Collisions in plasmas: the Landau equation

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Use of the entropy principle for specific equations

Spatially Homogeneous Kinetic equations:

1. Fokker-Planck: D. Bakry, M. Emery ; G. Toscani; A. Arnold, P. Markowich, G. Toscani, A. Unterreiter
2. Landau: LD, C. Villani
3. Boltzmann (Cercignani's conjecture): G. Toscani, C. Villani; C. Villani
4. Continuous coagulation-fragmentation: M. Aizenmann, T. Bak
5. Discrete coagulation-fragmentation: P.-E. Jabin, B. Niethammer

Parabolic equations:

1. Nonlinear diffusion: M. Del Pino, J. Dolbeault
2. Fourth order equations: M. Cáceres, J. Carrillo, G. Toscani
3. Reaction-Drift-Diffusion: A. Glitzky, K. Gröger, R. Hünlich; LD, K. Fellner

Boltzmann operator, abstract form

Unknown: $f := f(v) \geq 0$ density of a rarefied monoatomic gas w.r.t. the velocity $v \in \mathbb{R}^N$ ($N = 2, 3$).

Operator:

$$Q_{Bolt,B}(f,f)(v) = \int\int\int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N} \Big(f(v')f(v'_*) - f(v)f(v_*) \Big)$$

$$\times\, \delta_{v+v_*=v'+v'_*}\, \delta_{|v|^2+|v_*|^2=|v'|^2+|v'_*|^2}\, B\left(|v-v_*|, (v - \widehat{v_*, v'} - v'_*) \right) dv'\, dv'_*\, dv_*.$$

Remark: $Q_{Bolt}(M,M) = 0$ when $M(v) = \exp(a + b \cdot v - c\,|v|^2)$ is a Maxwellian function of v.

Boltzmann operator, parameter-form when $N = 2$

Boltzmann, Maxwell, ca 1860:

$$Q_{Bolt,B}(f,f)(v) = \int_{\mathbb{R}^2} \int_{-\pi}^{\pi} \Big(f(v')f(v'_*) - f(v)f(v_*) \Big) \, B(|v - v_*|, |\theta|) \, d\theta dv_*,$$

with

$$v' = \frac{v + v_*}{2} + R_\theta \left(\frac{v - v_*}{2} \right),$$

$$v'_* = \frac{v + v_*}{2} - R_\theta \left(\frac{v - v_*}{2} \right).$$

Grazing collisions asymptotics

Chapman-Cowling, circa 50

At the formal level,

$$Q_{Lan,\psi}(f,f)(v) = \lim_{\varepsilon \to 0} Q_{Bolt,B_\varepsilon}(f,f)(v),$$

when B_ε is a rescaled version of B (extended by 0):

$$B_\varepsilon(z,\theta) = \frac{1}{\varepsilon^3} B\left(z, \frac{\theta}{\varepsilon}\right),$$

and

$$\psi(z) = Cst\,|z|^2 \int B(z,\theta)\,(1-\cos\theta)\,d\theta.$$

Rigorous results in the context of renormalized solutions: Alexandre-Villani, circa 2000.

Landau operator

Landau, 36: For $f := f(v) \geq 0$ density of charged particles of velocity $v \in \mathbb{R}^N$ in a plasma ($N = 3$),

$$Q_{Lan,\psi}(f,f)(v) = \nabla\cdot\left\{\int_{\mathbb{R}^N} a(v-w)\left(f(w)\,\nabla f(v) - f(v)\,\nabla f(w)\right) dw\right\},$$

with

$$a_{ij}(z) := \psi(|z|)\,\Pi_{ij}(z), \qquad \Pi_{ij}(z) := \delta_{ij} - \frac{z_i z_j}{|z|^2}.$$

Moreover $\psi(z) = |z|^{-1}$ in the Coulomb case (of charged particles).

Remark: $Q_{Lan}(M,M) = 0$ when $M(v) = \exp(a + b\cdot v - c\,|v|^2)$ is a Maxwellian function of v.

