Second order conformally invariant elliptic equations

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May 22, 2017. ICTP, Trieste, Italy

• Yamabe problem

(M^n, g), $n \geq 3$, Riemannian, compact, $\exists?$ $\tilde{g} \sim g(\tilde{g} = u^{\frac{4}{n-2}} g)$ such that

$$R_{\tilde{g}} \equiv \text{Constant}.$$

• PDE:

$$-\Delta_g u + \frac{n-2}{4(n-1)} R_g u = \bar{R} u^{\frac{n+2}{n-2}}, \quad u > 0, \quad \text{on M},$$

$\bar{R} = -1, 0, 1.$

- Einstein-Hilbert functional:

$$E(g) = Vol(g)^{\frac{2-n}{n}} \int_M R_g dv_g.$$

- Euler-Lagrange equation:

$$Ric_g = \lambda g.$$

- Restricting to a conformal class of metrics :

$$[g] := \{\tilde{g} = u^{\frac{4}{n-2}} g \mid u \in C^\infty(M),\ u > 0\}$$

$$Y_g(u) \equiv E(\tilde{g}) = \frac{\int_M \left(|\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2\right) dv_g}{\left(\int_M u^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{n}}}.$$

- Euler-Lagrange equation for $E|_{[g]}$:

$$R_{\tilde{g}} = \lambda \qquad\qquad (\tilde{g} = u^{\frac{4}{n-2}} g).$$

- A critical point u leads to a solution of the PDE in u:

$$-\Delta_g u + \frac{n-2}{4(n-1)} R_g u = \bar{R} u^{\frac{n+2}{n-2}}, \quad u > 0, \quad \text{on M},$$

$\bar{R} = -1, 0, 1.$

• Three mutually exclusive cases:

The conformal Laplacian:

$$-L_g := -\Delta_g + \frac{n-2}{4(n-1)} R_g.$$

$$-\Delta_g u + \frac{n-2}{4(n-1)} R_g u = \lambda_1(-L_g) u^{\frac{n+2}{n-2}}, \quad u > 0, \quad \text{on M.}$$

$\lambda_1(-L_g)$ —- the sign of $\lambda_1(-L_g)$.

• Solution of the Yamabe problem:
Yamabe(1960), Trudinger (1968), Aubin (1976), Schoen (1984).

- Solution space.
- If $\lambda_1(-L_g) = 0$,

$$-\Delta_g u + \frac{n-2}{4(n-1)} R_g u = 0, \quad u > 0, \qquad \text{on M.}$$

- Eigenspace for the first eigenvalue is $1-$dimensional.
- Existence is also clear.

- If $\lambda_1(-L_g) < 0$,

$$-\Delta_g u + \frac{n-2}{4(n-1)} R_g u = -u^{\frac{n+2}{n-2}}, \quad u > 0, \qquad \text{on M.}$$

- Uniqueness of solution — maximum principle.

- Existence is also clear: $\underline{u} = \epsilon > 0$ subsolution, $\bar{u} = \frac{1}{\epsilon}$ supersolution.

• If $\lambda_1(-L_g) > 0$,

$$-\Delta_g u + \frac{n-2}{4(n-1)} R_g u = u^{\frac{n+2}{n-2}}, \quad u > 0, \qquad \text{on M.}$$

• If $(M^n, g) \sim (S^n, g_0)$, all solutions are known, and unique modulo conformal diffeomorphism of (S^n, g_0).

• If (M^n, g) not (S^n, g_0), solution space more complicated.

• Question.

$$\{u \in C^\infty(M) \mid u \text{ solution}\} \text{ is bounded in } L^\infty(M)?$$

Much work has been done:

- Schoen (1991) Yes, if (M^n, g) is locally conformally flat.
- L. and Zhu (1999) Yes, if $n = 3$.
- Independent works by three groups:
- Druet (2004) Yes, if $n \leq 4$.
- Marques (2005) Yes, if $n \leq 7$.
- L. and Zhang (2005) Yes, if $n \leq 9$.

After the above:

- L. and Zhang (2006) Yes, if $n \leq 11$.
- Khuri-Marques-Schoen (2009) Yes, if $n \leq 24$.
- Brendle (2009) No, if $n \geq 52$.
- Brendle-Marques (2009) No, if $n \geq 25$.
- For $8 \leq n \leq 24$, "Yes" provided the Positive Mass Theorem.

