

Entropy method for hypocoercive & non-symmetric Fokker-Planck equations with linear drift

Anton ARNOLD

with Jan Erb; Franz Achleitner

Trieste, May 2017

degenerate Fokker-Planck equations with linear drift

evolution of probability density $f(x, t)$, $x \in \mathbb{R}^n$, $t > 0$:

$$\begin{aligned}f_t &= \operatorname{div}(\mathbf{D}\nabla f + \mathbf{C}x f) \\f(x, 0) &= f_0(x)\end{aligned}\tag{1}$$

$\mathbf{D} \in \mathbb{R}^{n \times n}$... symmetric, const in x , **degenerate**

$\mathbf{C} \in \mathbb{R}^{n \times n}$... const in x

degenerate Fokker-Planck equations with linear drift

evolution of probability density $f(x, t)$, $x \in \mathbb{R}^n$, $t > 0$:

$$\begin{aligned} f_t &= \operatorname{div}(\mathbf{D}\nabla f + \mathbf{C}x f) \\ f(x, 0) &= f_0(x) \end{aligned} \tag{1}$$

$\mathbf{D} \in \mathbb{R}^{n \times n}$... symmetric, const in x , **degenerate**

$\mathbf{C} \in \mathbb{R}^{n \times n}$... const in x

goals: existence & uniqueness of steady state $f_\infty(x)$;

convergence $f(t) \xrightarrow{t \rightarrow \infty} f_\infty$ with sharp rates;

complete theory for the equation class (1)

hypo coercive example – from plasma physics

kinetic Fokker-Planck equation for $f(x, v, t)$, $x, v \in \mathbb{R}^n$:

$$f_t + \underbrace{v \cdot \nabla_x f}_{\text{free transport}} - \underbrace{\nabla_x V \cdot \nabla_v f}_{\text{influence of potential } V(x)} = \underbrace{\sigma \Delta_v f}_{\text{diffusion, } \sigma > 0} + \underbrace{\nu \operatorname{div}_v(vf)}_{\text{friction, } \nu > 0}$$

steady state: $f_\infty(x, v) = c e^{-\frac{\nu}{\sigma} \left[\frac{|v|^2}{2} + V(x) \right]}$

$V(x)$... given *confinement potential*

hypo coercive example – from plasma physics

kinetic Fokker-Planck equation for $f(x, v, t)$, $x, v \in \mathbb{R}^n$:

$$f_t + \underbrace{v \cdot \nabla_x f}_{\text{free transport}} - \underbrace{\nabla_x V \cdot \nabla_v f}_{\text{influence of potential } V(x)} = \underbrace{\sigma \Delta_v f}_{\text{diffusion, } \sigma > 0} + \underbrace{\nu \operatorname{div}_v(vf)}_{\text{friction, } \nu > 0}$$

steady state: $f_\infty(x, v) = c e^{-\frac{\nu}{\sigma} \left[\frac{|v|^2}{2} + V(x) \right]}$

$V(x)$... given *confinement potential*

rewritten (with x, v variables):

$$f_t = \operatorname{div}_{x,v} \left[\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & \sigma \mathbf{I} \end{pmatrix}}_{=: \mathbf{D} \dots \text{diffusion}} \nabla_{x,v} f + \underbrace{\begin{pmatrix} -v \\ \nabla_x V + \nu v \end{pmatrix}}_{\text{drift}} f \right]$$

- \nexists explicit Green's function for V not quadratic!

Outline:

- 1 hypocoercivity, prototypic examples
- 2 review of standard entropy method for non-degenerate Fokker-Planck equations
- 3 decay of modified “entropy dissipation” functional
- 4 mechanism of new method

(hypo)coercivity 1

example 1: standard Fokker-Planck equation on \mathbb{R}^n :

$$f_t = \operatorname{div}(\nabla f + x f) =: Lf \quad \dots \text{ symmetric on } H := L^2(f_\infty^{-1})$$

$$f_\infty(x) = c e^{-\frac{|x|^2}{2}}, \quad \ker L = \operatorname{span}(f_\infty)$$

- L is dissipative, i.e. $\langle Lf, f \rangle_H \leq 0 \quad \forall f \in \mathcal{D}(L)$
- $-L$ is **coercive** (has a spectral gap), in the sense:

$$\langle -Lf, f \rangle_H \geq \|f\|_{L^2(f_\infty^{-1})}^2 \quad \forall f \in \{f_\infty\}^\perp$$

(hypo)coercivity 2

example 2:

$$f_t = \operatorname{div}(\mathbf{D}\nabla f + \mathbf{C}_x f) =: Lf \quad (2)$$

with degenerate \mathbf{D} is degenerate parabolic;
(symmetric part of) $-L$ is **not coercive**.

(hypo)coercivity 2

example 2:

$$f_t = \operatorname{div}(\mathbf{D}\nabla f + \mathbf{C}_x f) =: Lf \quad (2)$$

with degenerate \mathbf{D} is degenerate parabolic;
(symmetric part of) $-L$ is **not coercive**.

Definition 1 (Villani 2009)

Consider L on Hilbert space H with $\mathcal{K} = \ker L$; let $\tilde{H} \hookrightarrow \mathcal{K}^\perp$ (densely)
(e.g. $H \dots$ weighted L^2 , $\tilde{H} \dots$ weighted H^1).

$-L$ is called **hypocoercive** on \tilde{H} if $\exists \lambda > 0, c \geq 1$:

$$\|e^{Lt}f\|_{\tilde{H}} \leq c e^{-\lambda t} \|f\|_{\tilde{H}} \quad \forall f \in \tilde{H}$$

- typically $c > 1$

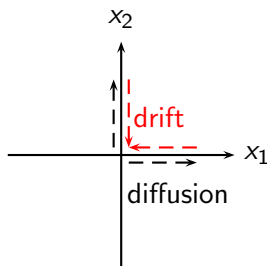
Steady state of (non)degenerate FP equations:

standard Fokker-Planck equation $f_t = \text{div}(\nabla f + x f)$:

unique steady state $f_\infty(x) = c e^{-|x|^2/2}$ as a balance of drift & diffusion;

sharp decay rate = 1

$n = 2$:



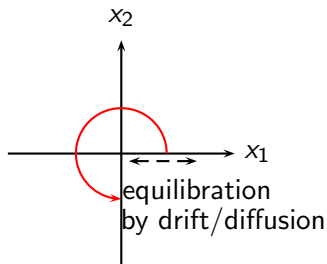
degenerate prototype:

