therefore

$$\rho_N(\lambda) = -\frac{2}{\pi N} \lim_{\varepsilon\to 0^+} \mathrm{Im} \frac{\partial}{\partial\lambda} \left\langle \mathrm{Log}\, Z(\lambda) \right\rangle \quad \checkmark.$$

To apply the E-J formula, we would thus need to compute

$$\left\langle \operatorname{Log} Z(\lambda) \right\rangle = \int dx_{11}\cdots dx_{NN}\ \mathbb{P}(x_{11},\dots,x_{NN}) \operatorname{Log}\left[\int_{\mathbb{R}^N} d\boldsymbol{y}\, \exp\left[-\frac{i}{2}\boldsymbol{y}^T(\lambda_\varepsilon \mathbb{1}-X)\boldsymbol{y}\right]\right],$$

which is very annoying (and potentially lethal!) as the Log is right in the middle!

One may think of carrying out the $\boldsymbol{y}$-integration first, compute the Log, and then carry out the $\boldsymbol{x}$-integration ("average over disorder"), however this obvious route would just run the E-J formula backwards, and we would just obtain the trivial identity

$$\rho_N(\lambda) = \rho_N(\lambda) \quad !!$$

The only way to extract meaningful information from the E-J formula would be to try to exchange the two integrations, and perform the average over disorder <u>first</u>. But how to kick the logarithm out of the way?

Two possible strategies $\begin{cases} \underline{\text{annealed}} \\ \underline{\text{quenched}} \end{cases}$

Quenched vs. Annealed

adj. made less severe or intense
subdued or overcome
allayed
squelched

to heat (metal or glass) and allow it to cool slowly, in order to remove internal stresses and toughen it.

Calling the multiple $\mathbf{y}$-integral $Z(\lambda)$ is intentional: we wish to interpret it as the partition function of an associated stat-mech model in the canonical ensemble

$$\underbrace{Z(\lambda) = \int_{\mathbb{R}^N} d\mathbf{y} \exp\left[-\frac{i}{2}\mathbf{y}^T(\lambda_\varepsilon \mathbb{1} - X)\mathbf{y}\right]}$$

"canonical" partition function
corresponding to a Gibbs-Boltzmann distribution

$$\mathbb{P}(y_1,\dots,y_N) = \frac{1}{Z(\lambda)} \exp\left[-\mathcal{H}(\mathbf{y}; X, \lambda)\right] \quad (*)$$

(*) this is only formal, as the energy function $\mathcal{H}$ is complex

If we interpret $Z(\lambda)$ this way, then $\operatorname{Log} Z(\lambda)$ is essentially the <u>free energy</u> of the $\mathbf{y}$-model, which is then averaged over the disorder, $\langle \operatorname{Log} Z(\lambda) \rangle$. This means that the two level of randomness - $\mathbb{P}(y_1,\dots,y_N)$ and the RM disorder - unfold on different timescales: <u>first</u>, the dynamical variables $\mathbf{y}$ need to equilibrate according to the Gibbs-Boltzmann distribution for a fixed instance of the random matrix X.

Then (only afterwards) the free energy is averaged over the disorder (different realizations of X). $\Longrightarrow$ Quenched route

$\Downarrow$

it is the correct way to proceed, but has the drawback that the logarithm is in the way....

A second strategy - which simplifies the calculations considerably - is to treat the disorder as annealed instead: this means that the associated stat-mech model is described in terms of the joint set of dynamical variables $\{\boldsymbol{y}, X\}$, leading to a partition function

$$Z^{(ann)}(\lambda) = \int dX\, d\boldsymbol{y}\, (\text{-----})\ .$$

The dynamical variables $\boldsymbol{y}$ are no longer integrated over at fixed value of the disorder X, but rather X and $\boldsymbol{y}$ fluctuate and thermalize together. A questionable but widespread way to describe in words an annealed average is

$$\operatorname{Log}\langle Z(\lambda)\rangle \quad \text{vs.} \quad \langle \operatorname{Log} Z(\lambda)\rangle$$

$\downarrow$

while we should write $\operatorname{Log} Z^{(ann)}(\lambda)$.

Clearly the annealed average is just an approximation (simpler to compute): it works well, though, for "dense" random matrix models.

Example: Recover the semicircle for GOE using E-J (in an annealed approximation).

$$\mathbb{P}[X] = \prod_{i=1}^{N} \frac{e^{-\frac{N}{2}x_{ii}^2}}{\sqrt{2\pi/N}} \prod_{i<j} \frac{e^{-Nx_{ij}^2}}{\sqrt{\pi/N}}$$

(rescaled the unit variance by a factor $1/N$ $\Rightarrow$ the eigenvalues are rescaled by $1/\sqrt{N}$).