A fully nonlinear Yamabe problem

• Schouten tensor:

$$A_g = (n-2)^{-1}(Ric_g - [2(n-1)]^{-1} R_g g),$$

Let

$$\lambda(A_g) = (\lambda_1, \cdots, \lambda_n) = \text{eigenvalues of } A_g.$$

Then

$$\lambda_1(A_g) + \cdots + \lambda_n(A_g) = R_g.$$

- Yamabe problem on (M, g) when $\lambda_1(-L_g) > 0$: Assume $\lambda_1(A_g) + \cdots + \lambda_n(A_g) > 0$, $\exists$? $\tilde{g} \sim g$ such that $\lambda_1(A_{\tilde{g}}) + \cdots + \lambda_n(A_{\tilde{g}}) = 1$.

- A more general question on (M, g): Assume $f(\lambda(A_g)) > 0$, $\exists$? $\tilde{g} \sim g$ such that $f(\lambda(A_{\tilde{g}})) = 1$.

- A second order fully nonlinear elliptic PDE:

$$\begin{aligned} A_{\tilde{g}} = \; & -\tfrac{2}{n-2} u^{-1} \nabla_g^2 u + \tfrac{2n}{(n-2)^2} \nabla_g u \otimes \nabla_g u \\ & - \tfrac{2}{(n-2)^2} u^{-2} |\nabla_g u|_g^2 g + A_g . \end{aligned}$$

- More precisely: Let

$$\Gamma \subset \mathbb{R}^n \text{ open,convex,symmetric cone,vertex at origin}$$

$$\Gamma_n \subset \Gamma \subset \Gamma_1$$

$$\Gamma_n := \{\lambda \in \mathbb{R}^n \mid \lambda_i > 0 \ \forall \ i\}, \quad \Gamma_1 := \{\lambda \in \mathbb{R}^n \mid \sum_{i=1}^{n} \lambda_i > 0\}$$

$$f \in C^1(\Gamma) \cap C^0(\overline{\Gamma}) \text{ symmetric function}$$

$$f_{\lambda_i} > 0 \text{ in } \Gamma \ \forall \ i, \quad f > 0 \text{ in } \Gamma, \quad f = 0 \text{ on } \Gamma$$

• Illuminating examples:

$(f, \Gamma) = (\sigma_k^{\frac{1}{k}}, \Gamma_k)$, $1 \leq k \leq n$

$$\sigma_k(\lambda) := \sum_{\lambda_{i_1} < \cdots < \lambda_{i_k}} \lambda_{i_1} \cdots \lambda_{i_k},$$

the k-th elementary symmetric function

Γ_k: the connected component of $\{\lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) > 0\}$ containing the positive cone $\Gamma_n = \{\lambda \in \mathbb{R}^n \mid \lambda_i > 0 \ \forall \ i\}$

- A fully nonlinear Yamabe problem Assume $\lambda(A_g) \in \Gamma$ on M, does there exist $\tilde{g} = u^{\frac{4}{n-2}} g$ such that

$$f(\lambda(A_{\tilde{g}})) = 1, \quad \lambda(A_{\tilde{g}}) \in \Gamma, \quad \text{on } M?$$

- If $(f, \Gamma) = (\sigma_1, \Gamma_1)$, the Yamabe problem.

- Answer is “**Yes**” if:

(i) (f, Γ), f concave, homogeneous of degree 1, (M, g) is locally conformally flat,
(ii) $(f, \Gamma) = (\sigma_k^{\frac{1}{k}}, \Gamma_k)$, and $k \geq \frac{n}{2}$,
(iii) $(f, \Gamma) = (\sigma_k^{\frac{1}{k}}, \Gamma_k)$, $k = 2$.

- Through work of many: [Alice Chang, Gursky, Paul Yang, 2002], [Pengfei Guan, Guofang Wang, 2003], [Aobing Li, L., 2003, 2005], [Gursky, Viaclovsky, 2004, 2007], [Yuxin Ge, Guofang Wang, 2006], [Weimin Sheng, Trudinger, Xujia Wang, 2007], [Luc Nguyen, L., 2014].

• "**Open**" in particular if: $(f,\Gamma) = (\sigma_k^{\frac{1}{k}}, \Gamma_k)$, $3 \leq k < \frac{n}{2}$.

• Answer would be "Yes" if can prove a priori estimates: For $\tilde{g} = u^{\frac{4}{n-2}} g$, $u > 0$,

$$f(\lambda(A_{\tilde{g}})) = 1 \text{ on M} \quad \text{implies} \quad u \leq C \text{ on M}.$$

• Taking $M = \mathbb{R}^n$, or rescaling a blow up sequence of solutions of the geometric equation leads to

$$f(\lambda(A^u)) = 1, \quad \text{in} \;\; \mathbb{R}^n,$$

where

$$A^u := -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \nabla^2 u + \frac{2n}{(n-2)^2} u^{-\frac{2n}{n-2}} \nabla u \otimes \nabla u$$

$$- \frac{2}{(n-2)^2} u^{-\frac{2n}{n-2}} |\nabla u|^2 I,$$

- Caffarelli, Nirenberg, Spruck, 1985:

Introduce (f, Γ) of such type, pioneering work on existence of smooth solutions for Dirichlet problem:

$$\begin{cases} f(\lambda(\nabla^2 u)) & = g(x), \quad \text{in } \Omega \subset \mathbb{R}^n, \\ u & = h(x) \quad \text{on} \partial\Omega. \end{cases}$$

- Equation $f(\lambda(A^u)) = 1$ resembles the above.
- Additional feature: conformal invariance of equation.