- degenerate diffusion (1D Fokker-Planck) + **rotation**

$$f_t = \operatorname{div} \left[\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{=D} \nabla f + \underbrace{\begin{pmatrix} x_1 - \omega x_2 \\ \omega x_1 \end{pmatrix}}_{=C x} f \right]$$

$$f_\infty(x) = c e^{-|x|^2/2} \quad \forall \omega \in \mathbb{R} \text{ (unique for } \omega \neq 0\text{);}$$

- sharp decay rate = $\frac{1}{2}$ ($= \min \Re \lambda_C$) for fast enough rotation ($|\omega| > \frac{1}{2}$)



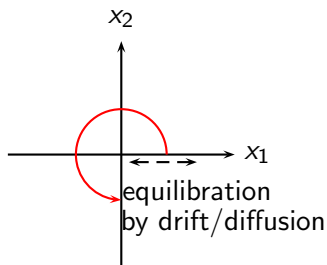
degenerate prototype:

- degenerate diffusion (1D Fokker-Planck) + **rotation**

$$f_t = \operatorname{div} \left[\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{=D} \nabla f + \underbrace{\begin{pmatrix} x_1 - \omega x_2 \\ \omega x_1 \end{pmatrix}}_{=C x} f \right]$$

$$f_\infty(x) = c e^{-|x|^2/2} \quad \forall \omega \in \mathbb{R} \text{ (unique for } \omega \neq 0\text{);}$$

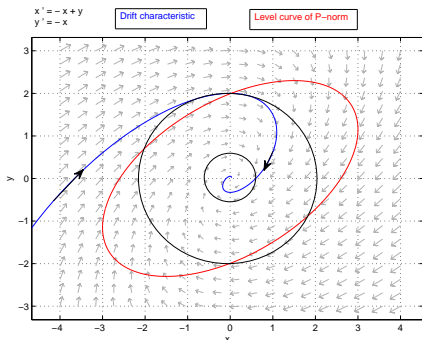
- sharp decay rate = $\frac{1}{2}$ ($= \min \Re \lambda_C$) for fast enough rotation ($|\omega| > \frac{1}{2}$)



- $|\omega| = \frac{1}{2}$: \mathbf{C} has a Jordan block \Rightarrow (sharp) decay rate = $\frac{1}{2} - \varepsilon$

degenerate prototype with $\omega = 1$:

- $f_t = \operatorname{div} \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \nabla f + \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} x f \right]$
- x_2 -axis: drift characteristics of $\dot{x} = -\mathbf{C}x$ tangent to level curve of $|x|$:



- level curve of “distorted” vector norm $\sqrt{x \cdot \mathbf{P} \cdot x}$; $\mathbf{P} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

Ref: [Dolbeault-Mouhot-Schmeiser] 2015

coefficients \mathbf{C} , \mathbf{D} in Fokker-Planck equation

$$f_t = \operatorname{div}(\mathbf{D} \nabla f + \mathbf{C} x f) =: Lf$$

Condition A: No (nontrivial) subspace of $\ker \mathbf{D}$ is invariant under \mathbf{C}^\top .
(equivalent: L is hypoelliptic.)

Proposition 1

Let Condition A hold.

- a) Let $f_0 \in L^1(\mathbb{R}^d) \Rightarrow f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$. [Hörmander 1969]
- b) Let $f_0 \in L^1_+(\mathbb{R}^d) \Rightarrow f(x, t) > 0, \forall t > 0$. (Green's fct > 0)

coefficients \mathbf{C} , \mathbf{D} in Fokker-Planck equation

$$f_t = \operatorname{div}(\mathbf{D} \nabla f + \mathbf{C} x f) =: Lf$$

Condition A: No (nontrivial) subspace of $\ker \mathbf{D}$ is invariant under \mathbf{C}^\top .
(equivalent: L is hypoelliptic.)

Proposition 1

Let Condition A hold.

- a) Let $f_0 \in L^1(\mathbb{R}^d) \Rightarrow f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$. [Hörmander 1969]
- b) Let $f_0 \in L^1_+(\mathbb{R}^d) \Rightarrow f(x, t) > 0, \forall t > 0$. (Green's fct > 0)

Condition B: Condition A + let \mathbf{C} be positively stable (i.e. $\Re \lambda_{\mathbf{C}} > 0$)
 $\rightarrow \exists$ confinement potential; drift towards $x = 0$.

- hypoelliptic + confinement = hypocoercive (for FP eq.)

steady state

$$f_t = \operatorname{div}(\mathbf{D}\nabla f + \mathbf{C}_x f) \quad (3)$$

Theorem 2

(3) has a unique (normalized) steady state $f_\infty \in L^1(\mathbb{R}^n)$ iff Condition B holds.

Then: $f_\infty(x) = c_K e^{-\frac{x^\top \mathbf{K}^{-1} x}{2}}$... non-isotropic Gaussian
 $0 < \mathbf{K} \in \mathbb{R}^{n \times n}$... unique solution of $2\mathbf{D} = \mathbf{C}\mathbf{K} + \mathbf{K}\mathbf{C}^\top$
(continuous Lyapunov equation)

normalization of Fokker-Planck equations

$$f_t = \operatorname{div}(\mathbf{D}\nabla f + \mathbf{C}_x f) \quad \text{with } f_\infty(x) = c_K e^{-\frac{x^\top \mathbf{K}^{-1} x}{2}}$$

transformations:

$$\textcircled{1} \quad y := \sqrt{\mathbf{K}}^{-1} x \Rightarrow$$

$$g_t = \operatorname{div}_y(\tilde{\mathbf{D}}\nabla_y g + \tilde{\mathbf{C}}_y g) \quad \text{with } g_\infty(x) = c e^{-\frac{|y|^2}{2}},$$
$$\tilde{\mathbf{D}} = \tilde{\mathbf{C}}_S$$

normalization of Fokker-Planck equations

$$f_t = \operatorname{div}(\mathbf{D}\nabla f + \mathbf{C}_x f) \quad \text{with } f_\infty(x) = c_K e^{-\frac{x^\top \mathbf{K}^{-1} x}{2}}$$

transformations:

① $y := \sqrt{\mathbf{K}}^{-1} x \Rightarrow$

$$g_t = \operatorname{div}_y(\tilde{\mathbf{D}}\nabla_y g + \tilde{\mathbf{C}}_y g) \quad \text{with } g_\infty(x) = c e^{-\frac{|y|^2}{2}},$$
$$\tilde{\mathbf{D}} = \tilde{\mathbf{C}}_S$$

② rotation of $y \Rightarrow \check{\mathbf{D}} = \operatorname{diag}(d_1, \dots, d_k, \underbrace{0, \dots, 0}_{n-k})$