Common formalism for Boltzmann and Landau kernels (presented here in 2D)

Proposition LD, Salvarani 01: We assume that:

- Q is bilinear and continuous from $\mathcal{S}(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2)$ in $C^\infty_{\rm temp}(\mathbb{R}^2)$;
- Q is invariant by translations and rotations, i.-e. $Q \circ \tau_h = \tau_h \circ Q$, and $Q \circ \Gamma_R = \Gamma_R \circ Q$, where $\Gamma_R \varphi(x) = \varphi(Rx)$;
- $Q(M, M) = 0$, when $M(v) = \exp(a + b \cdot v - c\,|v|^2)$;
- For all $f, g \in \mathcal{S}(\mathbb{R}^2)$, $v_0 \in \mathbb{R}^2$, such that $f, g \geq 0$ on $\mathbb{R}^2$ and $f(v_0) = 0$, one has $Q(f,g)(v_0) \geq 0$, i.-e. $e^{t\,Q(\cdot,g)} \geq 0$.

Common formalism for Boltzmann and Landau kernels (presented here in 2D)

Then there exists a positive measure μ such that

$$Q(f,g)(v) = \left\langle Pf\left(\frac{\mu(|v-v_*|,\theta)}{1-\cos\theta}\right)\right. ,$$

$$\left. f\left(\frac{v+v_*}{2} + R_\theta\left(\frac{v-v_*}{2}\right)\right) g\left(\frac{v+v_*}{2} - R_\theta\left(\frac{v-v_*}{2}\right)\right)\right\rangle_{\theta,v_*} .$$

When the measure μ has an atomic part in 0 (and in other points), one finds the sum of a Boltzmann operator (with or without angular cutoff) and a Landau operator.

Weak formulation of Boltzmann operator

$$\int_{\mathbb{R}^N} Q_{Bolt,B}(f,f)(v)\,\varphi(v)\,dv$$

$$= \frac{1}{4} \iint_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^{N-1}} \Big(f(v')f(v'_*) - f(v)f(v_*) \Big)$$

$$\times \Big(\varphi(v) + \varphi(v_*) - \varphi(v') - \varphi(v'_*) \Big)$$

$$\times\, \delta_{v+v_*=v'+v'_*}\, \delta_{|v|^2+|v_*|^2=|v'|^2+|v'_*|^2}\, B\, dv'\, dv'_*\, dv_*.$$

Consequence: conservation of mass, momentum and energy:

$$\int_{\mathbb{R}^N} Q_{Bolt,B}(f,f)(v) \begin{pmatrix} 1 \\ v_i \\ |v|^2/2 \end{pmatrix} dv = 0.$$

Weak formulation of Landau operator

$$\int_{\mathbb{R}^N} Q_{Lan,\psi}(f,f)(v)\,\varphi(v)\,dv$$

$$= \frac{1}{2} \iint_{\mathbb{R}^N\times\mathbb{R}^N} f(v)\,f(w)\psi(|v-w|)\left(\frac{\nabla f(v)}{f(v)} - \frac{\nabla f(w)}{f(w)}\right)^T \Pi(v-w)$$

$$\left(\nabla\varphi(v) - \nabla\varphi(w)\right) dvdw.$$

Consequence: conservation of mass, momentum and energy:

$$\int_{\mathbb{R}^N} Q_{Lan,\psi}(f,f)(v) \begin{pmatrix} 1 \\ v_i \\ |v|^2/2 \end{pmatrix} dv = 0.$$

Entropy inequality (H theorem) for Boltzmann equation

Entropy production: $(f := f(v))$

$$D_{Bolt,B}(f) := -\int_{\mathbb{R}^N} Q_{Bolt,B}(f,f)(v) \ln f(v)\, dv$$

$$= \frac{1}{4} \iint_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N} \Big(f(v')f(v'_*) - f(v)f(v_*) \Big)$$

$$\times \Big(\ln(f(v')f(v'_*)) - \ln(f(v)f(v_*)) \Big)$$

$$\times\, \delta_{v+v_*=v'+v'_*}\, \delta_{|v|^2+|v_*|^2=|v'|^2+|v'_*|^2}\, B\, dv\, dv_* dv' dv'_* \geq 0$$

Entropy inequality (H theorem) for Landau equation

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$$D_{Lan,\psi}(f) := -\int_{\mathbb{R}^N} Q_{Lan,\psi}(f,f)(v) \ln f(v)\, dv$$

$$= \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} f(v)\, f(w)\, \psi(|v-w|)$$

$$\left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w)\right)^T \Pi(v-w) \left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w)\right) dvdw \geq 0$$

H Theorem, case of equality (1)