For the annealed calculation, we need to compute

$$Z^{(ann)}(\lambda) = \int_{\mathbb{R}^N} d\boldsymbol{y} \int \prod_{i\leq j} dx_{ij}\, \mathbb{P}[X]\, e^{-\frac{i}{2}\boldsymbol{y}^T(\lambda_\varepsilon \mathbb{1} - X)\boldsymbol{y}}.$$

$$\propto \int_{\mathbb{R}^N} d\boldsymbol{y}\, e^{-\frac{i}{2}\lambda_\varepsilon \sum_{i=1}^{N} y_i^2} \left\langle e^{\frac{i}{2}\sum_{i=1}^{N} x_{ii} y_i^2} \right\rangle_{P(\{x_{ii}\})} \left\langle e^{i\sum_{i<j} x_{ij} y_i y_j} \right\rangle_{P(\{x_{ij}\})}$$

Expanding $e^z \simeq 1 + z + z^2/2 + \ldots$

and using that $\langle x_{ij} \rangle = 0$ and $\langle x_{ij}^2 \rangle = \dfrac{1}{N(2-\delta_{ij})}$.

we have

$$\left\langle \exp\left[\frac{i}{2}\sum_{i=1}^{N} X_{ii}\, y_i^2\right]\right\rangle = \left\langle \prod_{i=1}^{N} \exp\left[\frac{i}{2} X_{ii}\, y_i^2\right]\right\rangle$$

$$\underset{\substack{\uparrow \\ \text{using} \\ \text{independence}}}{=} \prod_{i=1}^{N} \left\langle \exp\left[\frac{i}{2} X_{ii}\, y_i^2\right]\right\rangle = \prod_{i=1}^{N} \left\langle 1 + \frac{i}{2} X_{ii}\, y_i^2 - \frac{1}{8} X_{ii}^2\, y_i^4 + \ldots \right\rangle$$

$$= \prod_{i=1}^{N} \left(\langle 1 \rangle + \frac{i}{2} y_i^2 \overset{\substack{0 \\ \shortparallel}}{\langle X_{ii} \rangle} - \frac{1}{8} y_i^4 \overset{\substack{1/N \\ \shortparallel}}{\langle X_{ii}^2 \rangle} + \ldots \right)$$

$$= \prod_{i=1}^{N} \left(1 - \frac{1}{8N} y_i^4 + \ldots \right) \simeq \prod_{i=1}^{N} e^{-\frac{1}{8N} y_i^4} \quad .$$

Similarly (exercise)

$$\left\langle \exp\left[i \sum_{i<j} x_{ij}\, y_i y_j \right] \right\rangle \simeq \prod_{i<j}^{N} e^{-\frac{1}{4N} y_i^2 y_j^2} \quad .$$

Putting everything back together.

$$\left\langle \exp\left[\frac{i}{2}\sum_{i=1}^{N}{}' x_{ii}\, y_i^2\right]\right\rangle \left\langle \exp\left[i \sum_{i<j}{}' x_{ij}\, y_i\, y_j\right]\right\rangle$$

$$\simeq \exp\left[-\frac{1}{8N}\sum_{i,j}{}' y_i^2 y_j^2\right] = \exp\left[-\frac{1}{8N}\left(\sum_{i=1}^{N}{}' y_i^2\right)^2\right].$$

Therefore

$$Z^{(ann)}(\lambda) \propto \int_{\mathbb{R}^N} d\mathbf{y}\; e^{-\frac{i}{2}\lambda_\varepsilon \sum_{i=1}^{N}{}' y_i^2} \cdot e^{-\frac{1}{8N}\left(\sum_{i=1}^{N}{}' y_i^2\right)^2}$$

Introducing the Gaussian identity (Hubbard-Stratonovich transformation).

$$\int_{-\infty}^{+\infty} dq \exp\left[-\alpha q^2 + i\gamma q\right] \propto \exp\left(-\gamma^2/4\alpha\right)$$

with $\gamma = \sum_{i=1}^{N} y_i^2$ and $\alpha = 2N$

the square has disappeared! The price to pay is an extra integration over 'q'!