[normalization from now on assumed]

review of entropy method:

linear symmetric Fokker-Planck equations

evolution of probability density $f(x, t)$, $x \in \mathbb{R}^n$, $t > 0$:

$$f_t = \operatorname{div}(\mathbf{D} \cdot [\nabla f + f \nabla A(x)]) =: Lf$$

$$f(x, 0) = f_0(x); \quad f_0 \in L^1_+(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} f_0 dx = 1 \quad \Rightarrow \quad f(x, t) \geq 0$$

review of entropy method:

linear symmetric Fokker-Planck equations

evolution of probability density $f(x, t)$, $x \in \mathbb{R}^n$, $t > 0$:

$$f_t = \operatorname{div}(\mathbf{D} \cdot [\nabla f + f \nabla A(x)]) =: Lf$$

$$f(x, 0) = f_0(x); \quad f_0 \in L^1_+(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} f_0 dx = 1 \quad \Rightarrow \quad f(x, t) \geq 0$$

$f_\infty(x) = e^{-A(x)} \dots$ (unique) normalized steady state

$$Lf = \operatorname{div}\left(f_\infty \mathbf{D} \nabla \frac{f}{f_\infty}\right) \dots \text{symmetric in } L^2(\mathbb{R}^n, f_\infty^{-1})$$

$\mathbf{D} > 0$... positive definite matrix

$A(x)$... scalar confinement potential, i.e. $A(x) \rightarrow \infty$ as $|x| \rightarrow \infty$;

$$\text{idea : } A(x) \gtrsim c|x|^2$$

admissible relative entropies (for entropy method)

for probability densities $f_{1,2}$:

$$e_\psi(f_1|f_2) := \int_{\mathbb{R}^n} \psi\left(\frac{f_1}{f_2}\right) f_2 \, dx \geq 0 \quad \dots \text{relative entropy}$$

$\psi : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$... entropy generators

$$\psi \geq 0, \quad \psi(1) = 0, \quad \psi'' > 0, \quad (\psi''')^2 \leq \frac{1}{2}\psi''\psi^{IV}$$

admissible relative entropies (for entropy method)

for probability densities $f_{1,2}$:

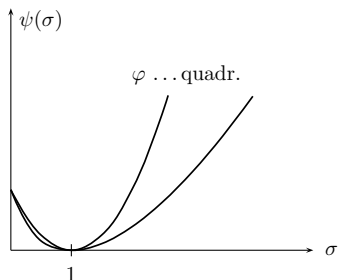
$$e_\psi(f_1|f_2) := \int_{\mathbb{R}^n} \psi\left(\frac{f_1}{f_2}\right) f_2 \, dx \geq 0 \quad \dots \text{relative entropy}$$

$\psi : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$... entropy generators

$$\psi \geq 0, \quad \psi(1) = 0, \quad \psi'' > 0, \quad (\psi''')^2 \leq \frac{1}{2}\psi''\psi^{IV}$$

examples: 1) $\psi_1(\sigma) = \sigma \ln \sigma - \sigma + 1$

2) $\psi_p(\sigma) = \sigma^p - 1 - p(\sigma - 1), \quad 1 < p \leq 2$



$$e_\psi(f_1|f_2) = 0 \iff f_1 = f_2$$

Lemma 1

Let $f(t)$ solve Fokker-Planck equation $f_t = \operatorname{div}(\mathbf{D} \cdot [\nabla f + f \nabla A(x)])$

$$\begin{aligned} \Rightarrow \frac{d}{dt} e_{\psi}(f(t)|f_{\infty}) &= - \int_{\mathbb{R}^n} \psi'' \left(\frac{f(t)}{f_{\infty}} \right) \nabla^{\top} \frac{f(t)}{f_{\infty}} \cdot \mathbf{D} \cdot \nabla \frac{f(t)}{f_{\infty}} f_{\infty} \, dx \\ &=: -I_{\psi}(f(t)|f_{\infty}) \leq 0 \dots \text{(negative) Fisher information} \end{aligned}$$

Step 1: exponent. decay of entropy dissipation for $\mathbf{D} \equiv \text{const}$

$$\mathbf{D} = \text{const. in } x, \quad f_\infty(x) = e^{-A(x)}$$

Theorem 3

Let $I_\psi(f_0|f_\infty) < \infty$. Let \mathbf{D}, A satisfy a

$$\boxed{\text{Bakry - Emery condition} \quad \frac{\partial^2 A(x)}{\partial x^2} \geq \underbrace{\lambda_1}_{>0} \mathbf{D}^{-1}} \quad \forall x \in \mathbb{R}^n \quad (4)$$

$$\Rightarrow I_\psi(f(t)|f_\infty) \leq e^{-2\lambda_1 t} I_\psi(f_0|f_\infty), \quad t \geq 0$$

A ... uniformly convex if $\mathbf{D} = \mathbf{I}$

Ref's: [Bakry-Emery] 1984/85;

[Arnold-Markowich-Toscani-Unterreiter] Comm. PDE 2001

- robust w.r.t. many nonlinear perturbations

Step 2: exponential decay of relative entropy for $\mathbf{D} \equiv \text{const}$

Theorem 4

Let \mathbf{D} , A satisfy BEC $\frac{\partial^2 A(x)}{\partial x^2} \geq \lambda_1 \mathbf{D}^{-1} \Rightarrow$

$$e_\psi(f(t)|f_\infty) \leq e^{-2\lambda_1 t} e_\psi(f_0|f_\infty), \quad t \geq 0$$

Step 2: exponential decay of relative entropy for $\mathbf{D} \equiv \text{const}$

Theorem 4

Let \mathbf{D} , A satisfy BEC $\frac{\partial^2 A(x)}{\partial x^2} \geq \lambda_1 \mathbf{D}^{-1} \Rightarrow$

$$e_\psi(f(t)|f_\infty) \leq e^{-2\lambda_1 t} e_\psi(f_0|f_\infty), \quad t \geq 0$$

Proof: from proof of Theorem 3 :

$$\frac{d}{dt} I(t) \leq -2\lambda_1 \underbrace{I(t)}_{=-e'(t)} \quad \Big| \int_t^\infty \dots dt$$

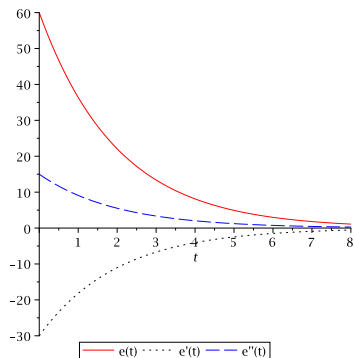
Since $I(t), e(t) \xrightarrow{t \rightarrow \infty} 0$:

$$\frac{d}{dt} e(t) \leq -2\lambda_1 e(t) \tag{5}$$

(+ density argument)

problem of entropy decay (cp. standard entropy method)