When $D_{Bolt,B}(f) = 0$, one has (if $B \neq 0$ a.e.)

$$v + v_* = v' + v'_* \qquad \text{and} \qquad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$$

$$\Rightarrow \qquad f(v)\, f(v_*) = f(v')\, f(v'_*),$$

so that finally

$$f(v)\, f(v_*) = T(v + v_*, |v|^2 + |v_*|^2).$$

H Theorem, case of equality (2)

We know that

$$f(v)\,f(w) = T(v+w, |v|^2 + |w|^2).$$

Then we use the operator:

$$L := (v-w) \times (\nabla_v - \nabla_w),$$

i.-e.

$$L_{ij} := (v_i - w_i)\,(\partial_{v_j} - \partial_{w_j}) - (v_j - w_j)\,(\partial_{v_i} - \partial_{w_i}),$$

and get

$$L\left[T(v+w, |v|^2 + |w|^2)\right] = 0.$$

H Theorem, case of equality (3)

Consequence: When $D_{Bolt,B}(f) = 0$, one has (for $i \neq j$)

$$q_{ij}^f(v,w) := (v_i - w_i)\left(\frac{\partial_j f(v)}{f(v)} - \frac{\partial_j f(w)}{f(w)}\right) - (v_j - w_j)\left(\frac{\partial_i f(v)}{f(v)} - \frac{\partial_i f(w)}{f(w)}\right) = 0,$$

i.-e.

$$\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w) \quad // \quad v - w.$$

We recall that for Landau equation:

$$D_{Lan,\psi}(f) = \frac{1}{2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(v)\, f(w)\, \psi(|v-w|)$$

$$\left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w)\right)^T \Pi(v-w) \left(\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(w)\right) dvdw.$$

Link between entropy dissipations

Proposition (Toscani-Villani 99, Villani 03): When $\psi(z) = |z|^2$ (Maxwell molecules), and $B(|z|, |\theta|) = |z|^2 \, \beta(|\theta|)$ (super hard spheres), one gets

$$D_{Bolt,B}(f) \geq Cst \int_0^{\infty} e^{-Cst\,\tau} \, D_{Lan,\psi}(e^{\tau\,L} f) \, d\tau,$$

where $e^{\tau\,L}$ is a semigroup generated by Fokker-Planck's operator:

$$Lf = \Delta f + \nabla \cdot (v\, f).$$

H Theorem, resolution of the case of equality (method of Boltzmann)

We know that (for $i \neq j$)

$$(v_i - w_i)\left(\frac{\partial_j f(v)}{f(v)} - \frac{\partial_j f(w)}{f(w)}\right) = (v_j - w_j)\left(\frac{\partial_i f(v)}{f(v)} - \frac{\partial_i f(w)}{f(w)}\right).$$

Using ∂_{v_i}, one gets

$$\frac{\partial_j f(v)}{f(v)} - \frac{\partial_j f(w)}{f(w)} + (v_i - w_i)\,\partial_{ij} \ln f(v) = (v_j - w_j)\,\partial_{ii} \ln f(v).$$

Using ∂_{w_j}, one gets $-\partial_{jj} \ln f(w) = -\partial_{ii} \ln f(v)$, and finally using ∂_{w_i}, one gets $-\partial_{ij} \ln f(w) - \partial_{ij} \ln f(v) = 0$.

At the end

$$f(v) = \exp(a + b \cdot v - c\,|v|^2).$$

H Theorem, resolution of the case of equality (method using integrals)

We know that (for $i \neq j$)

$$q_{ij}^f(v,w) = \left[v_i \frac{\partial_j f(v)}{f(v)} - v_j \frac{\partial_i f(v)}{f(v)} \right] - \frac{\partial_j f(w)}{f(w)} v_i + \frac{\partial_i f(w)}{f(w)} v_j$$

$$-w_i \frac{\partial_j f(v)}{f(v)} + w_j \frac{\partial_i f(v)}{f(v)} + \left[w_i \frac{\partial_j f(w)}{f(w)} - w_j \frac{\partial_i f(w)}{f(w)} \right] = 0.$$

