# Spectral Density for sparse random matrices: Bray-Rodgers integral equation.

$$\rho(x) = \frac{-2}{\pi N} \lim_{\varepsilon\to 0^+} \mathrm{Im} \frac{\partial}{\partial x} \left\langle \mathrm{Log}\, Z(x) \right\rangle .$$

$$Z(x) = \int_{\mathbb{R}^N} d\boldsymbol{y} \exp\left[ -\frac{i}{2} \boldsymbol{y}^T (x_\varepsilon \mathbb{1} - H) \boldsymbol{y} \right] .$$

$$\boxed{H_{ij} = c_{ij} K_{ij}} .$$

- $\mathbb{P}(\{c_{ij}\}) = \prod_{i<j} p(c_{ij})\, \delta_{c_{ij}, c_{ji}} \qquad p(c_{ij}) = \left(1 - \frac{c}{N}\right) \delta_{c_{ij},0} + \frac{c}{N} \delta_{c_{ij},1} .$
- Distribution of $\{K_{ij}\}$ at present unspecified.

Using the replica identity $\left\langle \mathrm{Log}\, Z(x) \right\rangle = \lim_{n\to 0} \frac{1}{n} \mathrm{Log} \left\langle Z(x)^n \right\rangle$

$$\left\langle Z(x)^n \right\rangle = \left\langle \int \prod_{i<j} dc_{ij}\, p(c_{ij}) \int_{\mathbb{R}^{Nn}} \prod_{a=1}^{n} d\boldsymbol{y}_a \exp\left[ -\frac{i}{2} \sum_{ij} \sum_{a=1}^{n} y_{ia} (x_\varepsilon \delta_{ij} - H_{ij}) y_{ja} \right] \right\rangle_K$$

$$= \int_{\mathbb{R}^{Nn}} \prod_{a=1}^{n} d\mathbf{y}_a \exp\left[-\frac{i}{2}x_\varepsilon \sum_{i=1}^{N}\sum_{a=1}^{n} y_{ia}^2\right] \int \prod_{i<j} dc_{ij} \left[\left(1-\frac{c}{N}\right)\delta_{c_{ij},0} + \frac{c}{N}\delta_{c_{ij},1}\right] \cdot$$

$$\left\langle \exp\left[ i \sum_{i<j}\sum_{a=1}^{n} y_{ia} \underbrace{H_{ij}}_{c_{ij}K_{ij}} y_{ja}\right]\right\rangle_K .$$

$$\int \prod_{i<j} dc_{ij} (\dots) = \prod_{i<j} \underbrace{\left\langle 1-\frac{c}{N}+\frac{c}{N}\exp\left[i \sum_a y_{ia}K_{ij}y_{ja}\right]\right\rangle_K}$$

$$\underbrace{\left\langle 1+\frac{c}{N}\left[\exp(\cdot \dots) - 1\right]\right\rangle_K}$$

$$\|$$

$$\exp\left\{\frac{c}{2N} \sum_{i,j}\left(\left\langle \exp(\underbrace{\dots}_{iK\sum_a y_{ia}y_{ja}})\right\rangle_K - 1\right)\right\}.$$

$$= \int_{\mathbb{R}^{Nn}} \prod_{a=1}^{n} d\mathbf{y}_a \; \exp\left[-\frac{i}{2} x_\varepsilon \sum_{i=1}^{N} \sum_{a=1}^{n} y_{ia}^2\right] \cdot \exp\left[\frac{c}{2N} \sum_{i,j} \left(\left\langle \exp\left(iK \sum_a y_{ia} y_{ja}\right)\right\rangle_K - 1\right)\right].$$

Introducing the replicated density.

$$\mu(\vec{y}) = \frac{1}{N} \sum_i \prod_a \delta(y_a - y_{ia}) .$$

$$\left[\vec{y} = (y_1, y_2, \dots, y_n)\right].$$

$$1 = \int \mathcal{D}\mu \, \mathcal{D}\hat{\mu} \; \exp\left[-i \int d\vec{y} \, \hat{\mu}(\vec{y}) \left(N \mu(\vec{y}) - \sum_i \prod_a \delta(y_a - y_{ia})\right)\right].$$