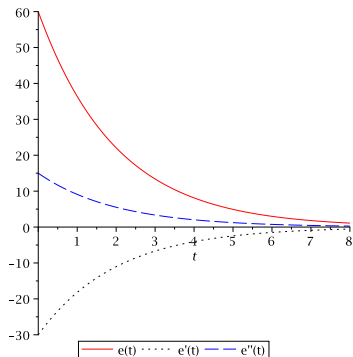
decay of quadratic entropy $e_2(t) = \|f(t) - f_\infty\|_{L^2(f_\infty^{-1})}^2$:



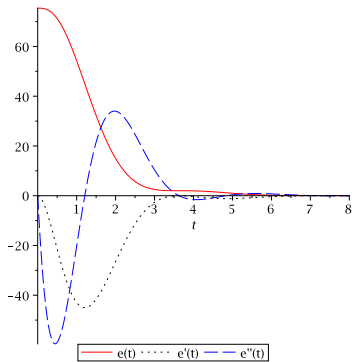
standard Fokker-Planck equation:
non-degenerate $\rightarrow e(t)$ is convex;
entropy dissip. $e'(t) < 0 \forall f \neq f_\infty$;
 $e' \leq -\mu e$ possible (with $\mu > 0$)

problem of entropy decay (cp. standard entropy method)

decay of quadratic entropy $e_2(t) = \|f(t) - f_\infty\|_{L^2(f_\infty^{-1})}^2$:



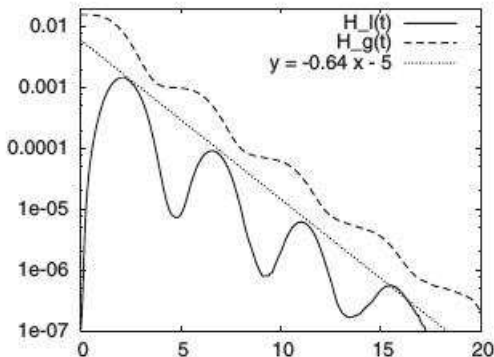
standard Fokker-Planck equation:
non-degenerate $\rightarrow e(t)$ is convex;
entropy dissip. $e'(t) < 0 \forall f \neq f_\infty$;
 $e' \leq -\mu e$ possible (with $\mu > 0$)



degenerate prototype ex.:
 $\rightarrow e(t)$ is not convex;
 $e'(t) = 0$ for some $f \neq f_\infty$;
 $e' \leq -\mu e$ wrong (in general)

entropy decay in inhomogeneous Boltzmann equation

- simulation of 1+2 D Boltzmann equation: wavy entropy decay
- — — — $H_g(t)$: relative entropy w.r.t. the global Maxwellian



Ref: [Filbet-Mouhot-Pareschi] 2006

new entropy method for degen. FP: $f_t = \operatorname{div}(\mathbf{D}\nabla f + \mathbf{C}_x f)$

- $e'(t) = 0$ for some $f \neq f_\infty \Rightarrow$ entropy dissipation:

$$\frac{d}{dt} e_\psi = - \int_{\mathbb{R}^n} \psi'' \left(\frac{f}{f_\infty} \right) \nabla^\top \frac{f}{f_\infty} \cdot \underbrace{\mathbf{D}}_{\geq 0} \cdot \nabla \frac{f}{f_\infty} f_\infty \, dx =: -I_\psi(f) \leq 0$$

is “useless” as Lyapunov functional.

new entropy method for degen. FP: $f_t = \operatorname{div}(\mathbf{D}\nabla f + \mathbf{C}_x f)$

- $e'(t) = 0$ for some $f \neq f_\infty \Rightarrow$ entropy dissipation:

$$\frac{d}{dt} e_\psi = - \int_{\mathbb{R}^n} \psi'' \left(\frac{f}{f_\infty} \right) \nabla^\top \frac{f}{f_\infty} \cdot \underbrace{\mathbf{D}}_{\geq 0} \cdot \nabla \frac{f}{f_\infty} f_\infty \, dx =: -I_\psi(f) \leq 0$$

is “useless” as Lyapunov functional.

\Rightarrow define **modified “entropy dissipation”** as auxiliary functional:

$$S_\psi(f) := \int_{\mathbb{R}^n} \psi'' \left(\frac{f}{f_\infty} \right) \nabla^\top \frac{f}{f_\infty} \cdot \underbrace{\mathbf{P}}_{> 0} \cdot \nabla \frac{f}{f_\infty} f_\infty \, dx \geq 0$$

goal: estimate between $S(f(t))$, $\frac{d}{dt} S(f(t))$ for “good” choice of $\mathbf{P} > 0$.

Then:

$$\mathbf{P} \geq c_P \mathbf{D} \quad \Rightarrow \quad S_\psi(f) \geq c_P I_\psi(f) \searrow 0$$

modified “entropy dissipation” $S_\psi(f)$: choice of \mathbf{P}

Lemma 2

Let \mathbf{Q} be positively stable, i.e. $\mu := \min\{\Re \lambda_{\mathbf{Q}}\} > 0$.

- 1 If all $\lambda_{\mathbf{Q}}^{\min} \in \{\lambda \in \sigma(\mathbf{Q}) \mid \Re \lambda = \mu\}$ are *non-defective* (i.e. geometric = algebraic multiplicity)

$$\Rightarrow \exists \mathbf{P} \in \mathbb{R}^{n \times n}, \mathbf{P} > 0 : \mathbf{P}\mathbf{Q} + \mathbf{Q}^\top \mathbf{P} \geq 2\mu \mathbf{P}.$$

modified “entropy dissipation” $S_\psi(f)$: choice of \mathbf{P}

Lemma 2

Let \mathbf{Q} be positively stable, i.e. $\mu := \min\{\Re \lambda_{\mathbf{Q}}\} > 0$.

- ① If all $\lambda_{\mathbf{Q}}^{\min} \in \{\lambda \in \sigma(\mathbf{Q}) \mid \Re \lambda = \mu\}$ are *non-defective* (i.e. geometric = algebraic multiplicity)

$$\Rightarrow \exists \mathbf{P} \in \mathbb{R}^{n \times n}, \mathbf{P} > 0 : \mathbf{P}\mathbf{Q} + \mathbf{Q}^\top \mathbf{P} \geq 2\mu \mathbf{P}.$$

- ② If (at least) one $\lambda_{\mathbf{Q}}^{\min}$ is *defective* \Rightarrow

$$\forall \varepsilon > 0 \quad \exists \mathbf{P} = \mathbf{P}(\varepsilon) > 0 : \mathbf{P}\mathbf{Q} + \mathbf{Q}^\top \mathbf{P} \geq 2(\mu - \varepsilon) \mathbf{P}.$$

modified “entropy dissipation” $S_\psi(f)$: choice of \mathbf{P}

Lemma 2

Let \mathbf{Q} be positively stable, i.e. $\mu := \min\{\Re \lambda_{\mathbf{Q}}\} > 0$.