Then (for $i \neq j$)

$$\frac{\partial_i f(v)}{f(v)} = \frac{Det \left(\int_{\mathbb{R}^N} f(w) \begin{bmatrix} 1 & q_{ij}^f(v,w) & w_i \\ w_i & q_{ij}^f(v,w)\, w_i + (v_j - w_j) & w_i^2 \\ w_j & q_{ij}^f(v,w)\, w_j - (v_i - w_i) & w_i\, w_j \end{bmatrix} dw \right)}{Det \left(\int_{\mathbb{R}^N} f(w) \begin{bmatrix} 1 & w_j & w_i \\ w_i & w_j\, w_i & w_i^2 \\ w_j & w_j^2 & w_i\, w_j \end{bmatrix} dw \right)}.$$

H Theorem, resolution of the case of equality (method using integrals)

Using the identity

$$q_{ij} f(v,w) = 0,$$

we end up with

$$\frac{\partial_i f(v)}{f(v)} = \frac{Det\left(\int_{\mathbb{R}^N} f(w) \begin{bmatrix} 1 & 0 & w_i \\ w_i & v_j - w_j & w_i^2 \\ w_j & -(v_i - w_i) & w_i\, w_j \end{bmatrix} dw \right)}{Det\left(\int_{\mathbb{R}^N} f(w) \begin{bmatrix} 1 & w_j & w_i \\ w_i & w_j\, w_i & w_i^2 \\ w_j & w_j^2 & w_i\, w_j \end{bmatrix} dw \right)},$$

so that

$$f(v) = \exp(a + b \cdot v - c\,|v|^2).$$

Cercignani's conjecture

Proposition (cf. LD-Villani 00): When $f \geq 0$ is such that $\int f \, dv = 1$, $\int f \, v \, dv = 0$, $\int f \, |v|^2/2 \, dv = N/2$,

$$D_{Lan, z \mapsto |z|^2}(f) \geq C(f) \int f(v) \left| \frac{\nabla f}{f}(v) + v \right|^2 dv$$

$$\geq C(f) \left(\int f \ln f - \int M \ln M \right),$$

where $C(f)$ only depends on (an upper bound of) the entropy $\int f \ln f$ of f, and

$$M(v) = (2\pi)^{-N/2} \exp(-|v|^2/2).$$

Consequence (Toscani-Villani 99, Villani 03): Cercignani's conjecture for the equation of Boltzmann (with "super hard spheres").

Idea of the proof

Link between $\frac{\partial_i f}{f}$ and the quantity in $D_{Lan,z\mapsto|z|^2}(f)$ (cf. LD 89):

$$D_{Lan,z\mapsto|z|^2}(f) = \frac{1}{4} \sum_{i,j=1,..,N} \int\int_{\mathbb{R}^N\times\mathbb{R}^N} f(v)\, f(w)\, \left|q_{ij}^f(v,w)\right|^2 \, dvdw,$$

$$q_{ij}^f(v,w) = (v_i - w_i)\left(\frac{\partial_j f}{f}(v) - \frac{\partial_j f}{f}(w)\right) - (v_j - w_j)\left(\frac{\partial_i f}{f}(v) - \frac{\partial_i f}{f}(w)\right).$$

Then for $i \neq j$

$$\frac{\partial_i f(v)}{f(v)} = \frac{Det\left(\int_{\mathbb{R}^N} f(w) \begin{bmatrix} 1 & q_{ij}^f(v,w) & w_i \\ w_i & q_{ij}^f(v,w)\, w_i + (v_j - w_j) & w_i^2 \\ w_j & q_{ij}^f(v,w)\, w_j - (v_i - w_i) & w_i\, w_j \end{bmatrix} dw\right)}{Det\left(\int_{\mathbb{R}^N} f(w) \begin{bmatrix} 1 & w_j & w_i \\ w_i & w_j\, w_i & w_i^2 \\ w_j & w_j^2 & w_i\, w_j \end{bmatrix} dw\right)}$$

Consequence

Estimate of convergence towards thermodynamic equilibrium in large time of the Landau equation with Maxwell molecules, and Boltzmann equation with super hard spheres

Theorem (cf. Toscani-Villani 99, LD-Villani 00, Villani 03): When $f_{in} \geq 0$ satisfies $\int f_{in}(v)\, dv = 1$, $\int f_{in}(v)\, v\, dv = 0$, $\int f_{in}(v)\, |v|^2/2\, dv = N/2$, and $\int f_{in} \ln f_{in} \leq \bar{H}$, the (unique) solution $f := f(t,v)$ of the (spatially homogeneous) Landau equation

$$\partial_t f = Q_{Lan, z \mapsto |z|^2}(f,f)$$

with Maxwell molecules, or the (spatially homogeneous) Boltzmann equation

$$\partial_t f = Q_{Bolt,B}(f,f)$$

with super hard spheres, satisfies

$$||f(t) - M||_{L^1(\mathbb{R}^3)} \leq Cst\, e^{-Cst\, t}.$$

where the constants only depend on $\bar{H}$.