(*)

$$\langle Z(x)^n \rangle = \int \mathcal{D}\mu \, \mathcal{D}\hat{\mu} \; \exp\Big[-iN \int d\vec{y} \, \hat{\mu}(\vec{y}) \mu(\vec{y}) + \frac{cN}{2} \int d\vec{y} \, d\vec{y}' \mu(\vec{y}) \mu(\vec{y}')$$

$$\cdot \left(\left\langle \exp\left(iK \sum_a y_a y'_a\right)\right\rangle_K - 1\right)\Big] \cdot \int_{\mathbb{R}^{Nn}} \prod_{a=1}^{n} d\mathbf{y}_a \; \exp\left[-\frac{i}{2} x_\varepsilon \sum_{i=1}^{N} \sum_{a=1}^{n} y_{ia}^2\right] \cdot$$

$$\exp\left[i \sum_i \int d\vec{y} \prod_a \delta(y_a - y_{ia}) \cdot \hat{\mu}(\vec{y})\right]$$

(**)

## Proof of (*).

$$\frac{cN}{2}\int d\vec{y}\, d\vec{y}'\, \mu(\vec{y})\,\mu(\vec{y}')\, f\left(\sum_a y_a y_a'\right)$$

$$= \frac{c\cancel{N}}{2}\cdot\frac{1}{\cancel{N}}\cdot\frac{1}{N}\sum_i\sum_j \int d\vec{y}\, d\vec{y}'\, \prod_a \delta(y_a - y_{ia})\prod_b \delta(y_b' - y_{jb})\, f\left(\sum_a y_a y_a'\right)$$

$$= \frac{c}{2N}\sum_{i,j} f\left(\sum_a y_{ia}\, y_{ja}\right). \qquad \checkmark$$

Let us now discuss the (**) bit.

This is just N identical copies of a single integral, therefore

$$(**) = \left[\int \underbrace{\cancel{\prod_{a=1}^{n} dy_a}}_{d\vec{y}_1} \exp\left[-\frac{i}{2}x_\varepsilon \sum_a y_{1a}^2 + i\int d\vec{y}\,\prod_a \delta\big(y_a - \overset{!!}{y_{1a}}\big)\,\hat{\mu}(\vec{y})\right]\right]^N$$

$$= \left[\int \underbrace{\cancel{\prod_{a=1}^{n} dy_a}}_{d\vec{y}} \exp\left[-\frac{i}{2}x_\varepsilon \sum_a y_a^2 + i\hat{\mu}(\vec{y})\right]\right]^N .$$

$$= \exp\left[N \operatorname{Log}\int \overset{d\vec{y}}{\cancel{\prod_a dy_a}} \cdots\right]$$

In summary:

$$\langle Z(x)^n \rangle = \int \mathcal{D}\mu \, \mathcal{D}\hat{\mu} \; \exp\left[ N \, S_n(\mu, \hat{\mu}; x) \right],$$

where

$$S_n(\mu, \hat{\mu}; x) = \frac{c}{2} \int d\vec{y} \, d\vec{y}\,' \mu(\vec{y}) \mu(\vec{y}\,') \left( \left\langle \exp\left(iK \sum_a y_a y'_a\right) \right\rangle_K - 1 \right)$$

$$- i \int d\vec{y} \, \hat{\mu}(\vec{y}) \mu(\vec{y}) + \mathrm{Log} \int \underbrace{\prod_a dy_a}_{d\vec{y}} \exp\left[ -\frac{i}{2} x_\varepsilon \sum_a y_a^2 + i \hat{\mu}(\vec{y}) \right].$$

Saddle-point approximation $N \to \infty$.

$$\frac{\delta S_n}{\delta \mu} = 0 \Rightarrow c \int d\vec{y}\,' \mu^*(\vec{y}\,') \left( \left\langle \exp\left(iK \overbrace{\sum_a y_a y'_a}^{\vec{y}\cdot\vec{y}\,'}\right) \right\rangle_K - 1 \right)$$

$$- i \hat{\mu}^*(\vec{y}) = 0$$

$$\frac{\delta S_n}{\delta \hat{\mu}} = 0 \Rightarrow -\cancel{i} \mu^*(\vec{y}) + \frac{\cancel{i} \exp\left[ -\frac{i}{2} x_\varepsilon \sum_a y_a^2 + i \hat{\mu}^*(\vec{y}) \right]}{\int \underbrace{\prod_a dy_a}_{d\vec{y}\,'} \exp\left[ -\frac{i}{2} x_\varepsilon \sum_a y_a^2 + i \hat{\mu}^*(\vec{y}\,') \right]} = 0.$$