$$Z^{(ann)}(\lambda) \propto \int_{-\infty}^{+\infty} dq\; e^{-2Nq^2} \underbrace{\int_{\mathbb{R}^N} d\mathbf{y} \exp\left[-\frac{i}{2}\lambda_\varepsilon \sum_{i=1}^{N}{}' y_i^2 + iq \sum_{i=1}^{N}{}' y_i^2\right]}_{\left[\int_{\mathbb{R}} dy \exp\left[\underbrace{-\frac{i}{2}\lambda_\varepsilon y^2 + iq y^2}_{-\frac{1}{2}\varepsilon y^2 - i\left(\frac{1}{2}\lambda - q\right) y^2}\right]\right]^N}$$

the integral is convergent as $\varepsilon > 0$

Writing: $X^N = \exp\left[N \operatorname{Log} X\right]$

$$Z^{(ann)}(\lambda) \propto \int_{-\infty}^{+\infty} dq \, \exp\left[-N \underbrace{\left(2q^2 - \frac{1}{2}\operatorname{Log}\left(\frac{2\pi}{\varepsilon + i(\lambda - 2q)}\right)\right)}_{\varphi_\lambda(q)}\right]$$

$\downarrow$

excellent for a Laplace approximation.

$$Z^{(ann)}(\lambda) \approx \exp\left(-N\,\varphi_\lambda(q^*)\right),$$

where q^* is such that

$$\varphi_\lambda'(q^*) = 0 \Longrightarrow$$

$$4q^* + \frac{1}{2q^* - \lambda_\varepsilon} = 0$$

$$\Downarrow$$

$$\boxed{q^* = \frac{1}{4}\left(\lambda_\varepsilon \pm \underbrace{\sqrt{\lambda_\varepsilon^2 - 2}}_{\downarrow}\right)}$$

the birth of a semicircle...

Applying now E-J formula in the annealed version and for $N \to \infty$.

$$\rho_{N\to\infty}(\lambda) = -\frac{2}{\pi N} \lim_{\varepsilon \to 0^+} \operatorname{Im} \frac{\partial}{\partial \lambda} \operatorname{Log} \underbrace{Z^{(ann)}(\lambda)}_{\simeq \exp(-N \varphi_\lambda(q^*))}$$

$$\simeq -\frac{2}{\pi N} \lim_{\varepsilon \to 0^+} \operatorname{Im} \frac{\partial}{\partial \lambda} \left[-N \varphi_\lambda(q^*) \right]$$

$$= \frac{2}{\pi} \lim_{\varepsilon \to 0^+} \operatorname{Im} \frac{\partial}{\partial \lambda} \varphi_\lambda(q^*) .$$

Using now the chain rule

$$\frac{\partial}{\partial \lambda} \varphi_\lambda(q^*) = q^{*\prime} \cdot \underbrace{\frac{\partial}{\partial q} \varphi_\lambda(q) \Big|_{q=q^*}}_{\substack{=0 \\ \text{by the stationary condition}}} + \frac{\partial}{\partial \lambda} \varphi_\lambda(q) \Big|_{q=q^*}$$

$$= \frac{\partial}{\partial \lambda} \left[\frac{1}{2} \operatorname{Log} \left(\varepsilon + i(\lambda - 2q) \right) \right] \Big|_{q=q^*}$$

$$= \frac{1}{2} \frac{i}{\varepsilon + i(\lambda - 2q)} = \frac{1}{2\lambda_\varepsilon - 4q^*} .$$

$$\Downarrow$$

$$\underset{N\to\infty}{\rho}(\lambda) = \frac{2}{\pi} \lim_{\varepsilon\to 0^+} \operatorname{Im} \frac{1}{\lambda_\varepsilon \mp \sqrt{\lambda_\varepsilon^2 - 2}} \, .$$

$$= \frac{1}{\pi} \lim_{\varepsilon\to 0^+} \operatorname{Im} \left[\lambda_\varepsilon \pm \sqrt{\lambda_\varepsilon^2 - 2} \right]$$

Using $\sqrt{a+ib} = p + iq$

with

$$p = \frac{1}{\sqrt{2}} \sqrt{\sqrt{a^2+b^2} + a}$$

$$q = \frac{\operatorname{sgn} b}{\sqrt{2}} \sqrt{\sqrt{a^2+b^2} - a}$$

for $a = \lambda^2 - \varepsilon^2$ and $b = -2\varepsilon x$ one obtains in the limit

$$\underset{N\to\infty}{\rho}(\lambda) = \frac{1}{\pi} \frac{1}{\sqrt{2}} \sqrt{|\lambda^2 - 2| - \lambda^2 + 2} = \begin{cases} \frac{1}{\pi}\sqrt{2-\lambda^2} & \text{if } |\lambda| < \sqrt{2} \\ 0 & \text{if } |\lambda| > \sqrt{2} \end{cases}$$