Proof (quantitative version of La Salle's principle)

Method of entropy-entropy dissipation: If

$$\partial_t f = Q_{Lan,z\mapsto|z|^2}(f,f),$$

then

$$-\partial_t \int f \ln f = -\int Q_{Lan,z\mapsto|z|^2}(f,f) \ln f$$

$$= D_{Lan,z\mapsto|z|^2}(f),$$

$$\geq C(f)\left(\int f \ln f - \int M \ln M\right).$$

We conclude with Gronwall's lemma and Csiszár-Kullback-Pinsker inequality.

Large time behavior for Landau equation in the Coulomb case

Theorem: Carrapatoso-LD-He 16; improved in Carrapatoso-Mischler 17
Let $f_{in} \in L^1(e^{\kappa\,|v|^{1/2}}) \cap L \ln L(\mathbb{R}^3)$, with $\kappa \in]0, 2/e[$.
Then there exists a global weak solution f for the (spatially homogeneous) Landau equation

$$\partial_t f = Q_{Lan,|\cdot|^{-1}}(f, f)$$

in the Coulomb case (with initial data f_{in}) such that

$$\forall\, t \geq 0, \qquad ||f(t,\cdot) - M||_{L^1(\mathbb{R}^3)} \leq Cst\, e^{-(1+t)^{\frac{1}{7}}\ \log(1+t)^{-\frac{6}{7}}}.$$

where Cst is a constant depending on the initial mass, energy, entropy and $||f_{in}||_{L^1(e^{\kappa\,|v|^{1/2}})}$. The rate can be improved under extra conditions on the initial datum.

Proposition: LD 14, Carrapatoso-LD-He 16 One can find $C := C(\bar{H}) > 0$ depending only on $\bar{H} > 0$ such that for all $f \geq 0$ satisfying

$$\int_{\mathbb{R}^3} f(v)\, dv = 1, \quad \int_{\mathbb{R}^3} f(v)\, v\, dv = 0, \quad \int_{\mathbb{R}^3} f(v)\, |v|^2\, dv = 3,$$

and also satisfying (an upper bound on the entropy)

$$H(f) := \int_{\mathbb{R}^3} f(v) \ln f(v)\, dv \leq \bar{H},$$

the following inequality holds:

$$D_{Lan,|\cdot|^{-1}}(f) \geq C(\bar{H}) \left(\int f\, |v|^5\, dv \right)^{-1} \int_{\mathbb{R}^3} f(v) \left| \frac{\nabla f(v)}{f(v)} + v \right|^2 (1+|v|^2)^{-3/2}\, dv.$$

Idea of the proof

Sobolev logarithmic inequality (Bakry-Emery 84): For $f \geq 0$ such that

$$\int_{\mathbb{R}^3} f(v)\, dv = \int_{\mathbb{R}^3} M(v)\, dv,$$

the following inequality holds:

$$\int_{\mathbb{R}^3} f(v) \left| \frac{\nabla f(v)}{f(v)} + v \right|^2 (1+|v|^2)^{-3/2}\, dv$$

$$\geq Cst \int_{\mathbb{R}^3} \left\{ f(v) \ln\left(\frac{Z_1}{Z_2(f)} \frac{f(v)}{M(v)} \right) + \frac{Z_2(f)}{Z_1} M(v) - f(v) \right\} (1+|v|^2)^{-3/2}\, dv,$$

with

$$Z_1 := \int_{\mathbb{R}^3} M(v)\,(1+|v|^2)^{-3/2}\, dv, \qquad Z_2(f) := \int_{\mathbb{R}^3} f(v)\,(1+|v|^2)^{-3/2}\, dv,$$

and

$$M(v) = (2\pi)^{-3/2} \exp(-|v|^2/2).$$

Idea of the proof

Weak version of **Cercignani's conjecture** for Landau equation with Coulomb potential: For $f \geq 0$ satisfying the normalization and such that $H(f) \leq \bar{H}$,