Setting $i\hat{\mu}^*(\vec{y}) = c\, g(\vec{y})$, we can combine the two equations to obtain:

$$\cancel{c}\, g(\vec{y}) = \frac{\cancel{c}\int d\vec{y}\,'\, f(\vec{y}\cdot\vec{y}\,')\exp\left[-\frac{i}{2}x_\varepsilon \sum_a y_a'^2 + c\, g(\vec{y}\,')\right]}{\int d\vec{y}\,'\exp\left[-\frac{i}{2}x_\varepsilon \sum_a y_a'^2 + c\, g(\vec{y}\,')\right]}$$

Bray-Rodgers integral equation.

where

$$f(z) = \left\langle e^{ikz} \right\rangle_k - 1 .$$

How to reconstruct the spectral density from $g(\vec{y})$?

From the E-J formula

$$\rho(x) = -\frac{2}{\pi N}\lim_{\varepsilon\to 0^+} \mathrm{Im}\,\frac{\partial}{\partial x}\lim_{n\to 0}\frac{1}{n}\mathrm{Log}\,\underbrace{\left\langle Z(x)^n \right\rangle}_{\substack{\downarrow N\to\infty \\ \exp\left[N S_n(\mu^*, \hat{\mu}^*; x)\right]}}$$

$$= -\frac{2}{\pi \cancel{N}} \lim_{\varepsilon \to 0^+} \mathrm{Im} \frac{\partial}{\partial x} \lim_{n\to 0} \frac{1}{n} \cancel{N}\, S_n(\mu^*, \hat{\mu}^*; x)$$

$$= -\frac{2}{\pi} \lim_{\varepsilon \to 0^+} \mathrm{Im} \lim_{n\to 0} \frac{1}{n} \frac{\partial}{\partial x} S_n(\mu^*, \hat{\mu}^*; x).$$

$$S_n = (1) + (2) + (3)$$

the only term depending explicitly on $x$.

Differentiating with respect to '$x$' the term (3)

$$\frac{\partial}{\partial x} S_n(\mu^*, \hat{\mu}^*; x) = \frac{-\frac{i}{2}\int d\vec{y} \left(\sum_a y_a^2\right) \exp\left[-\frac{i}{2} x_\varepsilon \sum_a y_a^2 + c\, g(\vec{y})\right]}{\int d\vec{y} \exp\left[-\frac{i}{2} x_\varepsilon \sum_a y_a^2 + c\, g(\vec{y})\right]}, \quad (\square)$$

where $g(\vec{y})$ satisfies the Bray-Rodgers equation.

Assuming now a replica-symmetric solution

$$g(y) = g(|\vec{y}|).$$

we can pass to spherical $n$-dim coordinates in $(\square)$

$$\rho(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \mathrm{Re} \lim_{n \to 0} \frac{1}{n} \frac{\displaystyle\int_0^\infty dy\, y^{n+1} \exp\left[-\frac{i}{2} x_\varepsilon y^2 + c\, g(y)\right]}{\underbrace{\displaystyle\int_0^\infty dy\, \underbrace{y^{n-1}}_{u'} \underbrace{\exp\left[-\frac{i}{2} x_\varepsilon y^2 + c\, g(y)\right]}_{v}}_{\text{by parts}}} .$$

$$\cancel{\frac{y^n}{n} \exp[\ldots]\Big|_0^\infty} - \frac{1}{n} \int_0^\infty dy\, y^n\, \partial_y \exp[\ldots]$$

$$\downarrow$$

$$\boxed{\rho(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \mathrm{Re} \frac{\displaystyle\int_0^\infty dy\, y \exp\left[-\frac{i}{2} x_\varepsilon y^2 + c\, g(y)\right]}{\displaystyle\int_0^\infty dy \exp\left[-\frac{i}{2} x_\varepsilon y^2 + c\, g(y)\right] \left(i x_\varepsilon y - c\, g'(y)\right)} .}$$

Which equation does $g(y)$ satisfy?

$$g(\vec{y}) = \frac{\displaystyle\int d\vec{y}'\, f(\vec{y} \cdot \vec{y}') \exp\left[-\frac{i}{2} x_\varepsilon \sum_a y_a'^2 + c\, g(\vec{y}')\right]}{\displaystyle\int d\vec{y}' \exp\left[-\frac{i}{2} x_\varepsilon \sum_a y_a'^2 + c\, g(\vec{y}')\right]} .$$