- ① If all $\lambda_{\mathbf{Q}}^{\min} \in \{\lambda \in \sigma(\mathbf{Q}) \mid \Re \lambda = \mu\}$ are *non-defective* (i.e. geometric = algebraic multiplicity)

$$\Rightarrow \exists \mathbf{P} \in \mathbb{R}^{n \times n}, \mathbf{P} > 0 : \mathbf{P}\mathbf{Q} + \mathbf{Q}^\top \mathbf{P} \geq 2\mu \mathbf{P}.$$

- ② If (at least) one $\lambda_{\mathbf{Q}}^{\min}$ is *defective* \Rightarrow

$$\forall \varepsilon > 0 \quad \exists \mathbf{P} = \mathbf{P}(\varepsilon) > 0 : \mathbf{P}\mathbf{Q} + \mathbf{Q}^\top \mathbf{P} \geq 2(\mu - \varepsilon) \mathbf{P}.$$

Proof: \mathbf{P} can be constructed explicitly; e.g. for \mathbf{Q} non-defective / diagonalizable:

$$\mathbf{P} := \sum_{j=1}^n z_j \otimes \bar{z}_j^\top ; \quad z_j \dots \text{eigenvectors of } \mathbf{Q}^\top$$

- \mathbf{P} not unique; but the decay rates μ (or $\mu - \varepsilon$) are independent of \mathbf{P} . □
- application with $\mathbf{Q} := \mathbf{C}$.

Step 1: exponential decay of auxiliary functional $S_\psi(f)$

$$S_\psi(f) := \int_{\mathbb{R}^n} \psi'' \left(\frac{f}{f_\infty} \right) \nabla^\top \frac{f}{f_\infty} \cdot \mathbf{P} \cdot \nabla \frac{f}{f_\infty} f_\infty \, dx \geq 0$$

Proposition 2

$\mu := \min\{\Re \lambda_C\}$. Let f_0 satisfy:

$$\int \psi'' \left(\frac{f_0}{f_\infty} \right) \left| \nabla \frac{f_0}{f_\infty} \right|^2 f_\infty \, dx < \infty \quad (\sim \text{weighted } H^1\text{-seminorm})$$

① If all λ_C^{\min} are non-defective $\Rightarrow S(f(t)) \leq e^{-2\mu t} S(f_0)$, $t \geq 0$;

Step 1: exponential decay of auxiliary functional $S_\psi(f)$

$$S_\psi(f) := \int_{\mathbb{R}^n} \psi'' \left(\frac{f}{f_\infty} \right) \nabla^\top \frac{f}{f_\infty} \cdot \mathbf{P} \cdot \nabla \frac{f}{f_\infty} f_\infty \, dx \geq 0$$

Proposition 2

$\mu := \min\{\Re \lambda_C\}$. Let f_0 satisfy:

$$\int \psi'' \left(\frac{f_0}{f_\infty} \right) \left| \nabla \frac{f_0}{f_\infty} \right|^2 f_\infty \, dx < \infty \quad (\sim \text{weighted } H^1\text{-seminorm})$$

- 1 If all λ_C^{\min} are non-defective $\Rightarrow S(f(t)) \leq e^{-2\mu t} S(f_0)$, $t \geq 0$;
- 2 If one λ_C^{\min} is defective $\Rightarrow S(f(t), \varepsilon) \leq e^{-2(\mu-\varepsilon)t} S(f_0, \varepsilon)$, $t \geq 0$.

Proof of Proposition 2 – modified entropy method

$$\frac{d}{dt} S(f(t)) = - \int \psi''\left(\frac{f}{f_\infty}\right) u^\top \underbrace{[\mathbf{P}\mathbf{C} + \mathbf{C}^\top \mathbf{P}]}_{\geq 2\mu\mathbf{P} \dots \text{replaces BEC}} u f_\infty dx$$

$$-2 \int \underbrace{\text{Tr}(\mathbf{X}\mathbf{Y})}_{\geq 0} f_\infty dx \leq -2\mu S(f(t)); \quad u = \nabla \frac{f}{f_\infty}$$

Proof of Proposition 2 – modified entropy method

$$\frac{d}{dt} S(f(t)) = - \int \psi''\left(\frac{f}{f_\infty}\right) u^\top \underbrace{[\mathbf{P}\mathbf{C} + \mathbf{C}^\top \mathbf{P}]}_{\geq 2\mu\mathbf{P} \dots \text{replaces BEC}} u f_\infty dx$$

$\geq 2\mu\mathbf{P} \dots$ replaces BEC

$$-2 \int \underbrace{\text{Tr}(\mathbf{X}\mathbf{Y})}_{\geq 0} f_\infty dx \leq -2\mu S(f(t)); \quad u = \nabla \frac{f}{f_\infty}$$

use $\mathbf{X} := \begin{pmatrix} \psi'' & \psi''' \\ \psi''' & \frac{1}{2}\psi^{IV} \end{pmatrix} \begin{pmatrix} f \\ f_\infty \end{pmatrix} \geq 0$,

since $\det \mathbf{X} = \frac{1}{2}\psi''\psi^{IV} - (\psi''')^2 \geq 0$ (for admissible rel. entropies) ;

Proof of Proposition 2 – modified entropy method

$$\frac{d}{dt} S(f(t)) = - \int \psi''\left(\frac{f}{f_\infty}\right) u^\top \underbrace{[\mathbf{P}\mathbf{C} + \mathbf{C}^\top \mathbf{P}]}_{\geq 2\mu\mathbf{P} \dots \text{replaces BEC}} u f_\infty dx$$