$$D_{Lan,z\mapsto|z|^{-1}}(f) \geq C(\bar{H}) \left(\int f \, |v|^5 \, dv \right)^{-1}$$

$$\times \int_{\mathbb{R}^3} \left\{ f(v) \, \ln \left(\frac{Z_1}{Z_2(f)} \, \frac{f(v)}{M(v)} \right) + \frac{Z_2(f)}{Z_1} \, M(v) - f(v) \right\} (1+|v|^2)^{-3/2} \, dv,$$

with

$$Z_1 := \int_{\mathbb{R}^3} M(v) \, (1+|v|^2)^{-3/2} \, dv, \qquad Z_2(f) := \int_{\mathbb{R}^3} f(v) \, (1+|v|^2)^{-3/2} \, dv,$$

and

$$M(v) = (2\pi)^{-3/2} \, \exp(-|v|^2/2).$$

Idea of the proof: polynomial moments are propagated

Polynomially weighted moments:

$$M_\ell(f) = \int_{\mathbb{R}^3} f(v)\,(1+|v|^2)^{\ell/2}\,dv.$$

Proposition: Carrapatoso-LD-He 16 Assume that $f_{in} := f_{in}(v) \ge 0$ lies in $L^1_\ell \cap L \ln L(\mathbb{R}^3)$, $\ell \ge 2$, and that f is an H/weak solution in $L^\infty(\mathbb{R}_+;\, L^1_2(\mathbb{R}^3))$ (with initial datum f_{in}) of the spatially homogeneous Landau equation in the Coulomb case.

Then there exists $C = C(M_2(0), \int f_{in}(v) \ln f_{in}(v)\,dv, M_\ell(0), \ell) > 0$ such that

$$\forall t \ge 0, \qquad M_\ell(f(t)) \le C\,(1+t).$$

Idea of the proof: stretched exponential moments are propagated

Exponentially weighted moments:

$$M_{s,\kappa}(f) = \int_{\mathbb{R}^3} f(v) \exp(\kappa\,|v|^s)\, dv.$$

Proposition: Carrapatoso-LD-He 16 Assume that $f_{in} := f_{in}(v) \geq 0$ lies in $L^1(e^{\kappa\,|v|^s})$ with $\kappa > 0$ and $s \in]0, 1/2[$ or $\kappa \in]0, 2/e[$ and $s = 1/2$ and in $L \ln L(\mathbb{R}^3)$, and that f is an H-solution in $L^\infty(\mathbb{R}_+;\, L^1_2(\mathbb{R}^3))$ of the spatially homogeneous Landau equation in the Coulomb case (with initial datum f_{in}). Then, there exists $C = C(M_2(0), \int f_{in}(v) \ln f_{in}(v)\, dv, M_{s,\kappa}(0)) > 0$ such that

$$\forall t \geq 0, \qquad M_{s,\kappa}(f(t)) \leq C(1 + t).$$

Idea of the proof

Interpolation: For any $\kappa_1 > \kappa$ and $\kappa_2 > 3/2\,\kappa$ there holds

$$\int e^{\kappa|v|^s} f(v)\,\ln f(v)\,dv \le Cst\,(M_{s,\kappa_1}(f) + M_{s,\kappa_2}(f)^{\frac{2}{3}}\,\|f\|_{L^3_{-3}} + 1)$$

$$\le Cst\,(M_{s,\kappa_1}(f) + M_{s,\kappa_2}(f)^{\frac{2}{3}}\,D_{Lan,z\mapsto|z|^{-1}}(f) + 1).$$

Use of a previous coercivity estimate

Proposition: LD 14 For all $f \geq 0$,

$$\|f\|_{L^3_{-3}} \leq Cst \int_{\mathbb{R}^3} |\nabla \sqrt{f(v)}|^2 \, (1 + |v|^2)^{-3/2} \, dv$$

$$\leq C \, (1 + D_{Lan, z \mapsto |z|^{-1}}(f)),$$

where

$$C := C\left(\int_{\mathbb{R}^3} f(v) \, dv, \int_{\mathbb{R}^3} f(v) \, v \, dv, \int_{\mathbb{R}^3} f(v) \, |v|^2/2 \, dv, \bar{H} \right),$$