Spherical n-dim coordinates.

$$g(y) = \frac{\displaystyle\int_0^\infty dr\, r^{n-1} \int_0^\pi d\phi\, (\sin\phi)^{n-2} f(y r \cos\phi) \exp\left[-\frac{i}{2} x_\varepsilon r^2 + c\, g(r)\right]}{\displaystyle\int_0^\infty dr\, r^{n-1} \int_0^\pi d\phi\, (\sin\phi)^{n-2} \exp\left[-\frac{i}{2} x_\varepsilon r^2 + c\, g(r)\right]} .$$

Specialising to Bray-Rodgers original choice

$$P(H_{ij}) = \left(1-\frac{c}{N}\right)\delta_{H_{ij},0} + \frac{c}{2N}\left[\delta_{H_{ij},1} + \delta_{H_{ij},-1}\right]$$

(sparse matrix, with symmetrically distributed nonzero entries),

this corresponds to a distribution for $K_{ij}$ of the form

$$P(\{K_{ij}\}) = \frac{1}{2}\delta_{K_{ij},1} + \frac{1}{2}\delta_{K_{ij},-1}$$

so

$$f(z) = \left\langle e^{iKz}\right\rangle_K - 1$$

$$= -1 + \frac{1}{2}\sum_{k=\pm 1} e^{ikz} = -1 + \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \cos(z) - 1.$$

In the end:

$$g(y) = \frac{\int_0^\infty dr\, r^{n-1} \overbrace{\int_0^\pi d\phi\, (\sin\phi)^{n-2}\left[\cos(yr\cos\phi) - 1\right]}^{\text{Angular integral}} \exp\left[-\frac{i}{2}x_\varepsilon r^2 + c\,g(r)\right]}{\int_0^\infty dr\, r^{n-1} \int_0^\pi d\phi\, (\sin\phi)^{n-2} \exp\left[-\frac{i}{2}x_\varepsilon r^2 + c\,g(r)\right]}$$

Using the formula

$$\int_0^{\pi} d\phi \, (\sin\phi)^{n-2} \left[ \cos(yr\cos\phi) - 1 \right]$$

$$= \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)}{2} \left\{ (-1)^{\frac{3n}{2}} i^n \left(\frac{2}{yr}\right)^{n/2} \left[ n J_{\frac{n}{2}}(yr) - yr J_{\frac{n}{2}+1}(yr) \right] - \frac{2}{\Gamma(n/2)} \right\},$$

where $J_k$ is the Bessel function of the first kind.

Setting $G(r) = e^{-\frac{1}{2}\chi_\varepsilon r^2 + cg(r)}$, we can rewrite the denominator as

$$\int_0^\infty dr\, r^{n-1} G(r) \int_0^{\pi} d\phi\, (\sin\phi)^{n-2} = -\frac{\sqrt{\pi}}{n} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma(n/2)} \int_0^\infty dr\, r^n G'(r)$$

doing one integration by parts.

Hence:

$$g(y) = -\frac{n\,\Gamma(n/2)}{2} \frac{\int_0^\infty dr\, r^{n-1} G(r) \left\{ (-1)^{\frac{3n}{2}} i^n \left(\frac{2}{yr}\right)^{n/2} \left[ n J_{\frac{n}{2}}(yr) - yr J_{\frac{n}{2}+1}(yr) \right] - \frac{2}{\Gamma(\frac{n}{2})} \right\}}{\int_0^\infty dr\, r^n G'(r)}$$

In the limit $n \to 0$:

- $n\,\Gamma(n/2) \to 2$

- $$\int_0^\infty dr\, r^n G'(r) \longrightarrow \int_0^\infty dr\, G'(r) = \underbrace{G(\infty)}_{=0} - \underbrace{G(0)}_{=e^{c\,g(0)}=0} = -1$$

- $r^{n-1}\left[\,\cdots\cdots\,\right] \longrightarrow -y\, J_1(yr)$,

We get eventually

$$\boxed{g(y) = -y \int_0^\infty dr\, J_1(yr)\, e^{-\frac{i}{2}x_\varepsilon r^2 + c\,g(r)}}$$

equivalent to eq. (18) of Bray-Rodgers paper.