$\geq 2\mu\mathbf{P} \dots$ replaces BEC

$$-2 \int \underbrace{\text{Tr}(\mathbf{X}\mathbf{Y})}_{\geq 0} f_\infty dx \leq -2\mu S(f(t)); \quad u = \nabla \frac{f}{f_\infty}$$

use $\mathbf{X} := \begin{pmatrix} \psi'' & \psi''' \\ \psi''' & \frac{1}{2}\psi^{IV} \end{pmatrix} \begin{pmatrix} f \\ f_\infty \end{pmatrix} \geq 0$,

since $\det \mathbf{X} = \frac{1}{2}\psi''\psi^{IV} - (\psi''')^2 \geq 0$ (for admissible rel. entropies) ;

$$\mathbf{Y} := \begin{pmatrix} \text{Tr}(\mathbf{D} \frac{\partial u}{\partial x} \mathbf{P} \frac{\partial u}{\partial x}) & u^\top \mathbf{D} \frac{\partial u}{\partial x} \mathbf{P} u \\ u^\top \mathbf{D} \frac{\partial u}{\partial x} \mathbf{P} u & (u^\top \mathbf{P} u) (u^\top \mathbf{D} u) \end{pmatrix} \geq 0,$$

with Cauchy-Schwarz

Step 2: exponential decay of relative entropy

Theorem 5

Let f_0 satisfy:

$$\int \psi'' \left(\frac{f_0}{f_\infty} \right) |u_0|^2 f_\infty dx < \infty .$$

$$\Rightarrow \boxed{e(f(t)|f_\infty) \leq c S(f(t)) \leq c e^{-2\mu t} S(f_0), \quad t \geq 0}$$

(reduced rate for a defective λ_C^{\min} : $2(\mu - \varepsilon)$)

Step 2: exponential decay of relative entropy

Theorem 5

Let f_0 satisfy:

$$\int \psi'' \left(\frac{f_0}{f_\infty} \right) |u_0|^2 f_\infty dx < \infty .$$

$$\Rightarrow \boxed{e(f(t)|f_\infty) \leq c S(f(t)) \leq c e^{-2\mu t} S(f_0), \quad t \geq 0}$$

(reduced rate for a defective λ_C^{\min} : $2(\mu - \varepsilon)$)

Proof: Consider non-degenerate (auxiliary) symmetric FP equation:

$$g_t = \operatorname{div} \left(\underbrace{\mathbf{P}}_{>0} \left(\nabla \frac{g}{f_\infty} \right) f_\infty \right); \quad g_\infty = f_\infty = c e^{-|x|^2/2} = c e^{-A(x)} \quad (6)$$

It satisfies the Bakry-Emery condition $\frac{\partial^2 A}{\partial x^2} = \mathbf{I} \geq \lambda_P \mathbf{P}^{-1}$.

$$\Rightarrow \text{convex Sobolev inequality: } e_\psi(g|f_\infty) \leq \frac{1}{2\lambda_P} S_\psi(g) \quad \forall g$$

Remark: $S_\psi(g)$ is the true entropy dissipation for (6) !

(parabolic) regularization of semigroup e^{Lt}

Proposition 3

The *Hörmander order* $m \in [1, n - k]$ ($k = \text{rank } \mathbf{D}$) is the minimum such that

$$\sum_{j=0}^m \mathbf{C}^j \mathbf{D} (\mathbf{C}^\top)^j \geq \kappa \mathbf{I} \quad \text{for some } \kappa > 0.$$

(Existence of m is equivalent to Condition A, i.e. hypoellipticity of L .)

$$\Rightarrow S_\psi(f(t)) \leq c t^{-(2m+1)} e_\psi(f_0 | f_\infty), \quad 0 < t \leq 1.$$

(parabolic) regularization of semigroup e^{Lt}

Proposition 3

The *Hörmander order* $m \in [1, n - k]$ ($k = \text{rank } \mathbf{D}$) is the minimum such that

$$\sum_{j=0}^m \mathbf{C}^j \mathbf{D} (\mathbf{C}^\top)^j \geq \kappa \mathbf{I} \quad \text{for some } \kappa > 0.$$

(Existence of m is equivalent to Condition A, i.e. hypoellipticity of L .)

$$\Rightarrow S_\psi(f(t)) \leq c t^{-(2m+1)} e_\psi(f_0 | f_\infty), \quad 0 < t \leq 1.$$

Ref's:

Prop. 3 is generalization to all admissible relative entropies of:

[Hérau] JFA 2007;

[Villani] book 2009 (only for quadratic & logarithmic entropies)

exp. decay of rel. entropy for $f_t = \operatorname{div}(\mathbf{D}\nabla f + \mathbf{C}_x f) =: Lf$

combination of regularization for initial time with Th.5 (entropy decay) \Rightarrow

Theorem 6 (Arnold-Erb 2014)

Let L satisfy Condition B; $\mu := \min\{\Re \lambda_C\}$. $\Rightarrow \exists c > 0$:

$$e_\psi(f(t)|f_\infty) \leq c e^{-2\mu t} e_\psi(f_0|f_\infty), \quad t \geq 0$$

(reduced rate for a defective λ_C^{\min} : $2(\mu - \varepsilon)$)

exp. decay of rel. entropy for $f_t = \operatorname{div}(\mathbf{D}\nabla f + \mathbf{C}_x f) =: Lf$

combination of regularization for initial time with Th.5 (entropy decay) \Rightarrow

Theorem 6 (Arnold-Erb 2014)

Let L satisfy Condition B; $\mu := \min\{\Re \lambda_C\}$. $\Rightarrow \exists c > 0$:

$$e_\psi(f(t)|f_\infty) \leq c e^{-2\mu t} e_\psi(f_0|f_\infty), \quad t \geq 0$$

(reduced rate for a defective λ_C^{\min} : $2(\mu - \varepsilon)$)

Proof:

$$e(t) \stackrel{\text{CSI}}{\leq} \frac{1}{2\lambda_P} S(f(t)) \stackrel{\text{decay}}{\leq} \frac{1}{2\lambda_P} e^{-2\mu(t-\delta)} S(f(\delta)) \stackrel{\text{regularization}}{\leq} c(\delta) e^{-2\mu t} e(0)$$

□

exp. decay of rel. entropy for $f_t = \operatorname{div}(\mathbf{D}\nabla f + \mathbf{C}_x f) =: Lf$

combination of regularization for initial time with Th.5 (entropy decay) \Rightarrow

Theorem 6 (Arnold-Erb 2014)

Let L satisfy Condition B; $\mu := \min\{\Re \lambda_C\}$. $\Rightarrow \exists c > 0$:

$$e_\psi(f(t)|f_\infty) \leq c e^{-2\mu t} e_\psi(f_0|f_\infty), \quad t \geq 0$$

(reduced rate for a defective λ_C^{\min} : $2(\mu - \varepsilon)$)

Proof:

$$e(t) \stackrel{\text{CSI}}{\leq} \frac{1}{2\lambda_P} S(f(t)) \stackrel{\text{decay}}{\leq} \frac{1}{2\lambda_P} e^{-2\mu(t-\delta)} S(f(\delta)) \stackrel{\text{regularization}}{\leq} c(\delta) e^{-2\mu t} e(0)$$

□

Remark: Rate μ is sharp, but constant c is not.

