Second order conformally invariant elliptic equations

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Recall

$$\Gamma \subset \mathbb{R}^n \text{ open,convex,symmetric cone,vertex at origin}$$

$$\Gamma_n \subset \Gamma \subset \Gamma_1$$

$$\Gamma_n := \{\lambda \in \mathbb{R}^n \mid \lambda_i > 0 \ \forall \ i\}, \quad \Gamma_1 := \{\lambda \in \mathbb{R}^n \mid \sum_{i=1}^n \lambda_i > 0\}$$

$$f \in C^1(\Gamma) \cap C^0(\overline{\Gamma}) \text{ symmetric function}$$

$$f_{\lambda_i} > 0 \text{ in } \Gamma \ \forall \ i, \quad f > 0 \text{ in } \Gamma, \quad f = 0 \text{ on } \Gamma$$

We will prove:
Proposition. (Comparison principle) Assume $\Omega \subset \mathbb{R}^n$ bounded open, $\{P_1, \cdots, P_m\} \subset \Omega$.

$$0 < u \in C^2(\overline{\Omega}), \quad \lambda(A^u) \in R^n \setminus \Gamma \text{ in } \Omega,$$

$$0 < v \in C^2(\overline{\Omega} \setminus \{P_1, \cdots, P_m\}), \quad \lambda(A^v) \in \overline{\Gamma} \text{ in } \Omega \setminus \{P_1, \cdots, P_m\}.$$

$$v \geq u \text{ on } \partial\Omega.$$

Then

$$v \geq u \text{ in } \overline{\Omega} \setminus \{P_1, \cdots, P_m\}.$$

We will use

Lemma 2-1. Assume $0 \in \Omega \subset \mathbb{R}^n$, open connected, $n \geq 2$, $0 \leq w \in C^2(\overline{\Omega} \setminus \{0\})$, $\Delta w \leq 0$ in $\Omega \setminus \{0\}$. Then

$$\text{Either } w \equiv 0 \text{ or } \liminf_{x \to 0} w(x) > 0.$$

Lemma 2-2. Assume $0 \in \Omega \subset \mathbb{R}^n$, open, $n \geq 2$, $v \in C^2(\Omega \setminus \{0\})$, $w_1, w_2 \in C^1(\Omega)$,

$$v \geq \max\{w_1, w_2\} \text{ in } \Omega \setminus \{0\},$$

$$\liminf_{x \to 0} v(x) = w_1(0) = w_2(0).$$

Then

$$\nabla w_1(0) = \nabla w_2(0).$$

Recall

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$$f \in C^1(\Gamma) \cap C^0(\overline{\Gamma}) \text{ symmetric function}$$

$$f_{\lambda_i} > 0 \text{ in } \Gamma \ \forall\ i, \quad f > 0 \text{ in } \Gamma, \quad f = 0 \text{ on } \Gamma$$

$$A^u := -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \nabla^2 u + \frac{2n}{(n-2)^2} u^{-\frac{2n}{n-2}} \nabla u \otimes \nabla u - \frac{2}{(n-2)^2} u^{-\frac{2n}{n-2}} |\nabla u|^2 I,$$

Proof.

- **Step 1.** Assume

$$0 < a < u < b \text{ in } B_3.$$

Prove

$$|\nabla \log u| \leq C(a, b).$$

- **Bernstein type argument**

Write $v = -\frac{2}{n-2} \log u$.
Then

$$f\big(\lambda(W)\big) = 1 \quad \alpha \leq v \leq \beta, \quad \text{in } B_3,$$

$$W := (W_{ij}) = e^{2v}\left(v_{ij} + v_i v_j - \frac{|\nabla v|^2}{2}\delta_{ij}\right).$$

Need to prove

$$|\nabla v| \leq C(\alpha, \beta) \ \text{ in } B_1.$$

Let $\rho(x)$ be smooth satisfying

$$\rho(x) = 1 \text{ in } B_1, \quad \rho(x) = 0 \text{ outside } B_2, \quad \rho(x) \in (0,1) \text{ in } B_2 \setminus B_1.$$

Consider

$$G := \rho e^{\phi(v)} |\nabla v|^2,$$

where

$$-\frac{1}{2}\phi' \geq c_1 > 0, \quad \phi'' + \phi' - (\phi')^2 \geq 0, \quad \text{on } [\alpha, \beta].$$

Only need to prove

$$G \leq C \ \text{ in } B_2.$$

Fix $x_0 \in B_2$,

$$G(x_0) = \max_{\overline{B}_2} G.$$

Then

$$G_i(x_0) = 0. \tag{1}$$

— involving $\phi(v(x_0)), \phi', \nabla v(x_0), \nabla^2 v(x_0)$.

$$(G_{ij}(x_0)) \leq 0.$$

— involving $\phi(v(x_0)), \phi', \phi'', \nabla v(x_0), \nabla^2 v(x_0), \nabla^3 v(x_0)$.
Apply ∂_k to the equation $f(\lambda(W)) = 1$ and evaluate it at x_0:

$$f_{\lambda_i}(\lambda)\partial_k W_{ii} = 0 \ \text{ at } x_0. \tag{2}$$

— involving $\phi(v(x_0)), \phi', \phi'', \nabla v(x_0), \nabla^2 v(x_0), \nabla^3 v(x_0)$.
Using $f_{\lambda_i} > 0$:

$$f_{\lambda_i} G_{ii}(x_0) \leq 0. \tag{3}$$

— involving $\phi(v(x_0)), \phi', \phi'', \nabla v(x_0), \nabla^2 v(x_0), \nabla^3 v(x_0)$.

Replacing $\nabla^3 v(x_0)$ in (3) by terms upto two derivatives of v, and also use (1) in (3), and using the properties of ϕ, we have

$$c_1 \rho |\nabla v|^2 \leq C + C\sqrt{\rho |\nabla v|^2}.$$

This gives

$$G(x_0) = \rho e^{\phi(v)} |\nabla v|^2 \leq C.$$

Done.