In this series of lectures, we

- Study estimates and raise open problems

A Liouville type Theorem

- **Theorem 1.** (L., 2006; L., Luc Nguyen, Bo Wang, 2016) $0 < u \in C^0(\mathbb{R}^n \setminus \{0\})$, viscosity solution of $\lambda(A^u) \in \partial\Gamma$ in $\mathbb{R}^n \setminus \{0\}$. Then u is radially symmetric about $\{0\}$.
- **Corollary 1.** $0 < u \in C^0(\mathbb{R}^n)$, viscosity solution of $\lambda(A^u) \in \partial\Gamma$ in $\mathbb{R}^n$. Then $u \equiv u(0)$.

$$\Gamma \subset \mathbb{R}^n \text{ open,convex,symmetric cone,vertex at origin}$$

$$\Gamma_n \subset \Gamma \subset \Gamma_1$$

$$\Gamma_n := \{\lambda \in \mathbb{R}^n \mid \lambda_i > 0 \ \forall \ i\}, \quad \Gamma_1 := \{\lambda \in \mathbb{R}^n \mid \sum_{i=1}^n \lambda_i > 0\}$$

$$A^u := -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \nabla^2 u + \frac{2n}{(n-2)^2} u^{-\frac{2n}{n-2}} \nabla u \otimes \nabla u - \frac{2}{(n-2)^2} u^{-\frac{2n}{n-2}} |\nabla u|^2 I,$$

We will prove:

Proposition 1. Assume $\Omega \subset \mathbb{R}^n$ bounded open, $\{P_1, \cdots, P_m\} \subset \Omega$.

$$0 < u \in C^2(\overline{\Omega}), \quad \lambda(A^u) \in R^n \setminus \overline{\Gamma} \text{ in } \Omega,$$

$$0 < v \in C^2(\overline{\Omega} \setminus \{P_1, \cdots, P_m\}), \quad \lambda(A^v) \in \overline{\Gamma} \text{ in } \Omega \setminus \{P_1, \cdots, P_m\}.$$

$$v \geq u \text{ on } \partial\Omega.$$

Then

$$v \geq u \text{ in } \overline{\Omega} \setminus \{P_1, \cdots, P_m\}.$$

We first prove:

Proposition 1 with no singularity. Assume $\Omega \subset \mathbb{R}^n$ bounded open.

$$0 < u, v \in C^2(\overline{\Omega}), \quad \lambda(A^u) \in R^n \setminus \overline{\Gamma},\ \lambda(A^v) \in \overline{\Gamma} \ \text{ in } \Omega,$$

$$v \geq u \text{ on } \partial\Omega.$$

Then

$$v \geq u \text{ in } \overline{\Omega}.$$

So

$$\lambda(A_{w+\epsilon\varphi}) \in \Gamma.$$

Take $\epsilon_i = \delta_i \to 0$, and let

$$w + \epsilon_i \varphi_i = v_i^{-\frac{2}{n-2}}.$$

Then $\{v_i\}$ has the needed approximation property.

A calculation gives

$$A_{w+\epsilon\varphi} = A_w + \epsilon\left\{w\nabla^2\varphi + \varphi\nabla^2 w - \nabla w\cdot\nabla\varphi I\right\} + \epsilon^2 A_\varphi.$$

Replacing $\nabla^2 w$ by $w^{-1}\left(A_w + \frac{1}{2}|\nabla w|^2 I\right)$ in the above, we have

$$A_{w+\epsilon\varphi} = (1+\epsilon\frac{\varphi}{w})A_w + \epsilon\left\{w\nabla^2\varphi + \frac{|\nabla w|^2}{2w}\varphi I - \nabla w\cdot\nabla\varphi I\right\} + \epsilon^2 A_\varphi.$$

Since

$$\nabla\varphi(y) = 2\delta\varphi(y)y, \quad \nabla^2\varphi(y) = 2\delta\varphi(y)I + 4\delta^2\varphi(y)y\otimes y.$$

$$w\nabla^2\varphi + \frac{|\nabla w|^2}{2w}\varphi I - \nabla w\cdot\nabla\varphi I \geq \delta\varphi w I.$$