kinetic Fokker-Planck eq. with non-quadratic potential

$$f_t + v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f = \sigma \Delta_v f + \nu \operatorname{div}_v(vf); \quad x, v \in \mathbb{R}^n$$

steady state factors in x, v : $f_\infty(x, v) = c e^{-\frac{\nu}{\sigma} \left[\frac{|v|^2}{2} + V(x) \right]}$

kinetic Fokker-Planck eq. with non-quadratic potential

$$f_t + v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f = \sigma \Delta_v f + \nu \operatorname{div}_v(vf); \quad x, v \in \mathbb{R}^n$$

steady state factors in x, v : $f_\infty(x, v) = c e^{-\frac{\nu}{\sigma} \left[\frac{|v|^2}{2} + V(x) \right]}$

$$f_t = \operatorname{div}_{x,v} \left[\begin{pmatrix} 0 & -\frac{\sigma}{\nu} \mathbf{I} \\ \frac{\sigma}{\nu} \mathbf{I} & \sigma \mathbf{I} \end{pmatrix} \nabla_{x,v} \left(\frac{f}{f_\infty} \right) f_\infty \right]$$

kinetic Fokker-Planck eq. with non-quadratic potential

$$f_t + v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f = \sigma \Delta_v f + \nu \operatorname{div}_v(vf); \quad x, v \in \mathbb{R}^n$$

steady state factors in x, v : $f_\infty(x, v) = c e^{-\frac{\nu}{\sigma} \left[\frac{|v|^2}{2} + V(x) \right]}$

$$f_t = \operatorname{div}_{x,v} \left[\begin{pmatrix} 0 & -\frac{\sigma}{\nu} \mathbf{I} \\ \frac{\sigma}{\nu} \mathbf{I} & \sigma \mathbf{I} \end{pmatrix} \nabla_{x,v} \left(\frac{f}{f_\infty} \right) f_\infty \right]$$

Theorem 7 (Arnold-Erb, Achleitner-Arnold 2015)

Let $n = 1$; $V(x) = \frac{\omega_0^2}{2}|x|^2 + \tilde{V}(x) \dots$ given confinement potential with $\sqrt{\max V''(x)} - \sqrt{\min V''(x)} \leq \nu$

$$\Rightarrow e_\psi(f(t)|f_\infty) \leq c S_\psi(f_0) e^{-2\kappa t}, \quad t \geq 0$$

kinetic Fokker-Planck eq. with non-quadratic potential

$$f_t + v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f = \sigma \Delta_v f + \nu \operatorname{div}_v(vf); \quad x, v \in \mathbb{R}^n$$

steady state factors in x, v : $f_\infty(x, v) = c e^{-\frac{\nu}{\sigma} \left[\frac{|v|^2}{2} + V(x) \right]}$

$$f_t = \operatorname{div}_{x,v} \left[\begin{pmatrix} 0 & -\frac{\sigma}{\nu} \mathbf{I} \\ \frac{\sigma}{\nu} \mathbf{I} & \sigma \mathbf{I} \end{pmatrix} \nabla_{x,v} \left(\frac{f}{f_\infty} \right) f_\infty \right]$$

Theorem 7 (Arnold-Erb, Achleitner-Arnold 2015)

Let $n = 1$; $V(x) = \frac{\omega_0^2}{2}|x|^2 + \tilde{V}(x) \dots$ given confinement potential with $\sqrt{\max V''(x)} - \sqrt{\min V''(x)} \leq \nu$

$$\Rightarrow e_\psi(f(t)|f_\infty) \leq c S_\psi(f_0) e^{-2\kappa t}, \quad t \geq 0$$

- $\tilde{V} \dots \mathcal{O}(1)$ perturbation
- Greens function not explicit

local vs. global decay rate in non-symmetric FP equations

decay of logarithmic entropy for the non-symmetric FP equation

$$f_t = \operatorname{div} \left[\begin{pmatrix} 1/4 & 0 \\ 0 & 1 \end{pmatrix} \nabla f + \begin{pmatrix} 1/4 x_1 - 4 x_2 \\ 4 x_1 + x_2 \end{pmatrix} f \right]$$

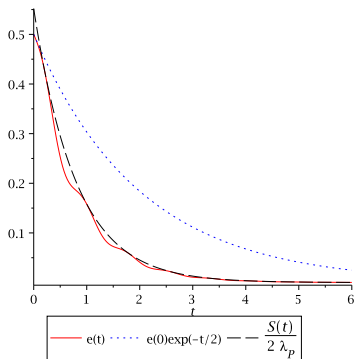
with $f_\infty(x) = c e^{-|x|^2/2} = e^{-A(x)}$: find $e(f(t)|f_\infty) \leq c e(0) e^{-\lambda t}$

local vs. global decay rate in non-symmetric FP equations

decay of logarithmic entropy for the non-symmetric FP equation

$$f_t = \operatorname{div} \left[\begin{pmatrix} 1/4 & 0 \\ 0 & 1 \end{pmatrix} \nabla f + \begin{pmatrix} 1/4 x_1 - 4 x_2 \\ 4 x_1 + x_2 \end{pmatrix} f \right]$$

with $f_\infty(x) = c e^{-|x|^2/2} = e^{-A(x)}$: find $e(f(t)|f_\infty) \leq c e(0) e^{-\lambda t}$



standard entropy method:

BEC $\frac{\partial^2 A}{\partial x^2} = \mathbf{I} \geq \lambda \mathbf{D}^{-1}$ yields sharp

local decay rate: $\lambda_l = \frac{1}{4}$; $c = 1$

“Hypocoercive method” yields sharp

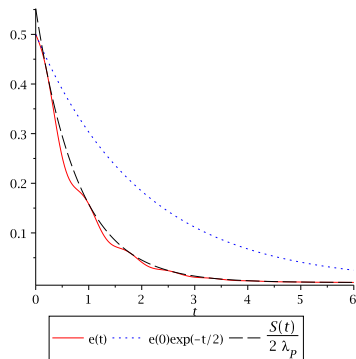
global decay rate: $\lambda_g = \frac{5}{8}$; $c > 1$

local vs. global decay rate in non-symmetric FP equations

decay of logarithmic entropy for the non-symmetric FP equation

$$f_t = \operatorname{div} \left[\begin{pmatrix} 1/4 & 0 \\ 0 & 1 \end{pmatrix} \nabla f + \begin{pmatrix} 1/4 x_1 - 4 x_2 \\ 4 x_1 + x_2 \end{pmatrix} f \right]$$

with $f_\infty(x) = c e^{-|x|^2/2} = e^{-A(x)}$: find $e(f(t)|f_\infty) \leq c e(0) e^{-\lambda t}$



standard entropy method:

BEC $\frac{\partial^2 A}{\partial x^2} = \mathbf{I} \geq \lambda \mathbf{D}^{-1}$ yields sharp

local decay rate: $\lambda_l = \frac{1}{4}$; $c = 1$

“Hypo-coercive method” yields sharp

global decay rate: $\lambda_g = \frac{5}{8}$; $c > 1$

Lemma: 2D, \forall admissible \mathbf{P} :

multiplicative constant $\frac{S(0)}{2\lambda_P}$ is sharp.