and

$$\bar{H} \geq \int_{\mathbb{R}^3} f(v) \, \ln f(v) \, dv.$$

Recent progresses on the subject

- Exponential convergence towards equilibrium for Boltzmann equation with angular cutoff and hard potentials Mouhot 06
- Exponential convergence towards equilibrium for Landau equation with (very) moderate soft potentials $\delta \in]1, 2[$ Carrapatoso 13/15
- Exponential convergence towards equilibrium for the (homogeneous or spatially inhomogeneous with initial data close to equilibrium) Boltzmann equation without angular cutoff and hard potentials Tristani 14, Héraud-Tonon-Tristani 17
- Stretched-exponential convergence towards equilibrium for the spatially inhomogeneous Landau equation with Coulomb potentials and initial data close to equilibrium Carrapatoso-Mischler 17

Based on estimates of spectral gaps in enlarged spaces Gualdani-Mischler-Mouhot 13

Open questions

- Optimal rate of convergence for general (non L^2) initial data
- Better smoothness estimates (possibly giving uniqueness)
- Possible link between the entropy dissipations of Boltzmann and Landau when the Coulomb case or other cases (different from Maxwell molecules) are considered
- Case of equality in the H theorem for generalized Boltzmann equations (relativistic, semiconductor-based, linked to weak turbulence)

Generalized Boltzmann equations, H theorem, case of equality (with M. Breden)

Abstract form of generalized Boltzmann's kernel:

$$Q_1'(f,f)(v) = \int\int\int \Big(f(v')f(v_*') - f(v)f(v_*) \Big)$$

$$\times\, \delta_{v+v_*=v'+v_*'}\, \delta_{\varepsilon(v)+\varepsilon(v_*)=\varepsilon(v')+\varepsilon(v')}\, B\, dv'\, dv_*'\, dv_*,$$

so that

$$D_1'(f) = \int\int\int\int \Big(f(v')f(v_*') - f(v)f(v_*) \Big)$$

$$\times \Big(\ln(f(v')f(v_*')) - \ln(f(v)f(v_*)) \Big)$$

$$\times\, \delta_{v+v_*=v'+v_*'}\, \delta_{\varepsilon(v)+\varepsilon(v_*)=\varepsilon(v')+\varepsilon(v')}\, B\, dv dv'\, dv_*'\, dv_*.$$

Generalized Boltzmann equations, H theorem, case of equality

Consequence: When $D_1'(f) = 0$ (and $B \neq 0$ a.e.), then

$$f(v)\, f(v_*) = T(v + v_*, \varepsilon(v) + \varepsilon(v_*)).$$

One uses the operator:

$$L' := (\nabla \varepsilon(v) - \nabla \varepsilon(v_*)) \times (\nabla_v - \nabla_{v_*}),$$

and gets

$$L' \Big[T(v + v_*, \varepsilon(v) + \varepsilon(v_*)) \Big] = 0.$$

Generalized Boltzmann equations, H theorem, case of equality

Consequence: When $D_1'(f) = 0$ (and $B \neq 0$ a.e.), then (for $i \neq j$)

$$(\partial_i \varepsilon(v) - \partial_i \varepsilon(v_*)) \left(\frac{\partial_j f(v)}{f(v)} - \frac{\partial_j f(v_*)}{f(v_*)} \right)$$

$$= (\partial_j \varepsilon(v) - \partial_j \varepsilon(v_*)) \left(\frac{\partial_i f(v)}{f(v)} - \frac{\partial_i f(v_*)}{f(v_*)} \right),$$

that is

$$\frac{\nabla f}{f}(v) - \frac{\nabla f}{f}(v_*) \quad // \quad \nabla \varepsilon(v) - \nabla \varepsilon(v_*).$$

Generalized Boltzmann equations, H theorem, case of equality

Proposition: We assume that (true in the relativistic case)

- For all $a, b, c \in \mathbb{R}, i \neq j$,

$$|\{v, \quad a + b\,\partial_i\varepsilon(v) + c\,\partial_j\varepsilon(v) = 0\}| = 0;$$

- $$\{1, \quad \partial_i\varepsilon(v), \quad \partial_i\varepsilon(v)\,\partial_j\varepsilon(v)\}_{i\leq j}$$

  is a linearly independent family;

Then if $D_1'(f) = 0$, one has

$$f(v) = \exp(a + b \cdot v - c\varepsilon(v)).$$