Why does the “hypo-coercive method” work?

algebraic essence: **comparison of the spectral gaps** of a non-symmetric matrix \mathbf{Q} and its symmetric part $\mathbf{Q}_s := \frac{1}{2}(\mathbf{Q} + \mathbf{Q}^T)$.

motivation: entropy method operates mostly with quadratic functionals (e.g. $e_2(f|f_\infty)$; $I_\psi = \int \psi''(\frac{f}{f_\infty}) \nabla^T \frac{f}{f_\infty} \cdot \mathbf{D} \cdot \nabla \frac{f}{f_\infty} f_\infty \, dx$)

Why does the “hypocoercive method” work?

algebraic essence: **comparison of the spectral gaps** of a non-symmetric matrix \mathbf{Q} and its symmetric part $\mathbf{Q}_s := \frac{1}{2}(\mathbf{Q} + \mathbf{Q}^T)$.

motivation: entropy method operates mostly with quadratic functionals (e.g. $e_2(f|f_\infty)$; $I_\psi = \int \psi''(\frac{f}{f_\infty}) \nabla^T \frac{f}{f_\infty} \cdot \mathbf{D} \cdot \nabla \frac{f}{f_\infty} f_\infty \, dx$)

Lemma 3

Let \mathbf{Q} be positively stable, i.e. $\mu := \min\{\Re \lambda_Q\} > 0$.

$\Rightarrow \lambda_{\min}(\mathbf{Q}_s) \leq \mu$ (“typically” strict inequality)

Why does the “hypo-coercive method” work?

algebraic essence: **comparison of the spectral gaps** of a non-symmetric matrix \mathbf{Q} and its symmetric part $\mathbf{Q}_s := \frac{1}{2}(\mathbf{Q} + \mathbf{Q}^T)$.

motivation: entropy method operates mostly with quadratic functionals (e.g. $e_2(f|f_\infty)$; $I_\psi = \int \psi''(\frac{f}{f_\infty}) \nabla^T \frac{f}{f_\infty} \cdot \mathbf{D} \cdot \nabla \frac{f}{f_\infty} f_\infty \, dx$)

Lemma 3

Let \mathbf{Q} be positively stable, i.e. $\mu := \min\{\Re \lambda_Q\} > 0$.

$\Rightarrow \lambda_{\min}(\mathbf{Q}_s) \leq \mu$ (“typically” strict inequality)

Lemma 4 (= Lemma 2 for choice of \mathbf{P})

$\exists \mathbf{P} > 0$ such that the similar matrix $\tilde{\mathbf{Q}} := \sqrt{\mathbf{P}}\mathbf{Q}\sqrt{\mathbf{P}}^{-1}$ satisfies:

$$\begin{aligned} \lambda_{\min}(\tilde{\mathbf{Q}}_s) &= \mu \\ \left(\lambda_{\min}(\tilde{\mathbf{Q}}_s) &= \mu - \varepsilon \quad \text{in the defective case} \right) \end{aligned}$$

- So: \exists a similarity transformation such that \mathbf{Q} , $\tilde{\mathbf{Q}}$, $\tilde{\mathbf{Q}}_s$ have the same spectral gap.

Application to $f_t = \operatorname{div}(\mathbf{D}\nabla f + \mathbf{C}xf) = Lf$

- Decay estimate of the drift characteristics $\dot{x} = -\mathbf{C}x$:

Let $\|x\|_P^2 := \langle x, \mathbf{P}x \rangle$.

$$\begin{aligned} \frac{d}{dt} \|x\|_P^2 &= -2x^T \mathbf{P} \mathbf{C} x \\ &= -(\sqrt{\mathbf{P}}x)^T \underbrace{(\sqrt{\mathbf{P}} \mathbf{C} \sqrt{\mathbf{P}}^{-1} + \sqrt{\mathbf{P}}^{-1} \mathbf{C}^T \sqrt{\mathbf{P}})}_{:= 2\tilde{\mathbf{C}}_s \geq 2\mu I} (\sqrt{\mathbf{P}}x) \\ &\leq -2\mu \|x\|_P^2 \end{aligned}$$

- carries over to all invariant eigenspaces of L

Conclusion

- new entropy method for degenerate Fokker-Planck eq. (+ linear drift)
- key tool: modified “entropy dissipation” functional \rightarrow diff. inequal.

Conclusion

- new entropy method for degenerate Fokker-Planck eq. (+ linear drift)
- key tool: modified “entropy dissipation” functional \rightarrow diff. inequal.
- extension to kinetic Fokker-Planck eq. (with non-quadratic potential)
- extension to non-degenerate, non-symmetric Fokker-Planck equations \rightarrow sharp envelope for global decay
- 2017: $(1 + t^2)e^{-2\kappa t}$ decay for defective case [AA-Einav-Wöhrer]

Conclusion

- **new entropy method** for degenerate Fokker-Planck eq. (+ linear drift)
- key tool: modified “entropy dissipation” functional \rightarrow diff. inequal.
- extension to **kinetic Fokker-Planck** eq. (with non-quadratic potential)
- extension to non-degenerate, **non-symmetric Fokker-Planck** equations \rightarrow sharp envelope for global decay
- 2017: $(1 + t^2)e^{-2\kappa t}$ decay for **defective case** [AA-Einav-Wöhrrer]

References

- **A. Arnold, J. Erb**: Sharp entropy decay for hypocoercive and non-symmetric Fokker-Planck equations with linear drift, arXiv 2014
- **F. Achleitner, A. Arnold, D. Stürzer**: Large-time behavior in non-symmetric Fokker-Planck equations, Rivista di Matematica della Univ. di Parma, 2015
- **F. Achleitner, A. Arnold, E. Carlen**: On linear hypocoercive BGK models, Springer Proc. in Math. & Stat., 2016

References:

- [Villani] 2009: exponential decay in weighted H^1 , but no sharp rates
- [Dolbeault-Mouhot-Schmeiser] Trans. AMS 2015: kinetic models, exponential decay in modified L^2 -norm; $m = 1$
- [Gadat-Miclo] KRM 2013: sharp rates for 2 Fokker-Planck toy models
- [Baudoin] 2014: Γ_2 -formalism, includes auxiliary gradient functional